THE STEENROD ALGEBRA IS SELF-INJECTIVE, AND THE STEENROD ALGEBRA IS NOT SELF-INJECTIVE.

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Abstract. It is well-known that the Steenrod algebra A is self-injective as a graded ring. We make the observation that simply changing the grading on A can make it cease to be self-injective. We see also that A is *not* self-injective as an *ungraded* ring.

These observations follow from the failure of certain coproducts of injective A-modules to be injective. Hence it is natural to ask: which coproducts of graded-injective modules, over a general graded ring, remain graded-injective? We give a complete solution to that question by proving a graded generalization of Carl Faith's characterization of Σ-injective modules. Specializing again to the Steenrod algebra, we use our graded generalization of Faith's theorem to prove that the covariant embedding of graded A_*-comodules into graded A-modules preserves injectivity of bounded-above objects, but does not preserve injectivity in general.

1. Self-injectivity of the Steenrod algebra.

The following is a well-known theorem, originally due to Adams and Margolis [1] at the prime $p = 2$, and Moore and Peterson [10] at odd primes p:

Theorem 1.1. *The mod p Steenrod algebra is self-injective. More precisely: the mod p Steenrod algebra A, regarded as a free* graded *left A-module, is injective in the category of* graded *left A-modules.*

More generally, any bounded-below free graded left A-module is injective in the category of graded left A-modules. See Theorem 12 in section 13.3 of [8] for this result, as well as its converse: a bounded-below graded left A-module is injective if and only if it is free.

However, using some old results in ring theory, it is also easy to prove the following:

Theorem 1.2. *The Steenrod algebra is* not *self-injective. More precisely: the Steenrod algebra A, as a free left A-module, is* not *injective in the category of* ungraded *left A-modules.*

Proof. Recall that an injective module is said to be Σ*-injective* if every coproduct of copies of that module is also injective. The main theorem of Megibben's paper [9] establishes that all countable injective modules, over any ring, are Σ-injective. If A were injective in the category of (ungraded) left A-modules, then by Megibben's theorem, the coproduct $\coprod_{n\in\mathbb{Z}} A$ would also be an injective left A-module. The coproduct $\coprod_{n\in\mathbb{Z}} A$ is the underlying ungraded left A-module of the graded left A-module $\coprod_{n\in\mathbb{Z}} \Sigma^n A$, which is known to *not* be injective in the category of graded A-modules: this is a special case of Proposition 10 in section 13.2 of Margolis's book [8], which establishes that a free graded left A-module cannot be injective in the graded module category unless it is bounded below.

Finally, the functor U from the graded module category to the ungraded module category has the property that, if UM is injective, then M is also injective. This is classical, and holds for any nonnegatively-graded ring: see for example Corollary 3.3.10 of [13]. The argument is now complete: if A were self-injective in the ungraded module category, then the ungraded direct sum $\coprod_{n\in\mathbb{Z}} A$ would also be injective, by Megibben's theorem; hence $\coprod_{n\in\mathbb{Z}} \Sigma^n A$ would be injective in the graded module category, contradicting Margolis's result[1]. □

Despite this paper's tongue-in-cheek title, there is of course no contradiction implied by Theorems 1.1 and 1.2 being both true: it is entirely consistent for a graded ring to be self-injective in the graded sense, but not self-injective in the ungraded sense.

Corollary 1.3. *Changing the grading on the Steenrod algebra, or forgetting the grading altogether, can change whether or not the Steenrod algebra is self-injective.*

Proof. We have just shown that forgetting the grading on the Steenrod algebra results in a ring which is not self-injective. As a consequence, if we change the grading on A by putting all its elements in degree 0, then the free A-module A is not injective in the resulting graded module category. □

2. A graded version of Faith's criterion for Σ-injectivity.

In the proof of Theorem 1.2, it was useful to consider whether a coproduct of copies of an injective (graded or ungraded) module remains injective. An injective module M is called Σ-*injective* (respectively, *countably* Σ-*injective*) if the direct sum $\coprod_{s\in S} M$ is injective for all sets S (respectively, all countable sets S). The universal property of injective modules ensures that injectivity is preserved under *products,* but it does not directly imply anything about injectivity of *coproducts,* i.e., direct sums. The classical Bass-Papp theorem[2] states that, if R is a ring, then R is left Noetherian if and only if every coproduct of injective left R-modules is injective. So over a left Noetherian ring, all injective left modules are Σ-injective.

