MODULAR FORMS SEMINAR, TALK 2: SPECIAL VALUES, PART 1.

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Last week we heard about the Riemann zeta-function. As Luca said, I will say a bit about the special values of the Riemann zeta-function at negative integers, that is, the values that $\zeta(n)$ takes when n is an integer. *This talk is supposed to survey some conjectural and some known properties of special values of L-functions. Sometimes I am going to talk about objects we have not yet defined in the seminar, like algebraic K-groups, and L-functions of modular forms. I will always try to make statements in which you can safely, temporarily treat any object not yet defined as a "black box." If you want more details on any of these "black boxes," please speak up in the seminar and I will gladly say more immediately, instead of putting off careful definitions until later!* Also, I wrote these notes in one day, so I apologize for all the typos that there probably are; but I did put some care into the table (0.2), below, and I hope I was successful in stamping out any typos that might be in there.

By a contour integration argument one shows that, when n is a positive integer, $\zeta(1-n)$ is equal to $\frac{-1}{n}$ times the nth Maclaurin coefficient of $\frac{te^t}{e^t-1}$. (I won't try to talk about this contour integration argument today, but I would like to talk about it in a later talk in this seminar, because it's an essential point whether one can make a similar argument, replacing the Riemann zeta-function with the L-function of a Maass form; the big question is what the Bernoulli numbers would be replaced by, in that setting.) The Maclaurin coefficients of $\frac{te^t}{e^t-1}$ have a name: they are called the *Bernoulli numbers*, and they arise in many places in mathematics. I will write B_n for the nth Bernoulli number, so that

$$\sum_{n\geqslant 0} \frac{B_n}{n!}t^n = \frac{te^t}{e^t-1}. \tag{0.1}$$

Historically, the Bernoulli numbers were first studied by Johann Faulhaber and Jakob Bernoulli, in trying to write a formula for the sum of the jth powers of the first k positive integers. This led to Bernoulli's 1713 formula

$$\sum_{n=1}^{k} n^j = \frac{1}{j+1}\sum_{n=0}^{j}\binom{j+1}{n}B_n k^{j+1-n}.$$

The values of the numbers B_n can be computed easily by power series methods, from (0.1). The first few values, along with the corresponding special values of $\zeta(1-n)$ and $\zeta(n)$ and, for reasons I haven't yet explained, the stable homotopy groups of spheres and the algebraic K-groups of the integers. For the rest of this talk I would like to sketch what is known and what is conjectured about the curious

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coincidences in this table (and generalizations of those curious coincidences).
(0.2)

n	B_n	$\zeta(1-n)$	$\zeta(n)$	$\pi^{st}_{2n-1}(S^0)$	$K_{2n-1}(\mathbb{Z})$	$K_{2n-2}(\mathbb{Z})$
0	1	(pole)	$\frac{-1}{2}$	0	0	0
1	$\frac{1}{2}$	$\frac{-1}{2}$	(pole)	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^\times \cong \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$
2	$\frac{1}{6}$	$\frac{-1}{12}$	$\frac{\pi^2}{6}$	$\mathbb{Z}/24\mathbb{Z}$	$\mathbb{Z}/48\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	0	0	irr. (Apery)	0	$\mathbb{Z}$	0 (Rognes)
4	$\frac{-1}{30}$	$\frac{1}{120}$	$\frac{\pi^4}{90}$	$\mathbb{Z}/240\mathbb{Z}$	$\mathbb{Z}/240\mathbb{Z}$	0
5	0	0	irr.?	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	0?
6	$\frac{1}{42}$	$\frac{-1}{252}$	$\frac{\pi^6}{945}$	$\mathbb{Z}/504\mathbb{Z}$	$\mathbb{Z}/1008\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
7	0	0	irr.?	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}$	0?
8	$\frac{-1}{30}$	$\frac{1}{240}$	$\frac{\pi^8}{9450}$	$\mathbb{Z}/480\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/480\mathbb{Z}$	0
9	0	0	irr.?	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	0?
10	$\frac{5}{66}$	$\frac{-1}{132}$	$\frac{\pi^{10}}{93555}$	$\mathbb{Z}/264\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/528\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
11	0	0	irr.?	0	$\mathbb{Z}$	0?
12	$\frac{-691}{2730}$	$\frac{691}{32760}$	$\frac{691\pi^{12}}{638512875}$	$\mathbb{Z}/8190\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/65520\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/691\mathbb{Z}$

