MODULAR FORMS SEMINAR, TALK 6: STACKS.

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Every elliptic curve (i.e., genus 1 smooth projective curve with at least one point) over a field k is isomorphic to the projectivization of the affine curve given by the equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for some $a_1, a_2, a_3, a_4, a_6 \in k$; this polynomial equation is called a *Weierstrass equation*, and a curve given by a Weierstrass equation is called a *Weierstrass curve.* A few useful quantities:

$$\begin{aligned} c_4 &= (a_1^2 + 4a_2)^2 - 24(2a_4 + a_1a_3), \\ c_6 &= -(a_1^2 + 4a_2)^3 + 36(a_1^2 + 4a_2)(2a_4 + a_1a_3) - 216(a_3^2 + 4a_6) \\ \Delta &= \frac{c_4^3 - c_6^2}{1728} \\ \omega &= \frac{dx}{2y + a_1x + a_3}. \end{aligned}$$

If k is of characteristic $\neq 2, 3$, then by a change of coordinates we can reduce to the simpler Weierstrass equation

$$y^2 = x^3 - 27c_4x - 54c_6, \qquad (0.1)$$

and then ω, which is an algebraically-nice choice of generator for the module of differential 1-forms on the elliptic curve, takes the especially simple form $\frac{dx}{2y}$.

While every elliptic curve is isomorphic to (the projectivization of) a Weierstrass curve, not every Weierstrass curve is an elliptic curve. The trouble is that a Weierstrass curve can have a singular point: a Jacobian calculation shows that you this happens if and only if $\Delta = 0$. So, for every field k, you have a surjection from $\hom_{\text{Comm Rings}}(\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}], k)$ to the set of elliptic curves over k.

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Better: let $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$, and let $\Gamma = A[r, s, t]$, with ring maps

$$\begin{aligned}
\eta_L &: A \to \Gamma \\
\eta_L(a_i) &= a_i \text{ for all } i \\
\epsilon &: \Gamma \to A \\
\epsilon(r) &= 0 \\
\epsilon(s) &= 0 \\
\epsilon(t) &= 0 \\
\eta_R &: A \to \Gamma \\
\eta_R(a_1) &= a_1 + 2s \\
\eta_R(a_2) &= a_2 - a_1 s + 3r - s^2 \\
\eta_R(a_3) &= a_3 + a_1 r + 2t \\
\eta_R(a_4) &= a_4 - a_3 s + 2a_2 r - a_1 t - a_1 rs - 2st + 3r^2 \\
\eta_R(a_6) &= a_6 + a_4 r + a_3 t + a_2 r^2 - a_1 rt - t^2 - r^3 \\
\Delta &: \Gamma \to \Gamma \otimes_A \Gamma \\
\Delta(s) &= s \otimes 1 + 1 \otimes s \\
\Delta(r) &= r \otimes 1 + 1 \otimes r \\
\Delta(t) &= t \otimes 1 + 1 \otimes t + s \otimes r
\end{aligned}$$

For any field k, $\mathcal{Ob}(k) := \hom_{\text{Comm Rings}}(A, k)$ and $\mathcal{Mor}(k) := \hom_{\text{Comm Rings}}(\Gamma, k)$ are each sets, and the maps η_L, η_R, and Δ induce maps

$$\begin{aligned}
\eta_L^* &: \mathcal{Mor}(k) \to \mathcal{Ob}(k) \\
\eta_R^* &: \mathcal{Mor}(k) \to \mathcal{Ob}(k) \\
\epsilon^* &: \mathcal{Ob}(k) \to \mathcal{Mor}(k) \\
\Delta^* &: \mathcal{Mor}(k) \times_{\mathcal{Ob}(k)} \mathcal{Mor}(k) \to \mathcal{Mor}(k),
\end{aligned}$$

which define the source, target, identity, and composition maps that form a groupoid with object set $\mathcal{Ob}(k)$ and morphism set $\mathcal{Mor}(k)$. The morphisms in this groupoid are, algebraically, exactly the isomorphisms of algebraic curves from one Weierstrass curve to another; that is, the isomorphism classes in the groupoid $(\mathcal{Ob}(k), \mathcal{Mor}(k))$ are precisely the isomorphism classes, in the category of algebraic varieties, between Weierstrass curves over k. It follows that $(A[\Delta^{-1}], \Gamma[\Delta^{-1}]$ represents a functor $\mathcal{F}$: Comm Rings $\to$ Groupoids such that, for each field k, the groupoid $\mathcal{F}(k)$ is equivalent to (but not necessarily *isomorphic* to!) the groupoid of elliptic curves over k and isomorphisms between elliptic curves.