The Steenrod algebra is not Noetherian on either side, so the Bass-Papp theorem does not apply. For non-Noetherian rings, the standard tool for determining whether a given injective module is Σ-injective is the following theorem of Faith, from [4]:

Theorem 2.1. *Suppose that R is a ring, and that M is an injective right R-module. Then the following conditions are equivalent:*

(1) *M is Σ-injective.*

[1]Alternatively, one can prove that the Steenrod algebra is not self-injective, as an ungraded ring, by simply appealing to the title result of Lawrence's paper [7] "A countable self-injective ring is quasi-Frobenius" and observing that the Steenrod algebra is countable and not quasi-Frobenius. Lawrence's argument is built upon the results of Faith's paper [4], whose main theorem we generalize below, in Theorem 2.4. Our point is that, whether we prove Theorem 1.2 by means of direct sums of injective modules (as in the argument given here) or by means of chains of annihilator ideals as in [7], either way we are led to the same investigations in this paper, leading to Theorem 2.4.

Thanks to Andy Baker and Ken Brown for pointing out Lawrence's paper to me.

[2]See Theorem 3.46 of [6] for a textbook treatment.

(2) *M is countably Σ-injective.*
(3) *R satisfies the ascending chain condition on its right ideals which are annihilators of subsets of M.*

Various generalizations of Faith's criterion are known. The most general that the author is aware of is Harada's, from [5]: given a Grothendieck category $\mathcal{C}$ with a generating set G of compact objects and an injective object Q of $\mathcal{C}$, for each left $\hom_{\mathcal{C}}(Q,Q)$-submodule N of $\hom_{\mathcal{C}}(S,Q)$, we can form the limit $\cap_{f\in N}\ker f$, which is a subobject of S. We call such a subobject of S an *annihilator ideal of S for Q*. Harada proves that Q is Σ-injective if and only if, for each $S \in G$, the partially-ordered set of annihilator ideals of S for Q satisfies the ascending chain condition.

While the category of graded A-modules is indeed Grothendieck, Harada's generalization of Faith's criterion does not make the fine distinctions that we will need in order to get a good grasp on the graded analogues of Σ-injectivity. Consider the following result of Margolis[3]:

Theorem 2.2. *Let M be a bounded-below free graded module over the Steenrod algebra (with its usual grading). Then M is gr-injective[4]. Furthermore:*

- *A direct sum of copies of M, without suspensions, remains gr-injective.*
- *More generally: a direct sum $\coprod_{s\in S}\Sigma^{d(s)}M$ of suspensions of copies of M remains gr-injective as long as there is a lower bound on the degrees $\{d(s) : s \in S\}$ of the suspensions.*
- *However, if there is no lower bound on the degrees $\{d(s) : s \in S\}$ of the suspensions, then the direct sum $\coprod_{s\in S}\Sigma^{d(s)}M$ is* not *gr-injective.*

It is clear from Theorem 2.2 that there are several natural but entirely distinct notions of Σ-injectivity in the graded setting. In the following definition, we give names to these notions of graded Σ-injectivity.

