PRODUCTS IN SPIN^c-COBORDISM

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Abstract. We calculate the mod 2 spin^c-cobordism ring up to uniform F-isomorphism (i.e., inseparable isogeny). As a consequence we get the prime ideal spectrum of the mod 2 spin^c-cobordism ring. We also calculate the mod 2 spin^c-cobordism ring "on the nose" in degrees ≤ 33. We construct an infinitely generated nonunital subring of the 2-torsion in the spin^c-cobordism ring. We use our calculations of product structure in the spin and spin^c cobordism rings to give an explicit example, up to cobordism, of a compact 24-dimensional spin manifold which is not cobordant to a sum of squares, which was asked about in a 1965 question of Milnor.

1. Introduction and summary of results

1.1. Spin^c cobordism. A spin^c-structure on a compact smooth n-dimensional manifold M is a reduction of its structure group from $O(n)$ to $Spin^c(n)$. We find the following perspective illuminating: a compact smooth manifold is

- orientable if its first Stiefel–Whitney class w_1 vanishes
- and admits a spin structure if its first two Stiefel–Whitney classes, w_1 and w_2, both vanish.

A spin^c-structure is intermediate between an orientation and a spin structure. Specifically, a compact smooth manifold M admits a spin^c structure if its first Stiefel–Whitney class w_1 vanishes, and its second Stiefel–Whitney class w_2 is a reduction of an integral class. That is, $w_2 \in H^2(M;\mathbb{F}_2)$ is in the image of the reduction-of-coefficients map $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{F}_2)$. For these and many other relevant facts, consult Stong's book [25].

The spin^c-cobordism ring, written $\Omega_*^{Spin^c}$, is the ring of spin^c-cobordism classes of compact smooth spin^c-manifolds. The addition is given by disjoint union of manifolds, while the multiplication is Cartesian product. There are several reasons to care about spin^c-cobordism: aside from its applications to mathematical physics, e.g. [7] and [29], spin^c-cobordism is of particular interest because it is one of the *complex-oriented* cobordism theories, and consequently there exists a one-dimensional group law on $\Omega_*^{Spin^c}$ which describes how the first Chern class in spin^c-cobordism behaves on a tensor product of complex line bundles. See [1] or [13] for these classical ideas, whose consequences for complex cobordism (as in [21]) have been enormous, but whose consequences for spin^c-cobordism have apparently never been fully explored[1].

[1]In future work, the authors hope to apply the results about the ring structure of the spin^c-cobordism ring obtained in this paper to the problem of describing the formal group law on the spin^c-cobordism ring in formal-group-law theoretic terms, similar to what Quillen did for complex and unoriented cobordism in [17],[18], what Baker–Morava did for 2-inverted symplectic bordism in [6], and what Buchstaber did for symplectic bordism [9]. It seems impossible to get

Since spin^c-cobordism is an example of a "(B,f)-cobordism theory" in the sense of Thom, the general results of [26] ensure that there exists a spectrum $MSpin^c$ such that $\pi_*(MSpin^c) \cong \Omega_*^{Spin^c}$. The homotopy type of the spectrum $MSpin^c$ is understood as follows.

Away from 2: The map $\pi : BSpin^c \longrightarrow BSO \times K(\mathbb{Z},2)$ is an odd-primary homotopy equivalence and induces an isomorphism $\Omega_*^{Spin^c}[\frac{1}{2}] \cong \Omega_*^{SO}(K(\mathbb{Z},2))[\frac{1}{2}]$, and consequently $MSpin^c[\frac{1}{2}] \cong MSO[\frac{1}{2}] \wedge \mathbb{C}P^\infty$.

At 2: In 1966, Anderson, Brown, and Peterson [2],[3] proved that $MSpin^c$ splits 2-locally as a wedge of suspensions of the connective complex K-theory spectrum ku and the mod 2 Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$:

$$MSpin^c_{(2)} \simeq Z \vee \coprod_J \Sigma^{4|J|} ku_{(2)} \tag{1}$$

where the coproduct (i.e., wedge sum) is taken over all partitions (i.e., unordered finite tuples of positive integers) J, and $|J|$ denotes the sum of the entries of J.

Not much is known about the summand Z in (1), other than that

- it is a coproduct of suspensions of copies of $H\mathbb{F}_2$,
- and from a Poincaré series [2], it is known how to solve inductively for the number of copies of $\Sigma^n H\mathbb{F}_2$ in Z, for each n.

In that sense, Z is understood *additively.*

This purely additive understanding of Z, and consequently of 2-local $\Omega_*^{Spin^c}$, is not entirely satisfying. To see the problem, consider the following table, which we reproduce from Bahri–Gilkey [5]:

n	$\dim_{\mathbb{F}_2} \pi_n Z$	n	$\dim_{\mathbb{F}_2} \pi_n Z$	n	$\dim_{\mathbb{F}_2} \pi_n Z$	n	$\dim_{\mathbb{F}_2} \pi_n Z$
0	0	8	0	16	0	24	2
1	0	9	0	17	0	25	0
2	0	10	1	18	3	26	9
3	0	11	0	19	0	27	0
4	0	12	0	20	1	28	4
5	0	13	0	21	0	29	1
6	0	14	1	22	5	30	14
7	0	15	0	23	0	31	1

TABLE 1.

The $\mathbb{F}_2$-linear dimension of $\pi_n Z$, as recorded in table 1, is equivalently the number of copies of $\Sigma^n H\mathbb{F}_2$ in 2-local $MSpin^c$, and equivalently the $\mathbb{F}_2$-rank of the 2-torsion subgroup of $\Omega_n^{Spin^c}$. Hence this table is telling us about the 2-torsion in the spin^c-cobordism ring. One has the sense that some deep pattern is present in the distribution of the 2-torsion, but whatever it is, it cannot be seen clearly from these $\mathbb{F}_2$-ranks, nor from the Poincaré series used to inductively compute them.

However, since $\pi_*(Z)$ is precisely the 2-torsion in $\Omega_*^{Spin^c}$, $\pi_*(Z)$ is not only a summand but also an *ideal* in $\Omega_*^{Spin^c}$. One wants to understand $\pi_*(Z)$ *multiplicatively,* i.e., one wants to be able to describe the ring structure on $\Omega_*^{Spin^c}$, including

much understanding of the formal group law of spin^c-cobordism without first coming to some understanding of the structure of its coefficient ring $\Omega_*^{Spin^c}$, which is the goal of this paper.

its 2-torsion elements. A reasonably clear description of $\Omega_*^{Spin^c}$ as a ring would yield a far more illuminating understanding of $\pi_*(Z)$ than the inductive formula for its $\mathbb{F}_2$-rank in each degree, which is presently all we have.

Fifty years after the additive structure of $MSpin^c$ was described by Anderson–Brown–Peterson, the problem of calculating the ring structure of $\Omega_*^{Spin^c}$ remains open. The purpose of this paper is to make progress towards a solution to this problem, restricting to the 2-local case, which is the most difficult[2].

1.2. **The mod 2 spin^c-cobordism ring in low degrees.** A traditional notation for the unoriented bordism ring $\Omega_*^O \cong MO_*$ is $\mathfrak{N}_*$. In the 1968 book [25, pg. 351], Stong asks:

> Open question: Can one determine these images nicely as subrings of $\mathfrak{N}_*$?

By "these images," Stong refers to the images of the natural maps $\Omega_*^{Spin} \to \mathfrak{N}_*$ and $\Omega_*^{Spin^c} \to \mathfrak{N}_*$. Our approach to understanding the mod 2 spin^c-cobordism ring begins by answering Stong's open question in a range of degrees. We use the Anderson–Brown–Peterson splitting [2], product structure in the Adams spectral sequences, and Thom's determination of $\mathfrak{N}_*$ using symmetric polynomials [26] to develop a method for calculating the image of the map $\Omega_*^{Spin^c} \to \mathfrak{N}_*$ through degree d, for any fixed choice of integer d. Our method gives a presentation for $\Omega^{Spin^c}/(2,\beta)$ through degree d, since the map $\Omega^{Spin^c}/(2,\beta) \longrightarrow \mathfrak{N}_*$ is injective. We carry out computer calculation using our method to obtain our first main theorem:

Theorem A (Theorem 3.4)**.** *The subring of the mod 2 spin^c-cobordism ring $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ generated by all homogeneous elements of degree ≤ 33 is isomorphic to*

$$\mathbb{F}_2[\beta, Z_4, Z_8, Z_{10}, Z_{12}, Z_{16}, Z_{18}, Z_{20}, Z_{22}, Z_{24}, Z_{26}, Z_{28}, Z_{32}, \\ T_{24}, T_{29}, T_{31}, T_{32}, T_{33}]/I$$

where I is the ideal generated by the relations:

- $\beta Z_i = 0$ *for each* $i \equiv 2 \mod 4$,
- *and* $\beta T_i = 0$ *and* $T_i^2 = U_{2i}$ *for* $i \in \{24, 29, 31, 32, 33\}$*, where each* U_i *is a particular polynomial in the generators* Z_j *with* $j \leq i - 20$*. The polynomial* U_i *is described explicitly preceding Theorem 3.3.*

The degrees of the generators are as follows: $\beta = [\mathbb{C}P^1]$ *is in degree* 2*, while* Z_i *and* T_i *are each in degree* i*.*

With Theorem A in hand, the patterns in table 1 become completely clear: in each degree in this range, one can see *why* the $\mathbb{F}_2$-linear dimension of the 2-torsion subgroup of $\Omega_*^{Spin^c}$ takes the particular value it takes, as follows. Since Anderson–Brown–Peterson proved that the 2-torsion coincides with the β-torsion in $\Omega_*^{Spin^c}$, in

[2]In principle, the ring structure on $\Omega_*^{Spin^c}$ *away* from 2 is understood, although only in a rather indirect way. Here is how it works: from the complex-orientability of MSO, one gets an isomorphism of rings $MSO[\frac{1}{2}]^*(\mathbb{C}P^\infty) \cong MSO[\frac{1}{2}]^*[[X]]$. The ring $MSO[\frac{1}{2}]^*[[X]]$ is also the "covariant bialgebra" of the formal group law of $MSO[\frac{1}{2}]_*$, in the sense of [12, Chapter 36]. Hence one can use the formal group law on $MSO[\frac{1}{2}]^*$ (whose universal property is given by [6]) to understand the coproduct on $MSO[\frac{1}{2}]^*[[X]]$, whose dual, in an appropriate sense, is responsible for the ring structure on $MSO[\frac{1}{2}]_*(\mathbb{C}P^\infty) \cong \Omega_*^{Spin^c}[\frac{1}{2}]$. This gives a means of understanding the ring $\Omega_*^{Spin^c}[\frac{1}{2}]$, although we know of nowhere in the literature where this has been carried out in any further detail.