Applying Spec[1], we have a groupoid object $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ in schemes. This is very nearly the same thing as an algebraic stack. Given a base scheme X, a *Grothendieck topology on* Aff/X is a collection τ of families of maps $\{Y_i \to Y\}_i$ in Aff/X, called *covers in* τ, which is closed under composition and pullback and such that, for every isomorphism $Y' \xrightarrow{\sigma} Y$ in Aff/X, the one-element family $\{Y' \xrightarrow{\sigma} Y\}$ is in τ. Motivating examples are:

- the Zariski topology, where a family of maps $\{Y_i \xrightarrow{f_i} Y\}_i$ in Aff/X is a cover iff each f_i is an open immersion and $\cup_i \operatorname{im} f_i = Y$ (in other words, Zariski covers are ordinary, plain old covers of schemes by open subschemes),
- the étale topology, where a family of maps $\{Y_i \xrightarrow{f_i} Y\}_i$ in Aff/X is a cover iff each f_i is finite-type and étale with open image in Y, and $\cup_i \operatorname{im} f_i = Y$ (in other words, étale covers are Zariski covers in which you're allowed to then take a finite-to-one unramified cover of any of your open subschemes),
- and the fpqc topology, where a family of maps $\{Y_i \xrightarrow{f_i} Y\}_i$ in Aff/X is a cover iff each f_i is quasicompact (i.e., preimages of affine opens in the codomain are unions of finitely many affine opens in the domain) and the universal map $f : \coprod_i Y_i \to Y$ is faithfully flat. (That is, the induced pullback functor on quasicoherent sheaves $f^* : \operatorname{Mod}(\mathcal{O}_Y) \to \operatorname{Mod}(\mathcal{O}_{\coprod_i Y_i})$ is faithful and exact.)

The individual maps $f_i : Y_i \to Y$ in covers in a Grothendieck topology τ are called the *opens* in τ. Given a Grothendieck topology τ on Aff/X, a presheaf of abelian groups $\mathcal{F} : (\operatorname{Aff}/X)^{\mathrm{op}} \to \mathrm{Ab}$ is a *sheaf* if, for every cover $\{Y_i \to Y\}_i$ in τ, the sequence

$$\mathcal{F}(Y) \longrightarrow \prod_i \mathcal{F}(Y_i)[r] \longrightarrow \prod_{i,j} \mathcal{F}(Y_i \times_Y Y_j) \tag{0.2}$$

is an equalizer sequence, that is, $\mathcal{F}(Y)$ is the kernel of the difference of $\mathcal{F}$ applied to the two projection maps on $Y_i \times_Y Y_j$. When τ is the Zariski topology, this agrees with the usual notion of a sheaf of abelian groups on X. (Actually, this is a slight lie: you have to restrict from $\operatorname{Aff}(X)$ to just the affine schemes over X which are members of some Zariski cover of X, i.e., you have to restrict to the full subcategory of Aff/X generated by the opens. That's called the *small Zariski site*, as opposed to the *big Zariski site* which consists of a topology on all of Aff/X, as defined above. But the difference between small and big sites is unimportant for

[1] At this part in these notes I can't help but assume the audience has some very basic familiarity with schemes. I wrote a version of these notes which didn't assume that, and went through much of the whole apparatus of differential forms, divisors, Riemann-Roch, and the genus classification for smooth algebraic curves over arbitrary fields, and then realized that if I start talking about these things in the seminar, it will take so long that we won't get to talk about much else. So I will assume the audience already knows what schemes are, and what their structure sheaves and quasicoherent modules are; and I'll assume that the audience either knows what the genus of a smooth algebraic curve over an arbitrary field is, or can accept that the notion of genus of smooth curves over $\mathbb{C}$ has a reasonable generalization to smooth curves over other fields. (The notion of genus is being used implicitly when we talk about elliptic curves over arbitrary fields: while you can define an elliptic curve over $\mathbb{C}$ as a quotient $\mathbb{C}/\Lambda$ with Λ a lattice in $\mathbb{C}$, it is very difficult to make sense of a definition like that if we replace $\mathbb{C}$ with a field like $\mathbb{F}_2(x)$. But the definition "an elliptic curve over k is a smooth algebraic curve over k, of genus 1, with at least one point" works just fine for any field k. You could also define an elliptic curve over k as a one-dimensional abelian variety over k, but this involves using the group structure on an elliptic curve, which we haven't quite gotten to in the seminar yet.)

most purposes: for example, the inclusion of sheaves of abelian groups on the small Zariski (or étale, or fpqc) site into the sheaves of abelian groups on the big site induces an isomorphism in all sheaf cohomology groups.) A sheaf of sets on X in the topology τ is defined the same way: the presheaf $\mathcal{F}$ just takes values in sets rather than abelian groups.