Definition 2.3. *Let R be a graded ring. We say that an gr-injective graded left R-module M is:*

- strictly Σ-injective *if the direct sum $\coprod_{s\in S} M$ is gr-injective for all sets S.*
- unboundedly Σ-injective *if the direct sum $\coprod_{s\in S}\Sigma^{d(s)}M$ is gr-injective for all sets S and all functions $d : S \to \mathbb{Z}$.*
- bounded-belowly Σ-injective *(respectively,* bounded-abovely Σ-injective*) if the direct sum $\coprod_{s\in S}\Sigma^{d(s)}M$ is gr-injective for all sets S and all functions $d : S \to \mathbb{Z}$ such that there is a lower bound (respectively, upper bound) on the values taken by d.*
- *If N is a set of integers, we say that M is* (Σ, N)-injective *if the direct sum $\coprod_{s\in S}\Sigma^{d(s)}M$ is gr-injective for all sets S and all functions $d : S \to N$.*

We have also the countable analogues of the above: for example, we say that M is countably strictly Σ-injective *if $\coprod_{s\in S} M$ is gr-injective for all countable sets S, and so on.*

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[3]See sections 13.2 and 13.3 of [8]. The first two parts of Theorem 2.2 are also consequences of the Moore-Peterson theorem, Theorem 2.7 from [10], establishing that bounded-below graded A-modules are free iff they are gr-injective.

[4]Here and from now on, we use a standard piece of terminology from graded ring theory: *gr-injective* means "graded in the injective module category."

Harada's theorem can be used to characterize the gr-injective graded modules that are strictly Σ-injective, or equivalently, $(\Sigma, \{0\})$-injective. There is another precedent in the literature for our study of graded Σ-injectivity: the paper [12] is about one version of graded Σ-injectivity, which its authors call "gr-Σ-injectivity." That notion of "gr-Σ-injectivity" is equivalent to what we call "unbounded Σ-injectivity." Năstăsescu and Raianu prove in [12] that, if a graded module is unboundedly Σ-injective, then its underlying ungraded module is Σ-injective. This is quite distinct from the notions of bounded-above Σ-injectivity and bounded-below Σ-injectivity, which as far as we know have not been studied before. In contrast to unbounded Σ-injectivity, as a consequence of Corollary 2.5, we will see that bounded-above Σ-injectivity and bounded-below Σ-injectivity are *not* preserved upon forgetting down to the ungraded module category.

The aim of this section is to prove a version of Faith's theorem which gives us a useful necessary and sufficient criterion for each of the various notions of graded Σ-injectivity.

In order to state our generalization of Faith's theorem, we introduce a graded version of some notation from [4]: given a graded left R-module M and a set X of homogeneous elements of R, we write $X^\perp$ for the set of homogeneous elements $m \in M$ such that $xm = 0$ for all $x \in X$. Given a set Y of homogeneous elements of M, we write $Y^\perp$ for the set of homogeneous elements $r \in R$ such that $ry = 0$ for all $y \in Y$.

We introduce two more simple pieces of notation, and one more piece of terminology:

- Given an integer m and a set N of integers, we will write $m + N$ for the set of integers $\{m + n : n \in N\}$. Similarly, $m - N$ will of course denote the set of integers $\{m - n : n \in N\}$
- Given a set N of integers, a graded ring R, and a graded left R-module M, we say that an ascending chain of homogeneous left ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ of R is *M-annihilator-stable for all degrees in N* if there exists some integer ℓ such that the submodule inclusions $I_\ell^\perp \supseteq I_{\ell+1}^\perp \supseteq I_{\ell+2}^\perp \dots$ are equalities in each grading degree d in N, i.e.,
$$(I_\ell^\perp)^d = (I_{\ell+1}^\perp)^d = (I_{\ell+2}^\perp)^d = \dots$$
for all $d \in N$.

Now we are ready for the main theorem. It is the graded generalization of the main theorem of Faith's paper [4]. Naturally our proof owes much to Faith's, although some of the ideas in our approach differ from Faith's, especially where care concerning the gradings is required.

Theorem 2.4. *Let R be a graded ring, let N be a set of integers, and let M be a gr-injective graded left R-module. Then the following are equivalent:*

1. *M is (Σ, N)-injective.*
2. *M is countably (Σ, N)-injective.*
3. *For each integer m, each ascending chain*
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \tag{2.1}$$
of homogeneous left ideals of R is M-annihilator-stable for all degrees in $m - N$.
4. *M is $(\Sigma, m + N)$-injective for every integer m.*

(5) *M is countably $(\Sigma, m+N)$-injective for every integer m.*

Proof.