degrees $n \leq 33$ the 2-torsion in $\Omega_n^{Spin^c}$ is simply the $\mathbb{F}_2$-linear combinations of the monomials in the generators Z_i, T_i such that at least one of the factors is β-torsion, i.e., at least one of the factors is either a generator Z_i with $i \equiv 2 \mod 4$, or a generator T_i. Here is the same table as table 1, but augmented with an $\mathbb{F}_2$-linear basis in each degree, using the multiplicative structure from Theorem A. We start in degree 10 since there is no nontrivial 2-torsion in $\Omega_*^{Spin^c}$ below degree 10.

n	$\dim_{\mathbb{F}_2} \pi_n Z$	$\mathbb{F}_2$-linear basis for $\pi_n Z$
10	1	Z_{10}
11,12,13	0	
14	1	$Z_4 Z_{10}$
15, 16, 17	0	
18	3	$Z_4^2 Z_{10}, Z_8 Z_{10}, Z_{18}$
19	0	
20	1	Z_{10}^2
21	0	
22	5	$Z_4^3 Z_{10}, Z_4 Z_8 Z_{10}, Z_{12} Z_{10}, Z_4 Z_{18}, Z_{22}$
23	0	
24	2	$Z_4 Z_{10}^2, T_{24}$
25	0	
26	9	$Z_4^4 Z_{10}, Z_4^2 Z_8 Z_{10}, Z_8^2 Z_{10}, Z_4 Z_{12} Z_{10}, Z_{16} Z_{10},$ $Z_4^2 Z_{18}, Z_8 Z_{18}, Z_4 Z_{22}, Z_{26}$
27	0	
28	4	$Z_4^2 Z_{10}^2, Z_8 Z_{10}^2, Z_{10} Z_{18}, Z_4 T_{24}$
29	1	T_{29}
30	14	$Z_4^5 Z_{10}, Z_4^3 Z_8 Z_{10}, Z_4 Z_8^2 Z_{10}, Z_4^2 Z_{12} Z_{10}, Z_8 Z_{12} Z_{10},$ $Z_4 Z_{16} Z_{10}, Z_{20} Z_{10}, Z_4^3 Z_{18}, Z_4 Z_8 Z_{18}, Z_{12} Z_{18},$ $Z_4^2 Z_{22}, Z_8 Z_{22}, Z_4 Z_{26}, Z_{10}^3$
31	1	T_{31}
32	8	$Z_4^3 Z_{10}^2, Z_4 Z_8 Z_{10}^2, Z_{12} Z_{10}^2, Z_4 Z_{10} Z_{18}, Z_{10} Z_{22},$ $Z_4^2 T_{24}, Z_8 T_{24}, T_{32}$
33	2	$Z_4 T_{29}, T_{33}$

TABLE 2.

One can also read off the product structure on the 2-torsion in $\Omega_*^{Spin^c}$ in degrees ≤ 33 from this table, since it is given by multiplication of monomials along with the relations from Theorem A.

It is evident from Theorem A that, in degrees ≤ 33, $\Omega_*^{Spin^c}$ has a subring generated by elements $Z_4, Z_8, Z_{12}, Z_{16}, \ldots$ and by elements Z_{2i} with i odd and not one less than a power of 2, subject to the relations $2Z_{2i} = 0 = \beta Z_{2i}$ for all odd i. We are able to show that this pattern extends into all degrees, and goes some way to describing the ideal $\pi_*(Z)$ of 2-torsion elements of $\Omega_*^{Spin^c}$ in multiplicative terms:

Theorem B (Theorem 4.3). *Consider the spin^c-cobordism ring as a graded algebra over the graded ring $S := \mathbb{Z}_{(2)}[\beta, Z_{2j} : j{+}1 \text{ not a power of } 2]/(\beta Z_{2j}, 2Z_{2j} \text{ for odd } j)$. Let J be the ideal of S generated by all the elements Z_{2j} with j odd. Then J embeds, as a non-unital graded S-algebra, into the 2-torsion ideal $\pi_*(Z)$ of the spin^c-cobordism ring.*

Theorem B describes the multiplicative structure of some, but not all, of the 2-torsion in $\Omega_*^{Spin^c}$. For example, in degrees ≤ 33, it accounts for precisely those monomials in table 2 which are *not* divisible by the elements T_i. In particular, the lowest-degree 2-torsion element of $\Omega_*^{Spin^c}$ which is not described by Theorem 4.3 is $T_{24} \in \Omega_{24}^{Spin^c}$.

1.3. **Milnor's 24-dimensional spin manifold.** We also calculate the image of the map $\Omega_*^{Spin} \to \mathfrak{N}_*$ through degree 31 in Proposition 3.2 via a similar method to the one used to calculate the image of the map $\Omega_*^{Spin^c} \to \mathfrak{N}_*$ in Theorem A. There is a noteworthy geometric consequence of Proposition 3.2. In the 1965 paper [15], Milnor asks this question:

> Problem. Does there exist a spin manifold Σ of dimension 24 so that $s_6(p_1, \dots, p_6)[\Sigma] \equiv 1 \mod 2$?

Here s_6 is a certain symmetric polynomial, and $p_1, \dots, p_6$ are Pontryagin classes. The reason for Milnor's question is that, in [15], Milnor proves that, for a compact smooth manifold M of dimension ≤ 23, the following conditions are equivalent:

(1) M is unorientedly cobordant to a spin manifold.
(2) The Stiefel–Whitney numbers of M involving w_1 and w_2 are all zero.
(3) M is unorientedly cobordant to $N \times N$, with N an orientable compact manifold.

Milnor points out that, if there exists a compact spin manifold Σ whose Pontryagin number $s_6(p_1, \dots, p_6)[\Sigma]$ is odd, then these conditions would fail to be equivalent in dimension 24. Anderson–Brown–Peterson [2],[3] established that, as a consequence of their splitting of 2-local $MSpin$, there does indeed exist such a compact spin manifold Σ. However, it seems that no explicit description of that 24-dimensional compact spin manifold has been given in the literature (or anywhere else, as far as we know).

In Theorem 3.6, we give an explicit formula for the unoriented bordism class of such a compact spin manifold Σ, as a disjoint union of products of real projective spaces and squares of Dold manifolds. We refer the reader to the Theorem 3.6 for a statement of that formula, which is lengthy. The formula is obtained using our calculation of the image of the map $\Omega_{24}^{Spin} \to \mathfrak{N}_{24}$ and the manifold representatives calculated in Proposition 3.5.

1.4. **Determination of the mod 2 spin^c-cobordism ring up to inseparable isogeny.** Thom's famous calculation [26] established that the unoriented bordism ring $\Omega_*^O = \mathfrak{N}_*$ is isomorphic to a polynomial algebra over $\mathbb{F}_2$. A theorem of Stong [24, Proposition 14] shows that the spin^c cobordism ring, reduced modulo torsion and then reduced modulo 2, is also isomorphic to a polynomial $\mathbb{F}_2$-algebra.

By contrast, the spin^c cobordism ring cannot itself be isomorphic to a polynomial algebra, since by [2], it has 2-torsion but is not an $\mathbb{F}_2$-algebra, hence it has nontrivial zero divisors. Similarly, since the mod 2 spin^c-cobordism ring has nontrivial β-torsion, it cannot be isomorphic to a polynomial $\mathbb{F}_2$-algebra.

It follows as a trivial consequence of Theorem A that the mod $(2, \beta)$ spin^c-cobordism ring *still* cannot be a polynomial $\mathbb{F}_2$-algebra. One can, with a bit of calculation, deduce the same fact from the additive structure of 2-local $MSpin^c$, by verifying that the Poincaré series of the mod $(2, \beta)$ spin^c-cobordism ring is not the Poincaré series of any polynomial algebra. This avoids the use of our multiplicative methods. The advantage of our multiplicative methods is that we are able to prove that $\Omega_*^{Spin^c}/(2,\beta)$ is instead *uniformly F-isomorphic* to a polynomial algebra.

As far as we know, the terms "F-isomorphism" (perhaps better known as "inseparable isogeny") and "uniform F-isomorphism" originated with Quillen [20]:

Definition 1.1. *Given a prime p, a homomorphism of $\mathbb{F}_p$-algebras $f : A \to B$ is said to be an F-*isomorphism *if*

- *for each $a \in \ker f$, some power a^n is zero, and*
- *for each element $b \in B$, some power b^{p^n} of b is in the image of f.*

The F-isomorphism f is said to be uniform *if n can be chosen independently of a and b.*

The notion of F-isomorphism is applied only to algebras over a field of positive characteristic, so we had better reduce modulo 2 in order to apply this idea to the spin^c-cobordism ring. We get a positive result:

Theorem C (Theorem 4.4)**.** *The mod 2 spin^c-cobordism ring $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ is uniformly F-isomorphic to the graded $\mathbb{F}_2$-algebra*

$$\mathbb{F}_2\left[\beta, y_{4i}, Z_{4j-2} : i \geq 1,\ j \geq 1,\ j \text{ not a power of } 2\right]/(\beta Z_{4j-2}), \tag{2}$$

with β the Bott element in degree 2, with y_{4i} in degree $4i$, and with Z_{4j-2} in degree $4j-2$.

Corollary D. *The mod $(2,\beta)$ spin^c-cobordism ring is uniformly F-isomorphic to a graded polynomial $\mathbb{F}_2$-algebra on*

- *a generator in degree $4i$ for all positive integers i,*
- *and a generator in degree $4j-2$ for all positive integers j such that j is not a power of 2.*

An F-isomorphism induces a homeomorphism on prime ideal spectra, so Theorem C yields a description of all prime ideals in the mod 2 spin^c cobordism ring. That is, we have

Corollary E (Corollary 4.5)**.** *The topological space* $\operatorname{Spec}\Omega_*^{Spin^c}/(2)$ *is homeomorphic to* Spec *of the $\mathbb{F}_2$-algebra* (2)*. The topological space* $\operatorname{Spec}\Omega_*^{Spin^c}/(2,\beta)$ *is homeomorphic to* Spec *of the $\mathbb{F}_2$-algebra described in Corollary D.*

1.5. **Conventions.**

- Given a ring R and symbols $x_1, \dots, x_n$, we write $R\{x_1, \dots, x_n\}$ for the free R-module with basis $x_1, \dots, x_n$.
- We write β for the Bott element in $\pi_2(ku)$, and also for its corresponding element $\beta = [\mathbb{C}P^1] \in \Omega_2^{Spin^c}$ under the Anderson–Brown–Peterson splitting of 2-local $MSpin^c$.

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2. Preliminaries

In this section we present an extended review of some well-known facts about spin and spin^c cobordism, including the relationships various cobordism spectra, their homotopy groups, homology and cohomology groups, including the Steenrod algebra action on cohomology and the Pontryagin product in homology. This background material is necessary in order to understand the proofs of the results in the rest of the paper. Readers confident in their knowledge of this background material can skip to section 3, where we begin proving new results.