Maybe now is a good time to mention for that any sheaf $\mathcal{F}$ on X, in any topology τ, we can evaluate $\mathcal{F}$ on X: we write $\Gamma(\mathcal{F})$ (read as *global sections of* $\mathcal{F}$) for this evaluation $\mathcal{F}(X)$, and Γ is then a functor $\Gamma : \mathrm{Ab}\,\mathrm{Sh}_\tau(X) \to \mathrm{Ab}$, from sheaves of abelian groups on X in the topology τ, to abelian groups. The *nth sheaf cohomology of* $\mathcal{F}$, written $H^n_\tau(X;\mathcal{F})$, is the nth right-derived functor of Γ on the category $\mathrm{Ab}\,\mathrm{Sh}_\tau(X)$, applied to $\mathcal{F}$. (Since $\mathrm{Ab}\,\mathrm{Sh}_\tau(X)$ is a Grothendieck category, it has enough injectives, so right-derived functors are defined.) Of course when τ is the Zariski topology, the resulting sheaf cohomology groups $H^n_{\mathrm{Zar}}(X;\mathbb{F})$ are just the classical sheaf cohomology groups $H^n(X;\mathbb{F})$. It's a nice exercise to work out how to extract a sheaves of abelian groups on the étale site of $\operatorname{Spec} k$ from every continuous k-representation of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, and to work out the isomorphism between the étale cohomology groups $H^n_{et}(\operatorname{Spec} k;\mathbb{F})$ of that sheaf, and the continuous group cohomology $H^n_c(\mathrm{Gal}(k^{\mathrm{sep}}/k);M)$ of the continuous k-representation M of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. So, for example, when k is a finite field, we get $k^{\mathrm{sep}} = \overline{k}$ and so $\mathrm{Gal}(k^{\mathrm{sep}}/k) \cong \mathrm{Gal}(\overline{k}/k) \cong \hat{\mathbb{Z}}$, and so we can really compute H^*_{et}, e.g.

$$H^1_{et}(\operatorname{Spec}\mathbb{F}_p;\mathbb{F}) \cong H^1_c(\mathrm{Gal}(\mathbb{F}_p^{\mathrm{sep}}/\mathbb{F}_p);M) \cong \operatorname{colim}_m H^1\left(\mathbb{Z}/m\mathbb{Z}; M^{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}\right)$$

$$\cong \left\{x \in M : \exists n \text{ s.t. } \mathrm{Fr}_p^n x + \mathrm{Fr}_p^{n-1} x + \dots + \mathrm{Fr}_p x + x = 0\right\} / \left\{x - \mathrm{Fr}_p x : x \in M\right\},$$

where $\mathrm{Fr}_p \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is the Frobenius (pth power) automorphism of $\overline{\mathbb{F}}_p$.

There are a few ways to define a *stack over X in the topology τ*. Here is a way which is pretty good if you have a topological perspective on things: you know what a sheaf of sets on X in the topology τ is, and of course it makes sense talk about presheaves of *groupoids* on Aff/X. Where it gets interesting is the sheaf axiom: if $\{Y_i \xrightarrow{f_i} Y\}_i$ is a cover in τ and $\mathcal{F} : (\mathrm{Aff}/X)^{\mathrm{op}} \to \mathrm{Groupoids}$ is a functor, you could ask for (0.2) to be an equalizer sequence in Groupoids, but this is too rigid: important examples (e.g. if $\mathcal{F}(\operatorname{Spec} R)$ is the groupoid of elliptic curves over R and their isomorphisms) fail to be sheaves of groupoids in this sense. Instead, we want to take advantage of the fact that groupoids have some "wobbliness": you can regard the isomorphisms in a groupoid as telling you about *homotopies* between objects in the groupoid—and consequently homotopies between morphisms between groupoids—and you can instead ask for the sheaf axiom (0.2) to hold *up to homotopy*. That's what a stack (in the topology τ) is: it's an up-to-homotopy sheaf of groupoids on Aff/X in the topology τ. More precisely: a presheaf of groupoids on Aff/X is a *stack in the topology* τ[2] if, for every cover $\{Y_i \xrightarrow{f_i} Y\}_i$ is a cover in