1 implies 2, 4 implies 5, 4 implies 1, and 5 implies 2: Immediate.

2 implies 3: Suppose that m is an integer, suppose that M is countably (Σ, N)-injective, and suppose we are given an ascending sequence (2.1) of homogeneous left ideals. Suppose we are given a function $d : \mathbb{N} \to -m+N$. For each nonnegative integer n, choose a homogeneous element $x_n \in I_n^{\perp}$ of degree $-d(n)$. For any given homogeneous element $r \in \bigcup_{n\geq 0} I_n$, there exists some integer q such that $r \in I_q$. Since $I_q \subseteq I_{q+1} \subseteq \dots$, we have also that $I_q^{\perp} \supseteq I_{q+1}^{\perp} \supseteq \dots$, and consequently $ri_n = 0$ for all $n > q$. Consequently all but finitely many of the components in

$$x(r) := (rx_0, rx_1, rx_2, rx_3, \dots) \in \prod_{n\geq 0} \Sigma^{d(n)+m} M$$

are zero. That is, $x(r)$ is a homogeneous element of the direct *sum* $\coprod_{n\geq 0} \Sigma^{d(n)+m} M$, not merely the direct *product*.

The degree of $x(r)$ is equal to m plus the degree of r. Consequently the right R-module homomorphism

$$\begin{aligned} x : \Sigma^m \left(\bigcup_{n\geq 0} I_n \right) &\to \coprod_{n\geq 0} \Sigma^{d(n)+m} M \\ r &\mapsto x(r) \end{aligned}$$

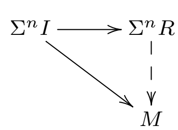
respects the grading. Each of the integers $d(n)+m$ is in the set N, so by the (Σ, N)-injectivity of M, the graded form of Baer's criterion[5] yields a graded right R-module homomorphism $g : \Sigma^m R \to \coprod_{n\geq 0} \Sigma^{d(n)+m} M$ such that $g(r) = x(r)$ for all $r \in \Sigma^m \left(\bigcup_{n\geq 0} I_n\right)$. Since the element $g(1)$ is an element of the direct sum $\coprod_{n\geq 0} \Sigma^{d(n)+m} M$, it must be zero in all but finitely summands. In particular, there must be some integer ℓ such that

$$\begin{aligned} g(1) &= (g_0, g_1, g_2, \dots, g_\ell, 0, 0, \dots) \\ &\in \coprod_{n\geq 0} \Sigma^{d(n)+m} M, \end{aligned}$$

and consequently

$$\begin{aligned} g(r) &= (rg_0, rg_1, rg_2, \dots, rg_\ell, 0, 0, \dots) \\ &= (rx_0, rx_1, rx_2, \dots, rx_\ell, 0, 0, \dots) \end{aligned}$$

[5]The graded Baer criterion is standard; see e.g. I.2.4 of [11]. It is as follows: for any graded ring R, a graded R-module M is gr-injective if and only if, for every graded left ideal I of R and every diagram



in the category of graded R-modules in which the top horizontal map is the canonical inclusion, a map of graded R-modules exists which fills in the dotted arrow and makes the diagram commute.

for each $r \in R$. Hence for every $n > \ell$, the element $x_n \in M$ is annihilated by $\bigcup_{n\geq 0} I_n$. Hence $x_n \in (\bigcup_{n\geq 0} I_n)^{\perp}$, which is a subset of $I_t^{\perp}$ for every integer t.

Now take stock of what we have just shown: we began with an arbitrary sequence $x_1, x_2, x_3, \dots$ of homogeneous elements of M whose degrees are in the set $m-N$, and such that each x_n is in $I_n^{\perp}$. We have just shown that, no matter how these choices are made, there exists some integer ℓ such that all the terms $x_\ell, x_{\ell+1}, x_{\ell+2}, \dots$ are in the *same* stage $I_\ell^{\perp}$. Hence what we have shown is that the sequence $I_1^{\perp} \supseteq I_2^{\perp} \supseteq I_3^{\perp} \supseteq \dots$ is eventually constant in each degree in $m-N$, exactly as we wanted to show.