Entries marked with "irr.?" are conjectured (but not known) to be irrational, and in fact transcendental. Entries marked 0? are conjectured (but not known) to be trivial; the vanishing of $K_{4n}(\mathbb{Z})$ for all integers $n > 0$ is equivalent to Vandiver's conjecture from number theory.

The appearance of 691 in $K_{22}(\mathbb{Z})$ and in the numerator of $\zeta(-11)$ is due to the irregularity of the prime 691, and Kummer's work on Fermat's Last Theorem. This is a story for another time!

- The relationship between the $\zeta(n)$ and the $\zeta(1-n)$ columns in (0.2) is simply given by the functional equation for $\zeta(s)$: unwinding the functional equation $\Lambda(s) = \Lambda(1-s)$ from Luca's talk, we have

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s), \tag{0.3}$$

  and when s is an odd negative integer, we can use $\Gamma(1-s) = (-s)!$ and $\zeta(1-s) = \frac{-B_s}{s}$ and (0.3) to express $\zeta(1-s)$ as a power of π times a rational number. When s is an even negative integer, the vanishing of $\sin\left(\frac{\pi s}{2}\right)$ tells us about the trivial zeroes of $\zeta(s)$, which Luca told us about; but the vanishing of $\sin\left(\frac{\pi s}{2}\right)$ also tells us that the functional equation for $\zeta(s)$ is not going to give us a formula for $\zeta(s)$ as a rational number times a power of π. It remains an open conjecture that $\zeta(s)$ is transcendental, when s is a positive integer > 1; the strongest known result in this direction is the irrationality of $\zeta(3)$, by a theorem of Apery from 1978 (but it is still not known that $\zeta(3)$ is transcendental!). (It is also expected that, unlike the case of even positive integers, $\zeta(s)$ is probably *not* be a power of π times a rational number, when s is an odd positive integer.)
- The relationship between the $\zeta(1-n)$ column and the $\pi^{st}_{2n-1}(S^0)$ column is given by Adams' computation of the image of the J-homomorphism, from 1965. Before I explain more, maybe I should mention that the nth stable homotopy group of a pointed topological space X is, by definition, the direct limit of the sequence

$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \dots, \tag{0.4}$$

where Σ is the operation on pointed topological spaces given by $\Sigma X = \frac{S^1 \times X}{S^1 \vee X}$; since you have a natural homeomorphism $\Sigma S^n \cong S^{n+1}$, for each map $f : S^n \to X$ you get a map $\Sigma f : S^{n+1} \cong \Sigma S^n \to \Sigma X$, and that's where the maps in (0.4) come from.

There is a certain homomorphism, called the *Whitehead J-homomorphism*, from the (unstable, classical) homotopy groups of the infinite special orthogonal group $SO = \cup_n SO(n)$ to the stable homotopy groups of S^0. The J-homomorphism is easy to define: given a continuous basepoint-preserving map $f : S^i \to SO(n)$, let $\hat{f} : S^i \times D^n \to D^n$ be given by $\hat{f}(x, y) = f(x)(y)$, where $f(x) : D^n \to D^n$ is the map given by simply applying the matrix $f(x) \in SO(n)$ to the points in D^n, by matrix multiplication. Now $S^i \times D^n$ sits inside S^{i+n} in a natural geometric way (you just have to glue the "end-caps" on to $S^i \times D^n$ to turn it into the smash product $S^i \wedge S^n$, which is homeomorphic to S^{i+n}) and $J(f) : S^{i+n} \to S^n$ is defined to be, on $S^i \times D^n \subseteq S^{i+n}$, the composite of $\hat{f} : S^i \times D^n \to D^n$ with the collapse-the-boundary projection $D^n \to S^n$; and $J(f)$ sends everything in S^{i+n} outside of $S^i \times D^n$ to the basepoint in S^n. (It is true, but not obvious, that $J(f)$ is indeed well-defined, and also homotopy-invariant: if f_0 is homotopic to f_1, then $J(f_0)$ is homotopic to $J(f_1)$.) Since spheres are compact, every continuous map $S^i \to SO$ factors through one of the subspaces $SO(n) \subseteq SO$, so the above construction in fact gives a map from $\pi_i(SO)$ to $\pi_{i+n}(S^n)$, and by composing it with the suspension maps