[2]Even this definition is an oversimplification, since a stack is really only a "sheaf up-to-homotopy" in two senses: in the sense that the sheaf axiom is replaced by an up-to-homotopy sheaf axiom (which is the totalization condition that I give in these notes,) but also in the sense that *even the presheaf itself is only defined up to homotopy*: given morphisms $U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3$ of schemes over the base scheme X, we ask for the morphism of groupoids $\mathcal{F}(g \circ f) : \mathcal{F}(U_3) \to \mathcal{F}(U_1)$ to be *homotopic* to $\mathcal{F}(f) \circ \mathcal{F}(g) : \mathcal{F}(U_3) \to \mathcal{F}(U_1)$, but we don't require them to be equal, so $\mathcal{F}$ is not actually a functor! The "presheaves which satisfy the sheaf axiom up to homotopy" approach I am taking in these notes is really only halfway between stacks and groupoid schemes

τ, the natural map from $\mathcal{F}(Y)$ to the totalization of the cosimplicial groupoid

$$\times_i \mathcal{F}(Y_i) \rightrightarrows \times_{i,j}\mathcal{F}(Y_i \times_Y Y_j) \rightrightarrows \times_{i,j,k}\mathcal{F}(Y_i \times_Y Y_j \times Y_k) \rightrightarrows \dots \tag{0.3}$$

is an equivalence (not necessarily an isomorphism!) of groupoids. A morphism of stacks is simply a natural transformation of presheaves of groupoids. The totalization of a cosimplicial set

$$X_0 \rightrightarrows X_1 \rightrightarrows X_2 \rightrightarrows \dots$$

is simply the equalizer of the two maps $X_0 \to X_1$, so if $\mathcal{F}$ is just a presheaf of *sets*, then to be a stack in the topology τ is simply to satisfy exactly the classical, familiar sheaf axiom, i.e., for (0.2) to be an equalizer sequence for every cover $\{Y_i \to Y\}_i$.

Every scheme Y represents a presheaf of sets $\hom_{\mathrm{Sch}/X}(-, Y) : (\mathrm{Aff}/X)^{\mathrm{op}} \to \mathrm{Sets}$, and we say that the topology τ is *subcanonical* if every presheaf of sets on $(\mathrm{Aff}/X)^{\mathrm{op}}$ representable by a scheme Y is in fact a sheaf in the topology τ. ("Subcanonical" because such topologies are exactly the topologies in which every cover is a universally effective epimorphic family, and the topology in which the covers are exactly the UEEFs is called the "canonical" topology.) If τ is subcanonical, then composing $\hom_{\mathrm{Sch}/X}(-, Y) : (\mathrm{Aff}/X)^{\mathrm{op}} \to \mathrm{Sets}$ with the inclusion $\mathrm{Sets} \subseteq \mathrm{Groupoids}$ gives us a stack in the topology τ, so sending Y to $\hom_{\mathrm{Sch}/X}(-, Y)$ embeds the category of schemes over X into the category of stacks over X in the topology τ. A stack $\mathcal{F}$ in the topology τ is called *algebraic* if there exists a morphism of stacks $\mathrm{Spec}\, A \to \mathcal{F}$ such that, for every affine scheme $\mathrm{Spec}\, R$ and every map of stacks $\mathrm{Spec}\, R \to \mathcal{F}$, the pullback $\mathrm{Spec}\, R \times_{\mathcal{F}} \mathrm{Spec}\, A$ is a scheme and the pullback map $\mathrm{Spec}\, R \times_{\mathcal{F}} \mathrm{Spec}\, A \to \mathcal{F}$ is smooth and quasicompact and is a cover in the topology τ. (There is some variation between different sources on the exact assumptions to be made here; for example, some sources assume that $\mathrm{Spec}\, R \times_{\mathcal{F}} \mathrm{Spec}\, A$ is only required to be an algebraic space in the topology τ. That's fine but requires we talk about algebraic spaces, which is not difficult, but is just another layer of abstraction that for the moment we can safely skirt.) Algebraic stacks in the étale topology are often called *Deligne-Mumford stacks*, or *DM stacks*; algebraic stacks in flat topologies (e.g. the fpqc topology) are often called *Artin stacks*.