3 implies 1: We first claim that condition 3 ensures that, for each homogeneous left ideal I of R, there exists a *finitely generated*[6] homogeneous left ideal I_1 of R contained in I such that $(I^{\perp})^d = (I_1^{\perp})^d$ for every degree d in the set $m-N$. The proof is as follows: consider the collection $\mathrm{Ideals}(M, I, m-N)$ of all finitely generated homogeneous left ideals of R contained in I. Preorder this collection by letting $J_1 \geq J_2$ if and only if $(J_1^{\perp})^d \subseteq (J_2^{\perp})^d$ for all d in $m-N$. Consequently the set of elements of $\mathrm{Ideals}(M, I, m-N)$ depends only on I, not on M, not on m, and not on N. However, the *preordering* on $\mathrm{Ideals}(M, I, m-N)$ *does* depend on all three of the variables M,I, and $m-N$.

Let $\pi_0 \mathrm{Ideals}(M, I, m-N)$ be the partially-ordered set of equivalence classes in the preordered set $\mathrm{Ideals}(M, I, m-N)$. We aim to show that every ascending sequence in $\pi_0 \mathrm{Ideals}(M, I, m-N)$ stabilizes. Given a pair of ideals $J_1, J_2 \in \mathrm{Ideals}(M, I, m-N)$ such that $J_1 \leq J_2$, it is routine to check that the sum $J_1 + J_2$ of the ideals satisfies both $J_2 \leq J_1 + J_2$ and $J_1 + J_2 \leq J_2$, i.e., we have an equivalence $J_1 + J_2 \sim J_2$ in $\mathrm{Ideals}(M, I, m-N)$. Consequently, given an ascending sequence

$$J_1 \leq J_2 \leq J_3 \leq J_4 \leq \dots \tag{2.2}$$

in $\mathrm{Ideals}(M, I, m-N)$, we have a sequence of containments of ideals

$$J_1 \subseteq J_1 + J_2 \subseteq J_1 + J_2 + J_3 \subseteq J_1 + J_2 + J_3 + J_4 \subseteq \dots,$$

each of which is in $\mathrm{Ideals}(M, I, m-N)$, and which stabilizes if and only if (2.2) stabilizes.

So we suppose that we have a sequence of containments of ideals as in (2.1), each of which is an element of $\mathrm{Ideals}(M, I, m-N)$. By the assumption of condition 3, the sequence (2.1) is M-annihilator stable for each degree in $m-N$. Consequently (2.1) stabilizes in $\pi_0 \mathrm{Ideals}(M, I, m-N)$, as desired.

Since the ascending chains in $\pi_0 \mathrm{Ideals}(M, I, m-N)$ all stabilize, Zorn's Lemma ensures that $\pi_0 \mathrm{Ideals}(M, I, m-N)$ has a maximal element. Let I' be an element of $\mathrm{Ideals}(M, I, m-N)$ representing a maximal element of $\pi_0 \mathrm{Ideals}(M, I, m-N)$. If x is a homogeneous element of I, then $I' + Rx$ is a member of $\mathrm{Ideals}(M, I, m-N)$ containing I'. By maximality of I', we must have $((I' + Rx)^{\perp})^d = ((I')^{\perp})^d$ for all $d \in m-N$. Since $x \in I' + Rx$, we must have $xm = 0$ for all $m \in (I' + Rx)^{\perp}$, hence $xm = 0$ for all $m \in ((I')^{\perp})^d$.

[6] To be clear, here and throughout this paper, whenever we say that a homogeneous ideal is "finitely generated," we shall always mean that it has a finite set of *homogeneous* generators.