$$\pi_{i+n}(S^n) \to \pi_{i+n+1}(S^{n+1}) \to \pi_{i+n+2}(S^{n+2}) \to \dots \tag{0.5}$$

we get a homomorphism from $\pi_i(SO)$ to the direct limit of (0.5), i.e., we get a homomorphism of abelian groups $J : \pi_i(SO) \to \pi_i^{st}(S^0)$.

In 1959, Bott famously computed the homotopy groups of SO, O, and U: these were the first "geometrically natural" topological spaces with nonvanishing homotopy groups in arbitrary large degrees for which the homotopy groups were successfully computed in all degrees. For the special orthogonal group, Bott's result is:

$$\pi_i(SO) \cong \begin{cases} \mathbb{Z} & \text{if } i \equiv 3 \mod 4 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i \equiv 0, 1 \mod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\pi_{4n-1}(SO) \cong \mathbb{Z}$, the image of the J-homomorphism $J_{4n-1} : \pi_{4n-1}(SO) \to \pi_{4n-1}^{st}(S^0)$ in any given degree congruent to 3 mod 4 is a cyclic group which could, a priori, be of any finite order; it cannot be infinite since Serre proved already in 1953 that $\pi_n^{st}(S^0)$ is finite for all $n > 0$. Adams computed the order of the image of J in those degrees, up to a factor of 2, and arrived at the formula

$$\#(\text{im } J_{4n-1}) = \text{denom}\, \zeta(1 - 2n). \tag{0.6}$$

(Adams also made a conjecture about what should happen with that factor of 2; this was the Adams conjecture, proven in the early 1970s by Quillen and independently by Sullivan.)

Adams' proof, however, only involved the Riemann zeta-function incidentally: the *von Staudt-Clausen theorem*, from 1840, identifies the denominator of B_n as the product of all the primes p such that $p - 1$ divides

n:

$$\mathrm{denom}(B_n) = \prod_{(p-1)\mid n} p.$$

Since $\zeta(1-n) = \frac{-B_n}{n}$, you get a similar formula for $\mathrm{denom}(\zeta(1-n))$ if you have some control over how factors of p in n could potentially cancel with factors of p in the *numerator* of B_n, when you form the quotient $\frac{B_n}{n}$. You get this control from an 1845 theorem of von Staudt: if $p-1$ does not divide n, then the p-adic valuation of B_n is at least as large as the p-adic valuation of n. This is exactly what you need, along with the von Staudt-Clausen theorem, to get the formula

$$\mathrm{denom}\,\zeta(1-n) = \prod_{(p-1)\mid n} p^{1+\nu_p(n)}.$$

(Remember that $\nu_p(n)$ is defined as the largest integer N such that p^N divides n.) Adams used algebraic methods to calculate the order of $\mathrm{im}\ J_{4n-1}$ localized at p, for each prime p; since $\mathrm{im}\ J_{4n-1}$ is a finite abelian group, its order is simply the product of the order of its p-localization at each prime p, and Adams calculated that the order of the p-localization of $\mathrm{im}\ J_{4n-1}$ was $p^{1+\nu_p(n)}$ for all odd primes p. So you get formula (0.6), up to a power of 2.