2.1. **Review of the cohomology of the spectra $MSpin^c$ and $MSpin$.** There is an exact sequence of Lie groups

$$1 \longrightarrow U(1) \longrightarrow Spin^c(n) \longrightarrow SO(n) \longrightarrow 1$$

that gives rise to the fiber sequence

$$BU(1) \longrightarrow BSpin^c \longrightarrow BSO. \tag{3}$$

Using this fibration, Harada and Kono [11] computed the mod 2 cohomology of the space $BSpin^c$:

Theorem 2.1. [11]

$$H^*(BSpin^c;\mathbb{F}_2) \cong \mathbb{F}_2(w_2, w_3, w_4, w_5, ...)/I \tag{4}$$

where I is the ideal $\langle w_3, \mathrm{Sq}^2(w_3), \mathrm{Sq}^2(Sq^4(w_3)), \mathrm{Sq}^8(Sq^4(Sq^2(w_3)), ...\rangle$.

The triviality of the ideal I in the cohomology of $BSpin^c$ is a consequence of the first d_2 differential in the Serre spectral sequence associated to the fiber sequence (3). It is not practical to write down a presentation for the $\mathbb{F}_2$-algebra $H^*(BSpin^c;\mathbb{F}_2)$ which is more explicit than (4), since the difficulty of calculating iterated Steenrod squares applied to the Stiefel–Whitney class w_3 grows rapidly as the number of Steenrod squares grows. For example, $\mathrm{Sq}^8(\mathrm{Sq}^4(\mathrm{Sq}^2(w_3))$ has 38 monomials when expressed as a polynomial in the Stiefel–Whitney classes.

There is also an exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Spin(n) \longrightarrow SO(n) \longrightarrow 1$$

which gives rise to the fiber sequence

$$B\mathbb{Z}/2\mathbb{Z} \longrightarrow BSpin(n) \longrightarrow BSO(n).$$

Using this, Quillen calculated:

Theorem 2.2. [19]

$$H^*(BSpin;\mathbb{F}_2) \cong \mathbb{F}_2(w_2, w_3, w_4, w_5, ...)/J$$

where J is the ideal $\langle w_2, w_3, \mathrm{Sq}^2(w_3), \mathrm{Sq}^2(Sq^4(w_3)), \mathrm{Sq}^8(Sq^4(Sq^2(w_3)), ...\rangle$.

By the Thom isomorphism, we have that $H^*(MSpin^c;\mathbb{F}_2) \cong H^*(BSpin^c;\mathbb{F}_2)\{U\}$ and $H^*(MSpin;\mathbb{F}_2) \cong H^*(BSpin;\mathbb{F}_2)\{U\}$ as graded $\mathbb{F}_2$-vector spaces. Since $H^*(BSpin^c;\mathbb{F}_2)$ and $H^*(BSpin;\mathbb{F}_2)$ are quotients of $H^*(BO;\mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, w_3, \dots]$, the action of Steenrod squares on $H^*(BSpin^c;\mathbb{F}_2)$ and $H^*(BSpin;\mathbb{F}_2)$ is determined by the Wu formula $\mathrm{Sq}^i w_j = \sum_{k=0}^{i} \binom{j+k-i-1}{k} w_{i-k}w_{j+k}$ and the Cartan formula. This, together with the formula $\mathrm{Sq}^n U = w^n U$ for the action of Steenrod squares on the Thom class U, determines the action of the Steenrod squares on the cohomology $H^*(MSpin^c;\mathbb{F}_2)$ of the spin^c-bordism spectrum $MSpin^c$.

2.2. **Review of MO_* and symmetric polynomials in the Stiefel–Whitney classes.** The following definitions are classical (see e.g. chapter 1 of [14]):

Definition 2.3. *Let n be a nonnegative integer.*

- *Suppose $\lambda = (a_1, \dots, a_n)$ is an unordered n-tuple of nonnegative integers. The* monomial symmetric polynomial associated to λ *is the symmetric polynomial*
$$m_\lambda(X_1, \dots, X_n) \in \mathbb{Z}[X_1, \dots, X_n]^{\Sigma_n}$$
which has the fewest nonzero monomial terms among all those which have $X_1^{a_1} X_2^{a_2} \dots X_n^{a_n}$ as a monomial term.
- *Given a nonnegative integer $m \leq n$, the* mth elementary symmetric polynomial *is the symmetric polynomial*
$$e_m(X_1, \dots, X_n) \in \mathbb{Z}[X_1, \dots, X_n]^{\Sigma_n}$$
given by
$$e_m(X_1, \dots, X_n) = \sum_{1 \leq d_1 < d_2 < \dots < d_m \leq n} X_{d_1} X_{d_2} \dots X_{d_m}.$$

The monomial symmetric polynomials form a $\mathbb{Z}$-linear basis for the ring of symmetric polynomials. The set of all finite products of elementary symmetric polynomials also famously (by Newton) forms a $\mathbb{Z}$-linear basis for the ring of symmetric polynomials. Consequently, for each λ, there exists a unique polynomial $P_\lambda(X_1, \dots, X_n)$ such that
$$P_\lambda \left(e_1(X_1, \dots, X_n), e_2(X_1, \dots, X_n), \dots, e_n(X_1, \dots, X_n)\right) = m_\lambda(X_1, \dots, X_n).$$
For more details about the polynomials $P_\lambda(X_1, \dots, X_n)$, see the material on the transition matrix $M(m, e)$ and its inverse $M(e, m)$ in section 1.6 of [14], particularly (6.7)(i).

See [25], particularly pages 71 and 96 and surrounding material, for a nice exposition of the following result, which dates back to Thom [26]: let Λ be the set of unordered finite-length tuples of *positive* integers, each of which is not equal to $2^a - 1$ for any integer a. Such integers are called "non-dyadic," and such partitions are called "non-dyadic partitions." For each $\lambda \in \Lambda$, write $|\lambda|$ for the length of λ, and write $||\lambda||$ for the sum of the elements of λ. Consider the polynomial
$$\begin{aligned} P_\lambda(w_1, \dots, w_{|\lambda|}) &\in H^*(BO; \mathbb{F}_2) \\ &\cong \mathbb{F}_2[w_1, w_2, \dots] \end{aligned}$$
in the Stiefel–Whitney classes $w_1, w_2, \dots$. Then (see page 96 of [25], or pages 301-302 of [27]) the set
$$\{P_\lambda(w_1, \dots, w_{|\lambda|})U) : \lambda \in \Lambda\}$$
is a homogeneous A-linear basis for the graded free A-module $H^*(MO; \mathbb{F}_2)$, where $U \in H^0(MO; \mathbb{F}_2)$ denotes the Thom class, and A is the mod 2 Steenrod algebra. Consequently, for each nonnegative integer n, $\pi_n(MO)$ is the $\mathbb{F}_2$-linear dual of the $\mathbb{F}_2$-vector space with basis the set
$$\left\{P_\lambda(w_1, \dots, w_{|\lambda|})U : \lambda \in \Lambda, ||\lambda|| = n\right\}. \tag{5}$$
Given two tuples $(a_1, \dots, a_m)$ and $(b_1, \dots, b_n)$, we have their concatenation
$$(a_1, \dots, a_m) \coprod (b_1, \dots, b_n) := (a_1, \dots, a_m, b_1, \dots, b_n).$$

The coproduct on $H^*(MO;\mathbb{F}_2)$ is then given by

$$\Delta(P_\lambda(w_1,\dots,w_{|\lambda|})U) = \sum_{\lambda',\lambda''\in\Lambda:\ \lambda=\lambda'\coprod\lambda''} P_{\lambda'}(w_1,\dots,w_{|\lambda'|})U\otimes P_{\lambda''}(w_1,\dots,w_{|\lambda''|})U.$$

Consequently, if we write $\{Y_\lambda\}$ for the basis of $\pi_*(MO)$ dual to the basis (5), then $Y_\lambda Y_{\lambda'} = Y_{\lambda\coprod\lambda'}$, and $\mathfrak{N}_*\cong\pi_*(MO)\cong\mathbb{F}_2[Y_{(2)},Y_{(4)},Y_{(5)},Y_{(6)},Y_{(8)},\dots]$, with $Y_{(i)}$ in degree i. We will sometimes write Y_i as an abbreviation for Thom's generator $Y_{(i)}$ of $\mathfrak{N}_*$.

2.3. **Maps between bordism theories.** The first stages of the Whitehead tower for the orthogonal group are:

$$BString = BO\langle 8\rangle \longrightarrow BSpin = BO\langle 4\rangle \longrightarrow BSO = BO\langle 2\rangle \longrightarrow BO.$$

While $BSpin^c$ does not fit into this sequence via a connective cover, the map $BSpin\longrightarrow BSO$ factors through $BSpin^c$. There is a commutative diagram whose rows and columns are fiber sequences:

$$\begin{array}{ccccc}
K(\mathbb{Z}/2\mathbb{Z},0) & \longrightarrow & Spin(n) & \longrightarrow & SO(n)\\
\downarrow & & \downarrow & & \downarrow\\
K(\mathbb{Z},1) & \longrightarrow & Spin^c(n) & \longrightarrow & SO(n)\\
\downarrow & & \downarrow & & \downarrow\\
K(\mathbb{Z},1)\cong U(1) & \longrightarrow & U(1) & \longrightarrow & *
\end{array}$$

On the level of spectra, we have maps

$$MString\longrightarrow MSpin\longrightarrow MSpin^c\longrightarrow MSO\longrightarrow MO. \tag{6}$$

The maps induced in homotopy give the maps of respective cobordism rings. We will specifically consider the images of $MSpin^c_*$ and $MSpin_*$ in MO_*.

By the Anderson–Brown–Peterson splitting of 2-local $MSpin^c$, the cohomology $H^*(MSpin^c;\mathbb{F}_2)$ splits as a direct sum of suspensions of $H^*(ku;\mathbb{F}_2)\cong A//E(1)$ and of $H^*(H\mathbb{F}_2;\mathbb{F}_2)\cong A$. Here we are using the standard notation A for the mod 2 Steenrod algebra, and $A//E(1)$ for its quotient $A\otimes_{E(1)}\mathbb{F}_2$, where $E(1)$ is the subalgebra of A generated by Sq^1 and by $Q_1=[\mathrm{Sq}^1,\mathrm{Sq}^2]$. Hence the $s=0$-line in the 2-primary Adams spectral sequence for $MSpin^c$,

$$\begin{aligned} E_2^{s,t}\cong \mathrm{Ext}_A^{s,t}\left(H^*(MSpin^c;\mathbb{F}_2),\mathbb{F}_2\right) &\Rightarrow \pi_{t-s}(MSpin^c)^\wedge_2\\ d_r: E_r^{s,t} &\to E_r^{s+r,t+r-1}\end{aligned} \tag{7}$$

is a direct sum of suspensions of $\mathbb{F}_2$, with one summand $\Sigma^t\mathbb{F}_2$ for each summand $\Sigma^t ku_{(2)}$ in $MSpin^c_{(2)}$, and also with one summand $\Sigma^t\mathbb{F}_2$ for each summand $\Sigma^t H\mathbb{F}_2$ in $MSpin^c_{(2)}$.