Now suppose you have an algebraic stack $\mathcal{F}$ in a reasonable[3] (e.g., Zariski or étale or fpqc) subcanonical topology τ, and suppose that the diagonal morphism $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is an affine morphism; this last condition is equivalent to saying that, for every (equivalently, any!) cover $\mathrm{Spec}\, A \to \mathcal{F}$, the pullback $\mathrm{Spec}\, R \times_{\mathcal{F}} \mathrm{Spec}\, A$ is an *affine* scheme. Consider what happens when $R = A$: we have affine schemes $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, A \times_{\mathcal{F}} \mathrm{Spec}\, A$, two projection maps $\mathrm{Spec}\, A \times_{\mathcal{F}} \mathrm{Spec}\, A \to \mathrm{Spec}\, A$, a diagonal map $\mathrm{Spec}\, A \to \mathrm{Spec}\, A \times_{\mathcal{F}} \mathrm{Spec}\, A$ coming from the universal property of the pullback, a factor swap map $\mathrm{Spec}\, A \times_{\mathcal{F}} \mathrm{Spec}\, A \to \mathrm{Spec}\, A \times_{\mathcal{F}} \mathrm{Spec}\, A$, and a

(e.g. commutative Hopf lalgebroids, explained below). If you want the real story on stacks, it's the book *Champs algébriques* by Laumon and Moret-Bailly.

[3]The technical meaning of "reasonable" here is that $\mathcal{F}$ needs to satisfy enough of Artin's additional axioms for Grothendieck topologies so that a presheaf will satisfy the sheaf axiom for all covers $\{Y_i \to \mathcal{F}\}_i$ in τ if it satisfies the sheaf axiom for for all one-element covers $\coprod_i Y_i \to Y$ in τ.

composite map

$$\begin{aligned}(\operatorname{Spec} A \times_{\mathcal{F}} \operatorname{Spec} A) \times_{\operatorname{Spec} A} (\operatorname{Spec} A \times_{\mathcal{F}} \operatorname{Spec} A) &\xrightarrow{\cong} \operatorname{Spec} A \times_{\mathcal{F}} \operatorname{Spec} A \times_{\mathcal{F}} \operatorname{Spec} A \\ &\rightarrow \operatorname{Spec} A \times_{\mathcal{F}} \mathcal{F} \times_{\mathcal{F}} \operatorname{Spec} A \\ &\xrightarrow{\cong} \operatorname{Spec} A \times_{\mathcal{F}} \operatorname{Spec} A\end{aligned}$$

Since $\operatorname{Spec} A$ and $\operatorname{Spec} A \times_{\mathcal{F}} \operatorname{Spec} A$ are affine schemes, we lose no information by taking their rings of global sections; I will call them A and Γ respectively. Then the above five maps give us ring homomorphisms

$$\begin{aligned}\eta_L &: A \rightarrow \Gamma \\ \eta_R &: A \rightarrow \Gamma \\ \epsilon &: \Gamma \rightarrow A \\ \chi &: \Gamma \rightarrow \Gamma \\ \Delta &: \Gamma \rightarrow \Gamma \otimes_A \Gamma,\end{aligned}$$

with η_L, η_R each smooth covers in τ, and these homomorphisms satisfy some compatibility properties with respect to one another: with a little work you find that they are exactly the compatibility properties to make the 7-tuple $(A, \Gamma, \eta_L, \eta_R, \epsilon, \chi, \Delta)$ into a *cogroupoid object in commutative rings,* that is, for every commutative ring R, the two sets $\hom_{\text{Comm Rings}}(A, R), \hom_{\text{Comm Rings}}(\Gamma, R)$ and the five ring homomorphisms connecting them are exactly the structure maps to define a groupoid, with object set $\hom_{\text{Comm Rings}}(A, R)$, morphism set $\hom_{\text{Comm Rings}}(\Gamma, R)$, domain map (i.e., the map that sends a morphism in the groupoid to its domain object) $\hom_{\text{Comm Rings}}(\eta_L, R)$, codomain map $\hom_{\text{Comm Rings}}(\eta_R, R)$, identity map (i.e., the map that sends an object in the groupoid to its identity morphism) $\hom_{\text{Comm Rings}}(\epsilon, R)$, inversion map (i.e., the map that sends a morphism in the groupoid to its inverse) $\hom_{\text{Comm Rings}}(\chi, R)$, and composition map the composite of the isomorphism $\hom_{\text{Comm Rings}}(\Gamma, R) \times_{\hom_{\text{Comm Rings}}(A,R)} \hom_{\text{Comm Rings}}(\Gamma, R) \xrightarrow{\cong} \hom_{\text{Comm Rings}}(\Gamma \otimes_A \Gamma, R)$ with $\hom_{\text{Comm Rings}}(\Delta, R)$. Such a 7-tuple is called a *Hopf algebroid.*