This argument applies for all homogeneous $x \in I$, so we have $(I^{\perp})^d \supseteq ((I')^{\perp})^d$. The reverse containment $I^{\perp} \subseteq (I')^{\perp}$ follows from I' being a subideal of I, so we have $(I^{\perp})^d = ((I')^{\perp})^d$ for all $d \in m - N$. So I' is the desired finitely generated subideal of I.

Now we use the gr-injectivity of M together with the graded Baer criterion. Given a graded left R-module homomorphism $f : \Sigma^m I \to \coprod_{s\in S} \Sigma^{d(s)} M$ with each $d(s)$ in N, we have a commutative diagram of graded left R-modules given by the solid arrows depicted below:

$$\begin{array}{ccccc} \Sigma^m I' & \longrightarrow & \Sigma^m I & \longrightarrow & \Sigma^m R \\ & & \downarrow f & & \downarrow \tilde{f} \\ & & \coprod_{s\in S} \Sigma^{d(s)} M & \longrightarrow & \prod_{s\in S} \Sigma^{d(s)} M. \end{array} \quad (2.3)$$

In (2.3), the map $\tilde{f}$ is obtained using the universal property of the gr-injective module $\prod_{s\in S} \Sigma^{d(s)} M$, and I' is a finitely generated homogeneous left ideal of R of the kind just constructed using Zorn's Lemma, i.e., I' is contained in I, and $(I')^{\perp}$ coincides with $I^{\perp}$ in every degree in $m - N$. The horizontal maps in (2.3) are each the natural subset inclusion maps.

Since I' is finitely generated, we may choose a finite homogeneous generating set $r_1, \dots, r_k$ for it. The images $\tilde{f}(r_1), \dots, \tilde{f}(r_k)$ of $r_1, \dots, r_k$ land in the direct sum $\coprod_{s\in S} \Sigma^{d(s)} M$, so there is a finite subset T of S such that the image of $\tilde{f}\mid_{I'}$ is contained in $\coprod_{s\in T} \Sigma^{d(s)} M \subseteq \prod_{s\in S} \Sigma^{d(s)} M$.

Given an element $z \in \prod_{s\in S} \Sigma^{d(s)} M$ and an element $s \in S$, write z_s for the component of z in the factor $\Sigma^{d(s)} M$ of $\prod_{s\in S} \Sigma^{d(s)} M$. Let $\hat{f} : \Sigma^m R \to \coprod_{s\in S} \Sigma^{d(s)} M$ be the graded left R-module map determined by

$$\hat{f}(1)_s = \begin{cases} \tilde{f}(1)_s & \text{if } s \in T \\ 0 & \text{if } s \notin T. \end{cases}$$

We have defined $\hat{f}$ so that $r_i \cdot \hat{f}(1) = r_i \cdot \tilde{f}(1)$ is true for all $i = 1, \dots, k$. Since $r_1, \dots, r_k$ generate I', we have $(\hat{f}(1) - \tilde{f}(1))_s \in (I')^{\perp}$ for each $s \in S$. In particular, $(\hat{f}(1) - \tilde{f}(1))_s$ is in degree $m - d(s)$ in $(I')^{\perp} \subseteq M$. Since $m - d(s) \in m - N$, we have $(\hat{f}(1) - \tilde{f}(1))_s \in I^{\perp}$ as well. Hence $r \cdot \hat{f} = r \cdot \tilde{f}(1)$ for all $r \in I$, i.e., $\hat{f}$ and $\tilde{f}$ agree on I. Consequently $\hat{f}$ fills in the dotted arrow in diagram (2.3) and makes the upper-left triangle commute. This is precisely the condition necessary to obtain gr-injectivity of $\coprod_{s\in S} \Sigma^{d(s)} M$ from the graded Baer criterion. Hence M is $(\Sigma, m - N)$-injective.

1 implies 4, and 2 implies 5: The suspension functor Σ is an automorphism of the graded module category, so a graded module is (Σ, N)-injective if and only if it is $(\Sigma, m + N)$-injective for all integers m.