Anderson–Brown–Peterson prove in [3] that all differentials in the Adams spectral sequence (7) are zero. Consequently the $s=0$-line $\hom_A(H^*(MSpin^c;\mathbb{F}_2),\mathbb{F}_2)$ is the reduction of $\pi_*(MSpin^c)$ modulo the ideal generated by 2 and by the Bott element $\beta\in\pi_2(ku)$.

This means we can calculate $\pi_*(MSpin^c)/(2,\beta)$ simply by calculating $\hom_A(H^*(MSpin^c;\mathbb{F}_2),\mathbb{F}_2)$, i.e., the A_*-comodule primitives $\mathbb{F}_2\Box_{A_*}H_*(MSpin^c;\mathbb{F}_2)$. The advantage of thinking in terms of comodule primitives is that the Adams spectral sequence respects ring structure: if we calculate the homology $H_*(MSpin^c;\mathbb{F}_2)$ as a ring, then by simply restricting to the comodule primitives in $H_*(MSpin^c;\mathbb{F}_2)$, we have calculated $\Omega_*^{Spin^c}/(2,\beta)$.

The same remarks apply *mutatis mutandis* for the spin bordism spectrum $MSpin$, the oriented bordism spectrum MSO or for the unoriented bordism spectrum MO in place of $MSpin^c$. The Anderson–Brown–Peterson splitting for $MSpin$ is as a wedge of suspensions of ko, $ko\langle 2\rangle$, and $H\mathbb{F}_2$. The analogue of the Anderson–Brown–Peterson splitting for MO is Thom's splitting of MO as a wedge of suspensions of $H\mathbb{F}_2$, while $MSO_{(2)}$ splits as a wedge of suspensions of $H\mathbb{Z}_{(2)}$ and $H\mathbb{F}_2$; as far as we know, the latter splitting was originally proven by Wall [27]. Since $H^*(MSpin^c;\mathbb{F}_2)$ is a quotient A-module of $H^*(MSO;\mathbb{F}_2)$, which is in turn a quotient A-module of $H^*(MO;\mathbb{F}_2)$, dualizing yields that $H_*(MSpin^c;\mathbb{F}_2)$ is a subcomodule of $H_*(MSO;\mathbb{F}_2)$, which is in turn a subcomodule of $H_*(MO;\mathbb{F}_2)$.

Hence our broad strategy for calculating $\Omega_*^{Spin^c}/(2,\beta)\cong MSpin^c_*/(2,\beta)$ and $\Omega_*^{Spin}/(2,\eta,\alpha,\beta)\cong MSpin_*/(2,\eta,\alpha,\beta)$, and the natural maps $MSpin_*/(2,\eta,\alpha,\beta)\to MSpin^c_*/(2,\beta)\to MO_*$, is to calculate the A_*-comodule primitives in $H_*(MSpin^c;\mathbb{F}_2)$ and in $H_*(MSpin;\mathbb{F}_2)$, regarding each as A_*-subcomodule algebras of $H_*(MO;\mathbb{F}_2)$. The resulting information will describe $\Omega_*^{Spin^c}/(2,\beta)$ as a subring of $\mathfrak{N}_*$. Details of this strategy are given in the description of the computational method in the proof of Proposition 3.1.

The relationships between the spin, spin^c, oriented, and unoriented cobordism rings and their homologies is summarized in the following diagram, in which hooked arrows represent one-to-one maps:

$$\begin{array}{ccc}
\mathrm{MSpin}_*/(2,\eta,\alpha,B) & \longrightarrow & H_*(\mathrm{MSpin};\mathbb{F}_2) \\
\big\downarrow & & \big\downarrow \\
\mathrm{MSpin}^c_*/(2,\beta) & \longrightarrow & H_*(\mathrm{MSpin}^c;\mathbb{F}_2) \\
\big\downarrow & & \big\downarrow \\
\mathrm{MSO}_*/(2) & \longrightarrow & H_*(\mathrm{MSO};\mathbb{F}_2) \\
\big\downarrow & & \big\downarrow \\
\mathrm{MO}_* & \hookrightarrow & H_*(\mathrm{MO};\mathbb{F}_2)
\end{array}$$

3. The $Spin^c$ bordism ring in low degrees

In the statement of Proposition 3.1, we use Thom's presentation $\mathbb{F}_2[Y_2,Y_4,Y_5,...]$ for the unoriented cobordism ring $\mathfrak{N}_*$.

Proposition 3.1. *The image of the map* $\Omega_*^{Spin^c} \longrightarrow \mathfrak{N}_*$ *agrees, in degrees* ≤ 33, *with the subring of* $\mathfrak{N}_*$ *generated by the elements*

$$Y_2^2, Y_4^2, Y_5^2, Y_6^2, Y_9^2, Y_{10}^2, Y_{11}^2, Y_{12}^2, Y_{13}^2, Y_{14}^2, Y_{15}^2, Y_{16}^2, T_{24}, T_{29}, T_{31}, T_{32}, \text{ and } T_{33},$$

where

$$\begin{aligned}
T_{24} = {}& Y_{14}Y_5^2 + Y_{13}Y_{11} + Y_{13}Y_9Y_2 + Y_{13}Y_6Y_5 + Y_{13}Y_5Y_2^3 + Y_{12}Y_5^2Y_2 + Y_{11}^2Y_2 \\
&+ Y_{11}Y_9Y_4 + Y_{11}Y_8Y_5 + Y_{11}Y_6Y_5Y_2 + Y_{11}Y_5Y_4^2 + Y_{11}Y_5Y_4Y_2^2 + Y_{10}Y_5^2Y_4 \\
&+ Y_{10}Y_5^2Y_2^2 + Y_9^2Y_4Y_2 + Y_9^2Y_2^3 + Y_9Y_8Y_5Y_2 + Y_9Y_6Y_5Y_4 + Y_9Y_6Y_5Y_2^2 \\
&+ Y_9Y_5^2 + Y_9Y_5Y_4^2Y_2 + Y_6^2Y_5^2Y_2 + Y_5^4Y_4 + Y_5^2Y_4^3Y_2 + Y_5^2Y_4^2Y_2^3,
\end{aligned} \tag{8}$$

$$\begin{aligned}
T_{29} = {}& Y_{19}Y_5^2 + Y_{17}Y_5^2Y_2 + Y_{14}Y_5^3 + Y_{13}Y_6Y_5^2 + Y_{13}Y_5^2Y_2^3 + Y_{11}Y_9^2 + Y_{11}Y_8Y_5^2 \\
&+ Y_{11}Y_5^2Y_4Y_2^2 + Y_{10}Y_9Y_5^2 + Y_{10}Y_5^3Y_2^2 + Y_9^3Y_2 + Y_9^2Y_6Y_5 + Y_9^2Y_5Y_2^3 \\
&+ Y_9Y_6Y_5^2Y_2^2 + Y_9Y_5^4 + Y_6^2Y_5^3Y_2 + Y_5^5Y_4 + Y_5^3Y_4^2Y_2^3,
\end{aligned}$$

$$\begin{aligned}
T_{31} = {}& Y_{21}Y_5^2 + Y_{19}Y_5^2Y_2 + Y_{17}Y_5^2Y_4 + Y_{17}Y_5^2Y_2^2 + Y_{16}Y_5^3 + Y_{13}Y_9^2 + Y_{13}Y_8Y_5^2 \\
&+ Y_{13}Y_5^2Y_4Y_2^2 + Y_{12}Y_9Y_5^2 + Y_{12}Y_5^3Y_2^2 + Y_{11}Y_{10}Y_5^2 + Y_{11}Y_9^2Y_2 + Y_{11}Y_6Y_5^2Y_2^2 \\
&+ Y_9^3Y_4 + Y_9^3Y_2^2 + Y_9^2Y_8Y_5 + Y_9^2Y_5Y_4^2 + Y_9^2Y_5Y_4Y_2^2 + Y_9^2Y_5Y_2^4 \\
&+ Y_9Y_8Y_5^2Y_2^2 + Y_9Y_6^2Y_5^2 + Y_9Y_5^2Y_4^2Y_2^2 + Y_8^2Y_5^3 + Y_6^2Y_5^3Y_4 + Y_5^3Y_4^3Y_2^2 + Y_5^3Y_4^2Y_2^4,
\end{aligned}$$

$$\begin{aligned}
T_{32} = {}& Y_{22}Y_5^2 + Y_{21}Y_{11} + Y_{21}Y_9Y_2 + Y_{21}Y_6Y_5 + Y_{21}Y_5Y_2^3 + Y_{20}Y_5^2Y_2 + Y_{19}Y_{13} \\
&+ Y_{19}Y_9Y_4 + Y_{19}Y_8Y_5 + Y_{19}Y_6Y_5Y_2 + Y_{19}Y_5Y_4^2 + Y_{19}Y_5Y_4Y_2^2 + Y_{18}Y_5^2Y_4 \\
&+ Y_{18}Y_5^2Y_2^2 + Y_{17}Y_{13}Y_2 + Y_{17}Y_{11}Y_4 + Y_{17}Y_8Y_5Y_2 + Y_{17}Y_6Y_5Y_4 + Y_{17}Y_6Y_5Y_2^2 \\
&+ Y_{17}Y_5^3 + Y_{17}Y_5Y_4^2Y_2 + Y_{16}Y_{11}Y_5 + Y_{16}Y_9Y_5Y_2 + Y_{14}Y_{13}Y_5 + Y_{14}Y_{11}Y_5Y_2 \\
&+ Y_{14}Y_9^2 + Y_{14}Y_9Y_5Y_4 + Y_{14}Y_9Y_5Y_2^2 + Y_{13}^2Y_6 + Y_{13}^2Y_2^3 + Y_{13}Y_{11}Y_6Y_2 \\
&+ Y_{13}Y_{10}Y_9 + Y_{13}Y_{10}Y_5Y_2^2 + Y_{13}Y_9Y_8Y_2 + Y_{13}Y_9Y_6Y_4 + Y_{13}Y_6^2Y_5Y_2 + Y_{13}Y_5^3Y_4 \\
&+ Y_{12}Y_{11}Y_9 + Y_{12}Y_{11}Y_5Y_2^2 + Y_{12}Y_{10}Y_5^2 + Y_{12}Y_9Y_6Y_5 + Y_{12}Y_5^4 + Y_{11}^2Y_{10} \\
&+ Y_{11}^2Y_8Y_2 + Y_{11}^2Y_6Y_2^2 + Y_{11}^2Y_4Y_2^3 + Y_{11}Y_{10}Y_6Y_5 + Y_{11}Y_9Y_8Y_4 + Y_{11}Y_9Y_6^2 \\
&+ Y_{11}Y_6^2Y_5Y_4 + Y_{11}Y_6Y_5^3 + Y_{10}^2Y_5^2Y_2 + Y_{10}Y_9Y_8Y_5 + Y_{10}Y_9Y_5Y_4^2 + Y_{10}Y_6^2Y_5^2 \\
&+ Y_9^3Y_5 + Y_9^2Y_8Y_6 + Y_9^2Y_6Y_4^2 + Y_9^2Y_5^2Y_4 + Y_9Y_8^2Y_5Y_2 + Y_9Y_8Y_5^3 + Y_9Y_6^3Y_5 \\
&+ Y_8^2Y_5^2Y_4Y_2 + Y_8^2Y_5^2Y_2^3, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
T_{33} = {}& Y_{23}Y_5^2 + Y_{21}Y_5^2Y_2 + Y_{19}Y_5^2Y_4 + Y_{18}Y_5^3 + Y_{17}Y_6Y_5^2 + Y_{14}Y_9Y_5^2 + Y_{13}Y_{10}Y_5^2 \\
&+ Y_{13}Y_5^2Y_4^2Y_2 + Y_{12}Y_{11}Y_5^2 + Y_{11}Y_{11}Y_{11} + Y_{11}Y_{11}Y_9Y_2 + Y_{11}Y_{11}Y_6Y_5 + Y_{11}Y_{11}Y_5Y_2^3 \\
&+ Y_{11}Y_5^2Y_4^2Y_4 + Y_{10}Y_5^3Y_4^2 + Y_9^2Y_5^3 + Y_9Y_6Y_5^2Y_4^2 + Y_8^2Y_5^3Y_2 + Y_5^5Y_4^2 + Y_5^3Y_4^4Y_2.
\end{aligned}$$