Conversely, given a Hopf algebroid $(A, \Gamma, \eta_L, \eta_R, \epsilon, \chi, \Delta)$ with η_L, η_R smooth covers in a reasonable subcanonical topology τ, the presheaf of groupoids on Aff sending each affine scheme X to the groupoid with object set $\hom_{\text{Sch}}(X, \operatorname{Spec} A)$, morphism set $\hom_{\text{Sch}}(X, \operatorname{Spec} \Gamma)$, and structure maps given by $\eta_L, \eta_R, \epsilon, \chi, \Delta$, is not necessarily a stack in the topology τ, but we can stackify it (the up-to-homotopy version of sheafification; if you write down the classical definition of sheafification of a presheaf as the sheaf whose values in an open Y is the colimit of the equalizer of (0.2) over finer and finer covers $\{Y_i \rightarrow Y\}_i$ of Y, and then replace the equalizer of (0.2) with the totalization of (0.3), you get stackification). That stack is algebraic, with affine cover given by $\operatorname{Spec} A$. In the end, for each commutative ring B, you get an equivalence between algebraic stacks over $\operatorname{Spec} B$ with affine diagonal, and Hopf algebroids of commutative B-algebras whose maps η_L, η_R are smooth covers in τ. Quasicoherent modules over the structure sheaf of the stack correspond to comodules over the Hopf algebroid, and consequently you can compute the sheaf cohomology $H^n_\tau(\mathcal{X}; \mathcal{F})$ of an algebraic stack $\mathcal{X}$ with affine diagonal, with coefficients in a quasicoherent sheaf $\mathcal{F}$, by calculating $\operatorname{Ext}^n_{(A,\Gamma)-\text{Comod}}(A, \mathcal{F}(A))$, where (A, Γ) is the Hopf algebroid associated to $\mathcal{X}$.

The Weierstrass Hopf algebroid $(\mathbb{Z}[a_1, a_2, a_3, a_4, a_6], \mathbb{Z}[a_1, a_2, a_3, a_4, a_6][r, s, t])$ defined above is an example of a Hopf algebroid: the maps η_L and η_R are smooth and faithfully flat, so the Weierstrass Hopf algebroid defines an Artin stack, i.e., an algebraic stack in the fpqc topology, with affine diagonal. (The quasicompactness condition in the definition of an fpqc cover is automatic in this case: under an affine morphism, the preimage of an affine open is an affine open, so affine morphisms are always quasicompact.) This is equivalent to the stack of Weierstrass curves, that is, the stack that sends a commutative ring R to the groupoids with objects the set of Weierstrass curves, and with morphisms the isomorphisms (of algebraic varieties) between Weierstrass curves. (The variables r, s, t in the Weierstrass Hopf algebroid are the generating algebraic changes-of-coordinate on Weierstrass curves, i.e., they generate the isomorphisms of algebraic varieties between Weierstrass curves.) Similarly, the Δ-inverted Weierstrass Hopf algebroid $\left(\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}], \mathbb{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}][r, s, t]\right)$ defined above is an example of a Hopf algebroid: the maps η_L and η_R are smooth and faithfully flat, so the Δ-inverted Weierstrass Hopf algebroid again defines an Artin stack. This is equivalent to the stack of Weierstrass curves with Δ invertible, which in turn is equivalent to the stack of *elliptic curves with good reduction*, that is, the elliptic curves over a ring R which remain elliptic curves (i.e., they remain smooth) upon reduction modulo every ideal in I. Of course, if R is a field, being an elliptic curves with good reduction is just the same thing as being an elliptic curve; but over $\mathbb{Z}$, for example, there are Weierstrass curves like $y^2 = x^3 + x$ which are smooth, and whose reduction to $\mathbb{F}_p$ is smooth for most primes p, but whose reduction to $\mathbb{F}_p$ is singular for some p: in the case $y^2 = x^3 + x$, over fields of characteristic 2 the Jacobian of the curve is $\begin{bmatrix} 3x^2 + 1 & -2y \end{bmatrix} = \begin{bmatrix} x^2 + 1 & 0 \end{bmatrix}$, so the curve is singular at the point $(1, 0)$.