Adams expressed that he thought that the appearance of the Riemann zeta-function in this topological context was just a fluke, and not due to any deep connection between the orders of stable homotopy groups and the number theory of zeta-functions; while I think Adams was right to take a skeptical attitude about the possibility of such connections without there being any further evidence of them, Lichtenbaum's conjecture from 1973 established a plausible deep link between orders of stable homotopy groups (as algebraic K-groups, in particular) and special values of zeta-functions more general than the Riemann zeta-function; I will explain Lichtenbaum's conjecture later in this talk. My understanding of the history is that in 1978 Quillen showed that one could reduce Lichtenbaum's conjecture to the Iwasawa main conjecture if one had an étale descent spectral sequence for algebraic K-theory and if one knew that étale K-theory agrees with algebraic K-theory in a certain range of degrees; this last statement was the *Quillen-Lichtenbaum conjecture*. Wiles proved the Iwasawa main conjecture (at least in the necessary cases: the totally real number fields) in 1990, Thomason built the spectral sequence in 1985, and Voevodsky, Rost, and collaborators proved the Quillen-Lichtenbaum conjecture in the mid-2000s, so the Lichtenbaum conjecture is now a theorem. The speaker has done work, and continues to do work, in the area of finding connections between stable homotopy groups of finite CW-complexes (rather than algebraic K-groups, which are indeed stable homotopy groups, but not stable homotopy groups of finite CW-complexes, e.g. manifolds) and special values of L-functions; in particular this has resulted in formulas for the KU-local stable homotopy groups of all finite CW-complexes with torsion-free homology in terms of special values of Hasse-Weil L-functions of certain toric varieties, as well as formulas for KU-local stable homotopy groups of Moore spaces in terms of special values of Dedekind zeta-functions of certain totally real abelian number fields; this then gets you a topological proof of some cases of the Leopoldt conjecture. (I am sorry that this paragraph is getting too deep into algebraic topology for how early in the seminar it is. Please feel free to skip reading it; I am only saying all this stuff at this point to make it clear that Adams' formula (0.6) is not a coincidence, but rather a special case of some deep but currently very poorly-understood connections between topology and number theory.)

In any case: the image of the J-homomorphism is not the entirety of $\pi_*^{st}(S^0)$. Homotopical localization methods filter $\pi_*^{st}(S^0)$ into infinitely many *periodic* families, of which the image of J is only the first. (In the language of Bousfield localization, the image of J agrees, up to 2-torsion, with the KU-local stable homotopy groups of spheres in positive degrees. Here KU is the spectrum representing complex K-theory.) One wants to be able to describe the other periodic families in $\pi_*^{st}(S^0)$ in terms of special values of L-functions; this is something the speaker works on, but currently none of the positive results in this direction are anywhere near as cleanly-stated as the KU-local results described in the previous paragraph. For reasons that I hope to explain as the seminar goes on, one expects that some of the L-functions whose special values will appear in this setting are the L-functions of *eigenvalue* $1/4$ *Maass forms*, a generalization of the L-functions of modular forms, which we haven't yet defined—but we're getting there!

- The relationship between the $\zeta(1-n)$ column and the last two columns in (0.2) is the subject of the Lichtenbaum conjecture, mentioned above. The Lichtenbaum conjecture for $\mathbb{Z}$ states that

$$\zeta(1-2n) = \pm\frac{\#(K_{4n-2}(\mathbb{Z}))}{\#(K_{4n-1}(\mathbb{Z}))}$$

up to multiplication by powers of 2. For example, reading (0.2), we have

$$\zeta(-11) = \frac{691}{32760} = 2\cdot\frac{691}{65520} = 2\cdot\frac{\#(K_{22}(\mathbb{Z}))}{\#(K_{23}(\mathbb{Z}))}.$$

More generally, if E is a totally real number field, Galois over $\mathbb{Q}$, with ring of integers $\mathcal{O}_E$, the Lichtenbaum conjecture states that

$$\zeta_E(1-2n) = \pm\frac{\#(K_{4n-2}(\mathcal{O}_E))}{\#(K_{4n-1}(\mathcal{O}_E))},$$

again up to powers of 2. This version of the Lichtenbaum conjecture is now (since the mid-2000s) known to be true, as explained above. (People sometimes also consider a more general version of the Lichtenbaum conjecture, which allows E to have complex embeddings; the statement then is roughly "the leading Taylor coefficient of $\zeta_E(s)$ at $s = 1-2n$ is equal to a certain transcendental number (a *higher regulator*) times $\frac{\#(K_{4n-2}(\mathcal{O}_E))}{\#(K_{4n-1}(\mathcal{O}_E))}$.")