Proof. This proposition is proven using computer calculation. We will describe our method for calculating im $(\Omega_*^{Spin^c} \to \Omega_*^O)$ in degrees $\leq d$, for any fixed choice of d. The first author wrote a Magma [8] program which implements this method, and we have made its source code available at `https://github.com/hassan-abdallah/spinc_cobordism`. Once the reader is convinced of the correctness of the method, the proof of this proposition consists of simply running the calculation through degree 33, either by using our software, or by writing their own software implementation of the method, if desired.

We freely use the relationship between the spin^c-cobordism ring, the unoriented cobordism ring, and the mod 2 cohomology of BO detailed in section 2. Let λ be a non-dyadic partition of a nonnegative integer n. We want to know whether its corresponding element $Y_\lambda \in MO_n$ is in the image of the map $MSpin^c_n \to MO_n$. The element $Y_\lambda \in MO_n$ has a Hurewicz image, i.e., the image of Y_λ under the Hurewicz map $\pi_n(MO) \to H_n(MO;\mathbb{F}_2)$. In section 2.2, we described the dual element $P_\lambda U \in H^n(MO;\mathbb{F}_2)$ to the Hurewicz image of Y_λ, using Thom's basis for MO_*. The element $P_\lambda U$ can be written as U times a polynomial in the Stiefel–Whitney classes by applying an appropriate transition matrix[3]. Once $P_\lambda U$ is calculated, we see that Y_λ is in the image if and only if, when reduced modulo w_1 and the relations in the $H^*(BO;\mathbb{F}_2)$-module $H^*(MSpin^c;\mathbb{F}_2)$, $P_\lambda U$ is an A-module primitive in $H^*(MSpin^c;\mathbb{F}_2)$.

Consequently our method for calculating $\mathrm{im}\,(\Omega_*^{Spin^c} \to \Omega_*^O)$ is merely a method for building up a basis for the $\mathbb{F}_2$-vector space of A-module primitives through some fixed degree d, *in terms of non-dyadic partitions.* We work one degree at a time, but via induction, assuming we have already completed the calculation at all lower degrees.

The induction begins at degree 0, where there is nothing to say: the empty partition $\varnothing$ yields the unique A-module primitive $\varnothing \cdot U$ in $H^0(MSpin^c;\mathbb{F}_2)$. For each integer $n \in [1,d]$, the product $\mathrm{Sq}^n \cdot \varnothing \cdot U \in H^n(MSpin^c;\mathbb{F}_2)$ is simply $w_n U$ modulo the relations in $H^*(MSpin^c;\mathbb{F}_2)$, by the classical formula $\mathrm{Sq}^n U = w_n U$ for the action of Steenrod squares on the Thom class in $H^0(MO;\mathbb{F}_2)$. Record the elements $\{\mathrm{Sq}^1 U, \mathrm{Sq}^2 U, \dots, \mathrm{Sq}^d U\}$ in an unordered list D. Here the symbol D stands for "decomposable," as we will use it to build up a list of A-module decomposables in $H^*(MSpin^c;\mathbb{F}_2)$ in degrees $\leq d$.

We are not done with the initial step in the induction: for each nonzero element $\mathrm{Sq}^n U$, we calculate $\{\mathrm{Sq}^1 \mathrm{Sq}^n U, \mathrm{Sq}^2 \mathrm{Sq}^n U, \dots, \mathrm{Sq}^{d-n} \mathrm{Sq}^n U\}$ using the Thom formula $\mathrm{Sq}^n U = w_n U$ and the Wu formula ([28], but see [16, pg. 94] for a textbook reference) for the action of Sq^n on Stiefel–Whitney classes, and we include the results in D. We keep going: calculate all the three-fold composites of Steenrod squares applied to U *landing in degrees* $\leq d$, then all the four-fold composites of Steenrod squares applied to U *landing in degrees* $\leq d$, and so on. We emphasize the phrase "landing in degrees $\leq d$" because it is what ensures that this calculation eventually terminates!

After we are done with that inductive calculation, D now contains an $\mathbb{F}_2$-linear basis for the A-submodule of $H^*(MSpin^c;\mathbb{F}_2)$ generated by the Thom class U, in all degrees $\leq d$.

Now we are ready for the inductive step:

Inductive hypothesis at the nth step: We have produced a list G of $\mathbb{F}_2$-linear combinations of non-dyadic partitions of degree $< n$, such that the $\mathbb{F}_2$-linear span of $\{P_\lambda U : \lambda \in G\}$ is a basis for the set of A-module primitives in $H^*(MSpin^c;\mathbb{F}_2)$ in all degrees $< n$. We have also produced a list D of $\mathbb{F}_2$-linear combinations of non-dyadic partitions of degree $\leq d$, such

[3]We remark that the computation of this transition matrix is one of the most computationally expensive parts of this process, despite its being a simple combinatorial problem. It is in fact the inverse of a Kostka matrix [23].

that the $\mathbb{F}_2$-linear span of $\{P_\lambda U : \lambda \in D\}$ is precisely the A-submodule of $H^*(MSpin^c; \mathbb{F}_2)$ generated by G in degrees $\leq d$.

Calculation for the nth inductive step: Write D_n for the $\mathbb{F}_2$-linear span of the degree n elements in D. Calculate an $\mathbb{F}_2$-linear basis B for $H^n(MSpin^c; \mathbb{F}_2)/D_n$. Let G' be $G \cup B$. Use the calculated transition matrix to convert the members of G' from the Thom/partition basis to the Stiefel–Whitney monomial basis, and then use the Thom formula and the Wu formula to calculate all Steenrod squares on the members of G', then all Steenrod squares on those, etc., in degrees $\leq d$. Use the transition matrix to convert back to the partition basis, and D' for the resulting list of linear combinations of non-dyadic partitions. Now we are ready to iterate, with G' in place of G, and with D' in place of D.

Once we complete the $n = d$ step, we have an $\mathbb{F}_2$-linear basis for the image of the map $\Omega_*^{Spin^c} \to \mathfrak{N}_*$ in all degrees $\leq d$, expressed in terms of Thom's partition basis for $\mathfrak{N}_*$. Consequently we have a description of $\Omega_*^{Spin^c}/(2, \beta)$, in all degrees $\leq d$, as a subring of $\mathfrak{N}_* \cong \mathbb{F}_2[Y_2, Y_4, Y_5, Y_6, Y_8, Y_9, \dots]$. □

In principle there is no obstruction to using the same method to make calculations of products in $\Omega_*^{Spin^c}$ in degrees > 33. We stopped at degree 33 simply because, around the time we completed degree 33, we could see enough of the ring structure of $\Omega_*^{Spin^c}$ to prove that the mod $(2, \beta)$ spin^c-cobordism ring is not a polynomial algebra, and to suggest the right statements for Proposition 4.2 and Theorem C. The products in $\Omega_*^{Spin^c}$ required for the proof of Theorem 3.6 are known as soon as one computes the ring structure through degree 24.

The same computational method described in the proof of Proposition 3.1, applied to $MSpin$ rather than $MSpin^c$, yields:

Proposition 3.2. *The image of the map $\Omega_*^{Spin} \longrightarrow \mathfrak{N}_*$ agrees, in degrees ≤ 31, with the subring of MO_* generated by the elements*

$$\langle Y_2^4, Y_5^2, Y_4^4, Y_9^2 + Y_5^2 Y_4^2, Y_{11}^2 + Y_9^2 Y_2^2 + Y_5^2 Y_4^2 Y_2^2, Y_{13}^2 + Y_{11}^2 Y_2^2 + Y_9^2 Y_4^2 + Y_8^2 Y_5^2, Y_6^4$$
$$T_{24} + Y_{12}^2 + Y_{10}^2 Y_2^2 + Y_8^2 Y_4^2 + Y_8^2 Y_2^4 + Y_6^2 Y_4^2 Y_2^2 + Y_5^4 Y_2^2 + Y_4^6, T_{29}\rangle.$$

Each of the elements T_i defined in Proposition 3.2 is a linear combination of monomials in $\mathfrak{N}_*$. Those monomials are generally not *individually* members of $\Omega_*^{Spin^c}$: for example, $Y_{14} Y_5^2 \in \mathfrak{N}_{24}$ does not lift to an element of $\Omega_{24}^{Spin^c}$, even though a linear combination of $Y_{14} Y_5^2$ with other monomials in degree 24 *does* lift to the element $T_{24} \in \Omega_{24}^{Spin^c}$.

However, $Y_i^2 \in \mathfrak{N}_{2i}$ lifts to the element $Z_{2i} \in \Omega_{2i}^{Spin^c}$, and consequently *the squares of each of the monomials in each of the elements T_i lift to $\Omega_*^{Spin^c}$*. For $i = 24, 29, 31, 32,$ and 33, let U_{2i} denote the element of $\Omega_*^{Spin^c}$ obtained by taking the definition of T_i in Proposition 3.1 and replacing each instance of Y_n with Z_{2n}.