□

Corollary 2.5. *Let R be a graded ring, and let M be a gr-injective graded left R-module. Then:*

- *M is strictly Σ-injective if and only if, for every integer d and every ascending chain*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad (2.4)$$

of homogeneous left ideals of R, the descending chain of graded submodules of M

$$I_0^{\perp} \supseteq I_1^{\perp} \supseteq I_2^{\perp} \supseteq \dots \tag{2.5}$$

is eventually constant in degree d.

- *M is unboundedly Σ-injective if and only if, for every ascending chain* (2.4) *of homogeneous left ideals of R, the descending chain* (2.5) *of graded submodules of M is eventually constant*[7].
- *M is bounded-abovely Σ-injective (respectively, bounded-belowly Σ-injective) if and only if, for every integer m and every ascending chain* (2.4) *of homogeneous left ideals of R, the descending chain* (2.5) *of graded submodules of M is eventually constant in all degrees $\geq m$ (respectively, all degrees $\leq m$).*

Corollary 2.6. *Let R be a graded ring. Every bounded-below gr-injective graded R-module is bounded-belowly Σ-injective. Furthermore, every bounded-above gr-injective graded R-module is bounded-abovely Σ-injective.*

3. Injective graded A^*-comodules and injective graded A-modules.

One special case of Corollary 2.6 is the theorem of Margolis (see sections 13.2 and 13.3 of [8]): a uniformly bounded-below direct sum of bounded-below gr-injective modules over a P-algebra (such as the Steenrod algebra) is gr-injective. Corollary 2.6 establishes that this is in fact a general feature of gr-self-injective rings, and does not require the ground ring to be a P-algebra.

Another special case of interest is the linear dual of the previous case. Let A be the mod p Steenrod algebra, and let dA denote the $\mathbb{F}_p$-linear dual graded A-module of A, regarded as a graded left A-module via the adjoint action[8] of A. It is easy to show that dA is gr-injective: this was Margolis's example of a demonstrably non-free gr-injective A-module, as in Proposition 12 in section 11.3 of [8]. We can now address the question of whether direct sums of copies of dA are also gr-injective:

Proposition 3.1. *The graded A-module dA is bounded-abovely Σ-injective, but dA is not bounded-belowly Σ-injective.*

Proof. Since dA is bounded above, it is a special case of Corollary 2.6 that dA is bounded-abovely Σ-injective. So all that remains is to show that dA is not bounded-belowly Σ-injective.

Consider the ascending chain of homogeneous left ideals

$$A(\mathrm{Sq}^1) \subseteq A(\mathrm{Sq}^1, \mathrm{Sq}^2) \subseteq A(\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4) \subseteq A(\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \mathrm{Sq}^8) \subseteq \dots$$

of A. (Here we work with the mod 2 Steenrod algebra for convenience of exposition, but an analogous argument works at odd primes.)

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[7]To be clear: the difference between this condition and the previous condition is that, for unbounded Σ-injectivity, we must have a single integer ℓ such that the sequence (2.5) is constant, in all degrees d, starting with the ℓth stage in the sequence. By contrast, for *strict* Σ-injectivity, the sequence (2.5) could stabilize at a different stage for each given degree, without there being a single stage by which (2.5) is constant in *every* degree.

[8]This is a standard construction; see e.g. section 11.3 of [8]. One particularly relevant way to think about the adjoint action of A on dA is as follows: regard dA as the A_*-comodule A_*, then apply the adjoint embedding of graded A_*-comodules into graded A-modules, discussed in this paper, below.

We consider the graded annihilator submodules of dA for each of these ideals. The annihilator submodule $(A(\mathrm{Sq}^1, \mathrm{Sq}^2, \dots, \mathrm{Sq}^{2^n}))^{\perp} \subseteq dA$ depends only on the action of $\mathrm{Sq}^1, \mathrm{Sq}^2, \dots, \mathrm{Sq}^{2^n}$ on dA, i.e., the structure of dA as a graded module over the subalgebra $A(n)$ of A generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \dots, \mathrm{Sq}^{2^n}$. It follows from generalities about P-algebras (see Theorem 12 in section 13.3 of [8]) that, for each n, the underlying $A(n)$-module of A is free on a set of homogeneous generators whose degrees are bounded below by zero, but are not bounded above.