I should explain at least a little bit what these things mean:

  - $\mathcal{O}_E$ is the set of roots, in E, of monic polynomials with integer coefficients. This forms a subring of E (in fact, a Dedekind domain in E) whose fraction field is E. You can think of $\mathcal{O}$ as something like an inverse operation to taking the field of fractions. The most familiar example is $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$.

– ζ_E, the Dedekind zeta-function of E, is the meromorphic function on $\mathbb{C}$ given to the right of the $\Re(s) = 1$ by the formulas

$$\zeta_E(s) = \sum_{I \subseteq \mathcal{O}_E} \frac{1}{\#(\mathcal{O}_E/I)^s} = \prod_{\mathfrak{m} \subseteq \mathcal{O}_E} \frac{1}{1 - \#(\mathcal{O}_E/\mathfrak{m})^{-s}}, \tag{0.7}$$

and given on the rest of $\mathbb{C}$ by analytic continuation. The sum in (0.7) is taken over all nonzero ideals I of $\mathcal{O}_E$, and the product is taken over all maximal ideals of $\mathcal{O}_E$. You can see from (0.7) that $\zeta_{\mathbb{Q}}(s)$ is the Riemann zeta-function.

– Given a commutative ring R, the algebraic K-groups of R can be defined in positive degrees as follows: let $GL(R) = \cup_n GL_n(R)$ be the infinite general linear group of R, and let $BGL(R)$ be the classifying space of principal $GL(R)$-bundles, i.e., $BGL(R)$ is a connected CW-complex such that $\pi_1(BGL(R)) \cong GL(R)$ and $\pi_n(BGL(R)) \cong 0$ for all $n > 1$. (This uniquely characterizes $BGL(R)$, up to homotopy equivalence.) Let $BGL(R)^+$ be the topological space constructed by the following two-step process due to Quillen (in 1972):
  * first, since S^1 is the boundary of a 2-cell D^2, for any element f of $GL(R) = \pi_1(BGL(R))$ we can attach a 2-cell to $BGL(R)$ along the image of f, which has the effect of killing off f in π_1 (and also of completely changing the higher homotopy groups). Attach 2-cells to $BGL(R)$ to kill off the commutators in $\pi_1(BGL(R))$.
  * Attaching those 2-cells has the effect of abelianizing π_1, but also introduces some new homology classes in H_2; now attach 3-cells to kill off those new homology classes. (It is not obvious that you can do this; you need to know that those homology classes are in the image of the Hurewicz transformation $\pi_2 \to H_2$. But they indeed are!)

  The resulting space—which is just $BGL(R)$ with 2-cells and 3-cells attached in a particular way—is called $BGL(R)^+$. We define $K_n(R)$ as $\pi_n(BGL(R)^+)$ when $n > 0$. By construction, $K_1(R)$ is the abelianization of $GL(R)$. It takes a bit more work but one also has a purely algebraic account of $K_2(R)$: for $n \geqslant 3$, let $\mathrm{St}_n(R)$ be the group with generating set $\{x_{i,j}(r) : r \in R, i, j \in \{1, \dots, n\}, i \neq j\}$ and with relations

$$x_{i,j}(r)x_{i,j}(s) = x_{i,j}(r+s)$$
$$[x_{i,j}(r), x_{k,l}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l \\ x_{i,l}(rs) & \text{if } j = k \text{ and } i \neq l \\ x_{k,j}(-sr) & \text{if } j \neq k \text{ and } i = l. \end{cases}$$

  The second K-group of R, $K_2(R)$, was defined by Milnor in 1967 as the center of the Steinberg group $\mathrm{St}(R) = \lim_{n\to\infty} \mathrm{St}_n(R)$. (It also agrees with the kernel of the map $\mathrm{St}(R) \to E(R)$ obtained by regarding $\mathrm{St}(R)$ as the universal central extension of the group $E(R)$ of elementary matrices with entries in R.)