For example, (8) yields that

$$\begin{aligned}U_{48} = {} & Z_{28}Z_{10}^2 + Z_{26}Z_{22} + Z_{26}Z_{18}Z_4 + Z_{26}Z_{12}Z_{10} + Z_{26}Z_{10}Z_4^3 + Z_{24}Z_{10}^2Z_4 + Z_{22}^2Z_4 \\ & + Z_{22}Z_{18}Z_8 + Z_{22}Z_{16}Z_{10} + Z_{22}Z_{12}Z_{10}Z_4 + Z_{22}Z_{10}Z_8^2 + Z_{22}Z_{10}Z_8Z_4^2 \\ & + Z_{20}Z_{10}^2Z_8 + Z_{20}Z_{10}^2Z_4^2 + Z_{18}^2Z_8Z_4 + Z_{18}^2Z_4^3 + Z_{18}Z_{16}Z_{10}Z_4 + Z_{18}Z_{12}Z_{10}Z_8 \\ & + Z_{18}Z_{12}Z_{10}Z_4^2 + Z_{18}Z_{10}^2 + Z_{18}Z_{10}Z_8^2Z_4 + Z_{12}^2Z_{10}^2Z_4 + Z_{10}^4Z_8 + Z_{10}^2Z_8^3Z_4 \\ & + Z_{10}^2Z_8^2Z_4^3.\end{aligned}$$

Then, as a consequence of Proposition 3.1, we have:

Theorem 3.3. *The subring of $\Omega_*^{Spin^c}/(2,\beta)$ generated by all homogeneous elements of degree ≤ 33 is isomorphic to:*

$$\mathbb{F}_2[Z_4, Z_8, Z_{10}, Z_{12}, Z_{16}, Z_{18}, Z_{20}, Z_{22}, Z_{24}, Z_{26}, Z_{28}, Z_{32}, T_{24}, T_{29}, T_{31}, T_{32}, T_{33}]/I,$$

where I is the ideal generated by $T_{24}^2 - U_{48}$, $T_{29}^2 - U_{58}$, $T_{31}^2 - U_{62}$, $T_{32}^2 - U_{64}$, and $T_{33}^2 - U_{66}$.

The relations $T_i^2 = U_{2i}$, with T_i indecomposable in $\Omega_i^{Spin^c}$ and with U_{2i} a polynomial in the indecomposable elements Z_n, immediately implies that $\Omega_*^{Spin^c}/(2,\beta)$ is not a polynomial algebra.

Theorem 3.4. *The subring of the mod 2 spin^c-cobordism ring $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ generated by all homogeneous elements of degree ≤ 33 is isomorphic to:*

$$\mathbb{F}_2[\beta, Z_4, Z_8, Z_{10}, Z_{12}, Z_{16}, Z_{18}, Z_{20}, Z_{22}, Z_{24}, Z_{26}, Z_{28}, Z_{32}, T_{24}, T_{29}, T_{31}, T_{32}, T_{33}]/I,$$

where I is the ideal generated by

- *βZ_i for each $i \equiv 2 \mod 4$,*
- *and βT_i and $T_i^2 - U_{2i}$ for $i = 24, 29, 31, 32, 33$.*

Proof. Let T denote the ideal of $\Omega_*^{Spin^c}$ consisting of 2-torsion elements, and let $\tilde{T}$ denote the kernel of the ring map

$$\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2 \to \left(\Omega_*^{Spin^c}/T\right) \otimes_{\mathbb{Z}} \mathbb{F}_2.$$

The ring $\left(\Omega_*^{Spin^c}/T\right) \otimes_{\mathbb{Z}} \mathbb{F}_2$ was calculated by Stong [24, Proposition 11]: it is a polynomial $\mathbb{F}_2[\beta]$-algebra on generators in degrees $4, 8, 12, 16, \dots$. Since $\tilde{T}$ is an ideal in $\Omega_*^{Spin^c}$, to calculate the product in the ring $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$, it suffices to calculate

- the products between generators of $\tilde{T}$,
- and the products between generators of $\tilde{T}$ and lifts, to $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ of generators of $\left(\Omega_*^{Spin^c}/T\right) \otimes_{\mathbb{Z}} \mathbb{F}_2$.

Both of these types of products land in $\tilde{T}$. Since $\tilde{T}$ maps injectively under the map $\Omega_*^{Spin^c} \to \mathfrak{N}_*$, we can embed $\Omega_*^{Spin^c}/(2,\beta)$ into $\mathfrak{N}_*$ and bring to bear our calculations of the image of this map, from Proposition 3.1. All we need to do is to determine, in our set of generators for $\Omega_*^{Spin^c}/(2,\beta)$ through degree 33, a maximal set of linear combinations of products of generators which generate β-torsion elements in $\Omega_*^{Spin^c}/(2)$, i.e., copies of $H\mathbb{F}_2$ rather than $ku_{(2)}$ in the Anderson–Brown–Peterson splitting.

As a consequence of the Anderson–Brown–Peterson splitting, generators of $\Omega_*^{Spin^c}/(2,\beta)$ that are 2-torsion, and hence β-torsion, in $\Omega_*^{Spin^c}$ are those whose corresponding A-module primitive in $H^*(MSpin^c;\mathbb{F}_2)$ is *not* Q_0 or Q_1 torsion. We identify such generators by re-running the entire process from the proof of Proposition 3.1, but with the following modification: at the start of the calculation, before the induction on degree, we begin by letting D be a list of *all Stiefel–Whitney monomials in* $H^*(MSpin^c;\mathbb{F}_2)$ *in degrees* $\leq d$ *which are* (Q_0,Q_1)*-torsion,* together with all words in the Steenrod squares applied to such (Q_0,Q_1)-torsion Stiefel–Whitney monomials[4], instead of letting D begin as the empty set. Consequently, as we proceed through the induction, D is not only the set of A-module decomposables, but also the set of Stiefel–Whitney monomials which generate copies of $A//E(1)$.

Re-running our inductive calculation from Proposition 3.1, but with this initial list for D, yields a set of A-module generators for $H^*(MSpin^c;\mathbb{F}_2)$ *modulo* (Q_0,Q_1)*-torsion*. Comparison of the lists produced by the first calculation and the second calculation, then using the translation matrix to translate back from the basis of Stiefel–Whitney monomials to the dual basis of partitions (i.e., Thom's basis for $\mathfrak{N}_*$), gives us a set of generators for im $\left(\Omega_*^{Spin^c}\to\mathfrak{N}_*\right)$ in degrees $0,1,\dots,n$, and tells us, for each generator, whether it corresponds to a copy of $ku_{(2)}$ or of $H\mathbb{F}_2$ under the Anderson–Brown–Peterson splitting.

In degrees ≤ 33, we find that an element $Y_\lambda\in\mathfrak{N}_*$ which is in the image of the map $\Omega_*^{Spin^c}\to\mathfrak{N}_*$ is 2-torsion as long as the partition λ includes an odd number. As described in section 1.1, the β-torsion elements of $\Omega_*^{Spin^c}$ are exactly the 2-torsion elements. This yields the presentation for $\Omega_*^{Spin^c}$ in degrees ≤ 33 in the statement of the theorem. □

[4]In principle, this step in the calculation could go wrong, failing to identify all the (Q_0,Q_1)-torsion in $H^*(MSpin^c;\mathbb{F}_2)$, as follows: suppose there is some $\mathbb{F}_2$-linear combination of Stiefel–Whitney monomials in $H^*(MSpin^c;\mathbb{F}_2)$ which is (Q_0,Q_1)-torsion, but *none of its summands are* (Q_0,Q_1)*-torsion*. If this occurs, it would *not* be noticed by the method we describe, since our method only checks the (Q_0,Q_1)-torsion status of Stiefel–Whitney monomials.

We handle this by a very simple idea: we make the calculation as described, and after making the full calculation, we "check our answer" by comparing to the known *additive* structure of $\Omega_*^{Spin^c}$, as follows. After running our method, we compare the rank of our calculated (Q_0,Q_1)-torsion in each degree to the expected rank, using the known Poincaré series for the 2-torsion in $\pi_*(MSpin^c)$. If our method has failed to notice a *linear combination* of Stiefel–Whitney monomials which was (Q_0,Q_1)-torsion despite its summands not being (Q_0,Q_1)-torsion, then the rank of the (Q_0,Q_1)-torsion from our calculation will be too small.

We have never observed this mismatched rank to happen, i.e., it does not happen through degree 33 in $MSpin^c_*$. If the rank mismatch were to ever occur, the fix is conceptually trivial, but computationally very hard: instead of populating D with all the homogeneous (Q_0,Q_1)-torsion Stiefel–Whitney monomials at the start of the torsion calculation, we simply populate D with the all the (Q_0,Q_1)-torsion Stiefel–Whitney *polynomials* at the start of the calculation, then re-run the calculation. This of course cannot miss any (Q_0,Q_1)-torsion in $H^*(MSpin^c;\mathbb{F}_2)$! Its disadvantage is simply that it is extremely computationally expensive, since the total number of homogeneous Stiefel–Whitney *polynomials* (not just monomials) in degrees $\leq d$ in $H^*(MSpin^c;\mathbb{F}_2)$ grows extremely quickly as d grows. We carry out the calculations in the way we describe—i.e., initially populating D by only the (Q_0,Q_1)-torsion Stiefel–Whitney *monomials*, rather than *polynomials*—to dramatically speed up calculation, and because we are able to check that the resulting answer in the end agrees with the answer we would have gotten with the much slower calculation using *all* the homogeneous Stiefel–Whitney polynomials.

Our next result determines explicit manifolds that represent some of the bordism classes whose powers occurs as ring-theoretic generators of $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ and of $\Omega_*^{Spin} \otimes_{\mathbb{Z}} \mathbb{F}_2$ in Proposition 3.1 and in Proposition 3.2, respectively. The following table was calculated through degree 6 by Thom [26]. We extend the calculation through degree 17. The symbol D_i denotes the i-dimensional Dold manifold, defined in [10].

Proposition 3.5 (Thom [26])**.** *Manifold representatives for elements Y_n in Thom's partition basis for $\mathfrak{N}_*$ are as follows:*

Element	*Manifold Representative*
Y_2	$\mathbb{R}P^2$
Y_4	$\mathbb{R}P^4 \sqcup \mathbb{R}P^2 \times \mathbb{R}P^2$
Y_5	D_5
Y_6	$\mathbb{R}P^6$
Y_8	$\mathbb{R}P^8 \sqcup (\mathbb{R}P^4)^2 \sqcup \mathbb{R}P^4 \times (\mathbb{R}P^2)^2 \sqcup (\mathbb{R}P^2)^4$
Y_9	$D_9 \sqcup D_5 \times \mathbb{R}P^4 \sqcup D_5 \times (\mathbb{R}P^2)^2$
Y_{10}	$\mathbb{R}P^{10} \sqcup (\mathbb{R}P^2)^5$
Y_{11}	$\mathbb{D}_{11} \sqcup \mathbb{D}_9 \times \mathbb{R}P^2$
Y_{12}	$\mathbb{R}P^{12} \sqcup (\mathbb{R}P^6)^2 \sqcup \mathbb{R}P^8 \times (\mathbb{R}P^2)^2 \sqcup (D_5^2 \times \mathbb{R}P^2 \sqcup (\mathbb{R}P^4)^3$
Y_{13}	$D_{13} \sqcup D_{11} \times \mathbb{R}P^2 \sqcup D_9 \times \mathbb{R}P^4 \sqcup D_8 \times \mathbb{R}P^5 \sqcup D_5 \times \mathbb{R}P^4 \times (\mathbb{R}P^2)^2$
Y_{14}	$\mathbb{R}P^{14}$
Y_{16}	$\mathbb{R}P^{16} \sqcup \mathbb{R}P^{12} \times (\mathbb{R}P^2)^2 \sqcup (\mathbb{R}P^8)^2 \sqcup \mathbb{R}P^8 \times (\mathbb{R}P^4)^2$ $\sqcup \mathbb{R}P^8 \times (\mathbb{R}P^2)^4 \sqcup (\mathbb{R}P^6)^2 \times \mathbb{R}P^4 \sqcup \mathbb{R}P^6 \times (D_5)^2$ $\sqcup (D_5)^2 \times (\mathbb{R}P^2)^3 \sqcup (\mathbb{R}P^4)^4 \sqcup (\mathbb{R}P^4)^2 \times (\mathbb{R}P^2)^4 \sqcup (\mathbb{R}P^2)^8$
Y_{17}	$D_{17} \sqcup D_{13} \times \mathbb{R}P^4 \sqcup D_{13} \times (\mathbb{R}P^2)^2 \sqcup \mathbb{R}P^{12} \times D_5 \sqcup D_{11} \times \mathbb{R}P^6$ $\sqcup D_{11} \times \mathbb{R}P^4 \times \mathbb{R}P^2 \sqcup D_{11} \times (\mathbb{R}P^2)^3 \sqcup D_9 \times \mathbb{R}P^8 \sqcup D_9 \times \mathbb{R}P^6 \times \mathbb{R}P^2$ $\sqcup \mathbb{R}P^8 \times D_5 \times (\mathbb{R}P^2)^2 \sqcup (\mathbb{R}P^6)^2 \times D_5 \sqcup (D_5)^3 \times \mathbb{R}P^2$ $\sqcup D_5 \times (\mathbb{R}P^4)^2 \times (\mathbb{R}P^2)^2 \sqcup D_5 \times \mathbb{R}P^4 \times (\mathbb{R}P^2)^4 \sqcup D_5 \times (\mathbb{R}P^2)^6$

Proof. Routine calculation using Stiefel–Whitney numbers. □

In [15], Milnor investigates whether every spin manifold is unorientedly cobordant to the square of an orientable manifold. He shows it is true for spin manifolds of dimension ≤ 23. The ambiguity in dimension 24 stems from the existence of an orientable manifold whose only nonzero Stiefel–Whitney numbers are $w_4w_6w_7^2$, w_6^4, w_4^6, $w_4^3w_6^2$, and $w_4^2w_8^2$. Milnor then poses the problem of whether a spin manifold of dimension 24 exists with these nonzero Stiefel–Whitney numbers. Anderson–Brown–Peterson stated two years later [3] that, as a corollary of their main theorem, the lowest dimension in which there exists an element of $\mathrm{Im}(\Omega_*^{Spin} \to \mathfrak{N}_*)$ which is not the square of an orientable manifold is 24 [3]. In Proposition 3.2 we calculated Ω_*^{Spin} in degrees through 31 in terms of Thom's partition basis, and in proposition 3.5 we have manifold representatives for ring-theoretic generators which suffice to generate everything in Ω_{24}^{Spin}. Solving for an element of Ω_{24}^{Spin} with Milnor's prescribed Stiefel–Whitney numbers, we find an explicit manifold of the kind Milnor asked for:

Theorem 3.6. *The cobordism class $T_{24} + Y_{12}^2 + Y_{10}^2Y_2^2 + Y_8^2Y_4^2 + Y_8^2Y_2^4 + Y_6^2Y_4^2Y_2^2 + Y_5^4Y_2^2 + Y_4^6 \in Im(\Omega_*^{Spin} \to \mathfrak{N}_*)$ has nonzero Stiefel–Whitney numbers $w_4w_6w_7^2$, w_6^4,*

w_4^6, $w_4^3 w_6^2$, and $w_4^2 w_8^2$, and is represented by the manifold:

$$\begin{aligned}
&(\mathbb{R}P^2)^6 \times (\mathbb{R}P^6)^2 \sqcup (\mathbb{R}P^4)^6 \sqcup \mathbb{R}P^2 \times (\mathbb{R}P^4)^3 \times (D^5)^2 \sqcup (\mathbb{R}P^2)^2 \times (\mathbb{R}P^4)^2 \times (\mathbb{R}P^6)^2 \\
&\quad \sqcup (\mathbb{R}P^2)^4 \times (\mathbb{R}P^8)^2 \sqcup (\mathbb{R}P^2)^3 \times \mathbb{R}P^4 \times D^5 \times (D^9) \sqcup (\mathbb{R}P^4)^2 \times (D^5)^2 \times (\mathbb{R}P^6) \\
&\quad \sqcup (\mathbb{R}P^2)^2 \times D^5 \times \mathbb{R}P^6 \times (D^9) \sqcup (\mathbb{R}P^6)^4 \sqcup (D^5)^2 \times \mathbb{R}P^6 \times (\mathbb{R}P^8) \\
&\quad \sqcup (\mathbb{R}P^4)^2 \times (\mathbb{R}P^8)^2 \sqcup (D^5)^3 \times (D^9) \sqcup \mathbb{R}P^4 \times (D^5)^2 \times (\mathbb{R}P^{10}) \\
&\quad \sqcup (\mathbb{R}P^2)^2 \times (\mathbb{R}P^{10})^2 \sqcup (\mathbb{R}P^4)^2 \times D^5 \times (D^{11}) \sqcup (\mathbb{R}P^2)^2 \times D^9 \times (D^{11}) \\
&\quad \sqcup \mathbb{R}P^2 \times (D^5)^2 \times (\mathbb{R}P^{12}) \sqcup \mathbb{R}P^2 \times \mathbb{R}P^4 \times D^5 \times (D^{13}) \sqcup D^5 \times \mathbb{R}P^6 \times (D^{13}) \\
&\quad \sqcup (D^5)^2 \times (\mathbb{R}P^{14}) \sqcup (\mathbb{R}P^{12})^2 \sqcup (D^{11}) \times (D^{13}) \sqcup (\mathbb{R}P^2)^6 \times (\mathbb{R}P^6)^2 \sqcup (\mathbb{R}P^4)^6 \\
&\quad \sqcup (\mathbb{R}P^2)^2 \times (\mathbb{R}P^4)^2 \times (\mathbb{R}P^6)^2 \sqcup (\mathbb{R}P^2)^4 \times (\mathbb{R}P^8)^2 \sqcup (\mathbb{R}P^6)^4 \\
&\quad \sqcup (\mathbb{R}P^4)^2 \times (\mathbb{R}P^8)^2 \sqcup (\mathbb{R}P^2)^2 \times (\mathbb{R}P^{10})^2 \sqcup (\mathbb{R}P^{12})^2.
\end{aligned}$$

4. The spin^c-cobordism ring in all degrees.

4.1. **A nonunital subring of the 2-torsion in the spin^c-cobordism ring.** In Theorem 3.4, we showed that, in degrees ≤ 33, the cobordism classes $Y_{(5,5)}, Y_{(9,9)}, Y_{(11,11)}$, and $Y_{(13,13)} \in \mathfrak{N}_*$ lift to indecomposable 2-torsion elements Z_{10}, Z_{18}, Z_{22}, and Z_{26} in $\Omega_*^{Spin^c}$. The elements $Y_{(5,5)}, Y_{(9,9)}, \dots$ are precisely those of the form $Y_i^2 \in \mathfrak{N}_*$ with i odd and non-dyadic. It is natural to ask whether this pattern extends above degree 33 as well. In Proposition 4.2, we show that something of this kind is true, if we use the Dold manifold D_i, from [10], rather than the cobordism class Y_i from Thom's partition basis for $\mathfrak{N}_*$. In low degrees, the algebraic relation between the Dold manifolds and the Thom generators for $\mathfrak{N}_*$ is as follows:

Proposition 4.1. *The squares of odd-dimensional Dold manifolds represent the following polynomials in Thom's generators $Y_2, Y_4, Y_5, Y_6, Y_8, \dots$ for $\mathfrak{N}_*$:*

Element	*Manifold Representative*
Y_5^2	D_5^2
$Y_9^2 + Y_5^2 Y_4^2$	D_9^2
$Y_{11}^2 + Y_9^2 Y_2^2 + Y_5^2 Y_4^2 Y_2^2$	D_{11}^2
$Y_{13}^2 + Y_{11}^2 Y_2^2 + Y_9^2 Y_4^2 + Y_8^2 Y_5^2$	D_{13}^2

Furthermore, each of these elements of Ω_*^O is in the image of the map $\Omega_*^{Spin} \to \Omega_*^O$.

Proof. The manifold representatives are straightforwardly calculated from Proposition 3.5. By Proposition 3.2, these elements are in the image of the map $\Omega_*^{Spin} \to \Omega_*^O$. □

Proposition 4.2. *For each odd integer i such that $i+1$ is not a power of 2, the Dold manifold D_i has the property that $D_i \times D_i$ lifts to an indecomposable $(2,\beta)$-torsion element of $\Omega_{2i}^{Spin^c}$. It furthermore lifts to an indecomposable 2-torsion element of Ω_{2i}^{Spin}.*

Proof. Dold [10] proves that there exists a minimal set of generators for the cobordism ring $\mathfrak{N}_*$ whose odd-degree elements are $D_i = P(2^r - 1, s2^r)$, where $i + 1 = 2^r(2s + 1)$ and i is odd. Milnor proved that the map $\Omega_*^U \longrightarrow \mathfrak{N}_*$ maps onto all squares of elements in $\mathfrak{N}_*$. This map factors through $\Omega_*^{Spin^c}$, so $D_i \times D_i$ lifts to

$\Omega_{2i}^{Spin^c}$ for all non-dyadic i. The mod 2 cohomology of the Dold manifold $P(m,n)$ is given as a graded ring by

$$H^*(P(m,n);\mathbb{F}_2) \cong \mathbb{F}_2[c,d]/(c^{m+1},d^{n+1}),$$

with $c \in H^1(P(m,n);\mathbb{F}_2)$ and $d \in H^2(P(m,n);\mathbb{F}_2)$. The total Stiefel–Whitney class of $P(m,n)$ is

$$w(P(m,n)) = (1+c)^m(1+c+d)^{n+1}.$$

Setting $m = 2^r - 1$ and $n = s2^r$, the total Stiefel–Whitney class of D_i as in the statement of the theorem is then given by:

$$w(D_i) = (1+c)^{2^r-1}(1+c+d)^{s2^r},$$

and in particular, $w_1 = 0$. Hence D_i is orientable.