  So the definition of $K_n(R)$ as $\pi_n(BGL(R)^+)$ agrees, when $n = 1$ and $n = 2$, with the definitions $K_1(R) = GL(R)/[GL(R), GL(R)]$ and $K_2(R) = \mathrm{center}(\mathrm{St}(R))$ of K-groups that people had before 1972, when there was no definition of K-groups above K_2. Today, there are many constructions of algebraic K-groups of rings, but all of them are inescapably as the homotopy groups of some space or spectrum constructed from the ring. (The only known possible exception is a recent construction of Grayson which is purely algebraic—it does not involve explicitly taking the homotopy groups of anything—but the groups arising from Grayson's construction have not been proven to agree with the classical K-groups of a ring.)

The purely algebraic definition of K_2 given above, in terms of generators and relations, actually plays a role in how the Lichtenbaum conjecture was formulated: before Lichtenbaum's conjecture, there was the 1970 conjecture of Birch and Tate: if E is a totally real number field, then $\zeta_E(-1) = \pm\frac{\#(K_2(\mathcal{O}_E))}{w_2(E)}$, where $w_2(E)$ is the largest integer N such that $\mathrm{Gal}(E(\zeta_N)/E)$ is an elementary abelian 2-group. For example, $\mathrm{Gal}(\mathbb{Q}(\zeta_{24})/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ but $\mathrm{Gal}(\mathbb{Q}(\zeta_{48})/\mathbb{Q})$ has an element of order 4, so $w_2(\mathbb{Q}) = 24$; meanwhile, $K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, so the Birch-Tate conjecture predicts that $\zeta(-1) = \pm\frac{2}{24}$, while the correct answer is $\zeta(-1) = \frac{-B_2}{2} = \frac{-1}{12}$. (The Birch-Tate conjecture was proven, up to a power of 2, as a consequence of Wiles' 1990 proof of the Iwasawa main conjecture.)

- Finally, the relationship between the $\pi_*^{st}(S^0)$ column in (0.2) and the $K_*(\mathbb{Z})$ columns in (0.2) is that the algebraic K-groups of a commutative ring R are not just the homotopy groups of $BGL(R)^+$: work of May and Segal in the 1970s showed that the K-groups of R are in fact the homotopy groups of an E_∞*-ring spectrum* $\mathcal{K}(R)$; these are topological gadgets that behave very much like commutative rings, but rather than every E_∞-ring spectrum being a commutative $\mathbb{Z}$-algebra, every E_∞-ring spectrum is a commutative algebra over the sphere spectrum, which we will call S. Consequently we have a unit map $S \to \mathcal{K}(R)$, and on taking homotopy groups, we have a map of graded rings $\pi_*^{st}(S^0) \to K_*(R)$ for every commutative ring R.

  The map $\pi_*^{st}(S^0) \to K_*(\mathbb{Z})$ is neither injective nor surjective, but it has a very meaningful image: from calculation, we know that the elements of $K_*(\mathbb{Z})$ that are in the image of the map $\pi_*^{st}(S^0) \to K_*(\mathbb{Z})$ are "essentially" (that is, up to applying an operation which topologists call "multiplying by the Bott class" and which number theorists call "twisting by the cyclotomic character") the ones which give rise, via the Lichtenbaum conjecture, to the denominators of $\zeta(1-n)$; and these are also "essentially" (that is, up to 2-torsion) those elements of $\pi_*^{st}(S^0)$ which are in the image of the J-homomorphism.

Natural places to go next would be the factorization of Dedekind zeta-functions into products of Artin L-functions, the relationship between degree 1 Artin L-functions and Dirichlet L-functions, special values of Dirichlet L-functions, and modularity conjectures and theorems for degree > 1 Artin L-functions; that leads us to L-functions of modular forms and Maass forms.