To show that $D_i \times D_i$ lifts to the spin cobordism ring, one can carry out an algebraic calculation to show that none of the nonzero Stiefel–Whitney numbers of $D_i \times D_i$ are divisible by w_2. This is not a difficult calculation, but it is simpler to invoke the main result of Anderson's paper [4]: the square of any orientable compact manifold is unorientedly cobordant to a spin manifold. Since w_1 vanishes on D_i, its square $D_i \times D_i$ must be in the image of the map $\Omega_{2i}^{Spin} \to \mathfrak{N}_{2i}$.

It follows easily from the structure of ko_*, and the Anderson–Brown–Peterson splitting of MSpin into a wedge of suspensions of ko, $ko\langle 2\rangle$, and $H\mathbb{F}_2$, that all elements of Ω_*^{Spin} in degrees $\equiv 2 \mod 4$ are 2-torsion. Hence, for odd i, any spin-cobordism class that lifts $D_i \times D_i \in \mathfrak{N}_{2i}$ must be 2-torsion.

Let $\widetilde{D_i^2}$ be a lift of $D_i \times D_i \in \mathcal{N}_{2i}$ to Ω_{2i}^{Spin}. The image of $\widetilde{D_i^2}$ under the map $\Omega_{2i}^{Spin} \to \Omega_{2i}^{Spin^c}$ is then a lift of D_i^2 to $\Omega_{2i}^{Spin^c}$, and it is 2-torsion since it is the image of a 2-torsion element. It is furthermore β-torsion in $\Omega_*^{Spin^c}$, since by the Anderson–Brown–Peterson splitting of 2-local $MSpin^c$, every 2-torsion element of $\Omega_*^{Spin^c}$ is also β-torsion.

To see that $D_i^2 \in \Omega_{2i}^O$ lifts to an *indecomposable* element of Ω_*^{Spin}, it is enough to observe that $\Omega_*^{Spin}/(2,\eta,\alpha,\beta)$ embeds into $\mathfrak{N}_*$, and since D_i has second Stiefel–Whitney class $d \neq 0$, D_i does not lift to Ω_*^{Spin}. Hence the unique lift of D_i^2 to $\Omega_*^{Spin}/(2,\eta,\alpha,\beta)$ is indecomposable, hence any lift of D_i^2 to Ω_*^{Spin} is indecomposable. A completely analogous argument establishes that any lift of D_i^2 to $\Omega_*^{Spin^c}$ is also indecomposable. □

Since Dold [10] showed that $D_i \in \mathfrak{N}_i$ can be written as Thom's generator Y_i plus decomposables in the same degree, Proposition 4.2 tells us that *some of* the patterns exhibited in table (2) are not limited to degrees ≤ 33, and indeed extend to *all* degrees. Namely, for odd non-dyadic i, if we write Z_{2i} for a lift of $D_i \times D_i \in \mathfrak{N}_{2i}$ to an indecomposable 2-torsion element of $\Omega_{2i}^{Spin^c}$ (guaranteed to exist by Proposition 4.2), and for even i we write Z_{2i} for a lift of $Y_i^2 \in \mathfrak{N}_{2i}$ to an element of $\Omega_{2i}^{Spin^c}$, then we have:

Theorem 4.3. *Consider the spin^c-cobordism ring as a graded algebra over the graded ring $S := \mathbb{Z}_{(2)}[\beta, Z_{2j} : j \geq 2,\ j$ non-dyadic$]/(\beta Z_{2j},\ 2Z_{2j}$ for odd $j)$. Then the ideal $(Z_{2j} : j$ odd$)$ of S embeds, as a non-unital graded S-algebra, into the 2-torsion ideal $\pi_*(Z)$ of the spin^c-cobordism ring.*

4.2. **Mod 2 spin^c-cobordism, up to uniform F-isomorphism.** There is a subring of the mod 2 spin^c-cobordism ring $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ which is generated by

- the large nonunital subring of the 2-torsion in $\Omega_*^{Spin^c}$ constructed in Theorem 4.3,
- and Stong's generators $y_4, y_8, y_{12}, y_{16}, \dots$ of the mod-torsion mod-2 spin^c-cobordism ring $(\Omega_*^{Spin^c}/tors) \otimes_{\mathbb{Z}} \mathbb{F}_2$.

This subring of $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ is strictly smaller than $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ itself. However, we will now show that this subring is *uniformly F-isomorphic* to $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$. See Definition 1.1 for the definition of a uniform F-isomorphism.

Theorem 4.4. *The mod 2 spin^c-cobordism ring is uniformly F-isomorphic to the graded $\mathbb{F}_2$-algebra*

$$\mathbb{F}_2\left[\beta, y_{4i}, Z_{4j-2} : i \geq 1,\ j \geq 1,\ 2j \text{ not a power of } 2\right] / (\beta Z_{4j-2}), \tag{9}$$

with β the Bott element in degree 2, with y_{4i} in degree $4i$, and with Z_{4j-2} in degree $4j-2$.

Proof. Write $tors$ for the ideal of $\Omega_*^{Spin^c}$ consisting of the 2-torsion elements. Stong [24, Proposition 14] proved that $\left(\Omega_*^{Spin^c}/tors\right) \otimes_{\mathbb{Z}} \mathbb{F}_2$ is a polynomial $\mathbb{F}_2$-algebra on generators y_2 and $y_4, y_8, y_{12}, \dots$. Since $\Omega_2^{Spin^c} \cong \mathbb{Z}$ generated by $\beta = [\mathbb{C}P^1]$, Stong's generator y_2 agrees modulo 2 with the Bott element β. Hence $\left(\Omega_*^{Spin^c}/tors\right) \otimes_{\mathbb{Z}} \mathbb{F}_2$ is isomorphic as a graded $\mathbb{F}_2$-algebra to $\mathbb{F}_2\left[\beta, y_{4i} : i \geq 1\right]$.

Now let B denote the graded subring of $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ generated by β, by y_{4i} for all $i > 1$, and by the mod 2 reductions of the $(2, \beta)$-torsion elements in $\Omega_*^{Spin^c}$ from Theorem 4.2 which lift $D_i \times D_i$ for all odd non-dyadic i. Since B contains all the squares of elements in $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$, the graded $\mathbb{F}_2$-algebra map

$$B \hookrightarrow \Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$$

is a uniform F-isomorphism.

Let $\tilde{T}$ denote the kernel of the ring map $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2 \to \left(\Omega_*^{Spin^c}/tors\right) \otimes_{\mathbb{Z}} \mathbb{F}_2$. Filter B by powers of the ideal $B \cap \tilde{T}$, i.e., equip B with the $(B \cap \tilde{T})$-adic filtration. By the Anderson–Brown–Peterson splitting and by Theorem 4.2, the associated graded ring $E_0 B$ is isomorphic to $\mathbb{F}_2[\beta]$ tensored with the image of B in $\mathfrak{N}_*$ and reduced modulo the relations $\beta \cdot x = 0$ for all $x \in B \cap \tilde{T}$, i.e., $E_0 B$ is isomorphic to (9).

We claim that B itself is isomorphic to (9). The $(B \cap \tilde{T})$-adic filtration on B is additively split, so $B \cong E_0 B$ as graded $\mathbb{F}_2$-vector spaces. In principle, the ring structure on $E_0 B$ could differ from the ring structure on B if the multiplication on B were to exhibit $(B \cap \tilde{T})$-adic filtration jumps, i.e., when we multiply two elements x, y of B, with x of $B \cap \tilde{T}$-adic filtration i and with y of $(B \cap \tilde{T})$-adic filtration j, we could perhaps get an element of $(B \cap \tilde{T})$-adic filtration $> i + j$.

However, even if filtration jumps occur, $E_0 B$ is still isomorphic to the graded $\mathbb{F}_2$-algebra with presentation (9). This is by a freeness argument similar to the classical argument that, if the associated graded of a filtered commutative k-algebra is a polynomial (i.e., free commutative) k-algebra, then the original filtered commutative k-algebra must also have been free commutative. The argument is as follows. Let $\mathcal{C}$ be the category of pairs (A, S), where A is a graded-commutative

$\mathbb{F}_2[\beta]$-algebra, and S is a set of homogeneous elements of A such that $\beta \cdot x = 0$ for all $x \in S$. There is a forgetful functor from $\mathcal{C}$ to the category $\mathcal{S}ub$ of pairs (S_0, S_1) in which S_0, S_1 are sets and $S_1 \subseteq S_0$. The forgetful functor $\mathcal{C} \to \mathcal{S}ub$ sends (A, S) to the underlying sets of A and of S. The graded $\mathbb{F}_2[\beta]$-algebra (9) is the free object of $\mathcal{C}$ on the pair

$$(\{y_4, y_8, y_{12}, y_{16}, \dots\} \cup \{Z_2, Z_6, Z_{10}, Z_{18}, Z_{22}, \dots\},\ \{Z_2, Z_6, Z_{10}, Z_{18}, Z_{22}, \dots\}).$$

By Proposition 4.2, the elements $\{Z_2, Z_6, Z_{10}, Z_{18}, Z_{22}, \dots\}$ are β-torsion in B, not merely in E_0B. Hence there are no relations on E_0B except those which make it an object of the category of $\mathcal{C}$, and B lives in $\mathcal{C}$ as well, i.e., E_0B and B are isomorphic in the category $\mathcal{C}$. Hence E_0B and B are isomorphic as graded $\mathbb{F}_2$-algebras, and hence $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ is uniformly F-isomorphic to the $\mathbb{F}_2$-algebra (9), as claimed. □

In section 3, we showed that $\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2$ is not isomorphic to a polynomial algebra. Nevertheless, since an F-isomorphism induces a homeomorphism on the prime spectra (see [20, Proposition B.8] or [22, Lemma 29.46.9]), we have:

Corollary 4.5. *The topological space* Spec $\left(\Omega_*^{Spin^c}/(2,\beta)\right)$ *is homeomorphic to* Spec *of a polynomial* $\mathbb{F}_2$*-algebra on countably infinity many generators.*

Furthermore, the topological space Spec $\left(\Omega_*^{Spin^c} \otimes_{\mathbb{Z}} \mathbb{F}_2\right)$ *is homeomorphic to* Spec *of the ring* (9)*.*

The last two sentences in Stong's 1968 book [25] before the appendices begin are:

> One may relate the pair $(Spin, Spin^c)$ through exact sequences in precisely the same way as (SU, U) are related (or as (SO, O) are related). Computationally this is not of much use since one has no way to nicely describe the torsion in $\Omega_*^{Spin^c}$.

We regard Theorems 4.3 and 4.4 as progress toward nicely describing the torsion in $\Omega_*^{Spin^c}$ by means of ring structure.

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