Since $A(n)$ is a Frobenius algebra, the $\mathbb{F}_2$-linear dual dA of A is also free as an $A(n)$-module, but it is free on a set of homogeneous generators whose degrees are bounded above by zero, but are not bounded below. The upshot is that, for each integer m, there exists an integer ℓ such that the sequence

(3.6)
$$\left(A(\mathrm{Sq}^1)\right)^{\perp} \supseteq \left(A(\mathrm{Sq}^1, \mathrm{Sq}^2)\right)^{\perp} \supseteq \left(A(\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4)\right)^{\perp} \supseteq \left(A(\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \mathrm{Sq}^8)\right)^{\perp} \supseteq \dots$$

is constant after its ℓth stage in all degrees *greater than* m. However, there is no integer ℓ such that (3.6) is constant after its ℓth stage in all degrees *less than* m.

Consequently, by Corollary 2.5, dA is not bounded-belowly Σ-injective. □

There is a good reason to consider the various forms of graded Σ-injectivity for the specific A-module dA. Recall that the category $\mathrm{gr}\,\mathrm{Comod}(A_*)$ of graded right comodules over the dual Steenrod algebra A_* admits a *covariant* embedding into the category $\mathrm{gr}\,\mathrm{Mod}(A)$ of graded left A-modules. This is a purely algebraic construction, quite classical: I do not know its earliest appearance, but as far as I know, [2] was its first mention in the context of algebraic topology. Given a graded right A_*-comodule M with coaction map $\psi : M \to M \otimes_{\mathbb{F}_p} A_*$, identify A with its double dual A_{**}, and then the action map

$$A \times M \xrightarrow{\cong} A_{**} \times M \to M$$

sends a pair $(f, m) \in A_{**} \times M$ to the image of m under the composite

$$M \xrightarrow{\psi} M \otimes_{\mathbb{F}_p} A_* \xrightarrow{M \otimes f} M \otimes_{\mathbb{F}_p} \mathbb{F}_p \xrightarrow{\cong} M. \tag{3.7}$$

This action of A on M is called the *adjoint action*. This construction yields a covariant, exact, faithful, full functor[9] $\iota : \mathrm{gr}\,\mathrm{Comod}(A_*) \to \mathrm{gr}\,\mathrm{Mod}(A)$. The book [3] is an excellent reference for these and other properties of the functor ι, not only for the dual Steenrod algebra A_*, but also for a general coalgebra in place of A_*.

As the covariant embedding ι of $\mathrm{gr}\,\mathrm{Comod}(A_*)$ into $\mathrm{gr}\,\mathrm{Mod}(A)$ is exact, one expects ι to preserve a great deal of cohomological information. If ι were to send injective objects to injective objects, then it would be easy to prove that right-derived functors in the category of graded A_*-comodules can be computed by first embedding the comodule category into graded A-modules, then calculating right derived functors there. Right derived functors in the comodule category arise in practice in algebraic topology[10], but module categories are far more familiar and

[9] Our presentation of the functor ι and its properties presumes that we are using cohomological gradings throughout, so that A is in nonnegative degrees and A_* is in nonpositive degrees. This ensures that the adjoint action indeed respects the gradings.

[10] E.g. the input term of the $H\mathbb{F}_p$-Adams spectral sequence for a non-finite-type spectrum, which generally only has a description in terms of Cotor_{A_*}, the derived functors of the cotensor product of comodules, rather than Ext_A; see chapter 2 of [14]. Also, the input term of the Sadofsky spectral sequence [15], which is comprised of the right derived functors of product in the category of graded A_*-comodules.

understood than comodule categories, so it is of obvious interest to know whether homological algebra in graded A_*-comodules can be described in terms of homological algebra in graded A-modules! Yet it seems this question—of whether ι sends gr-injective comodules to gr-injective modules—is not addressed in the literature. As far as the author knows, an answer is not known.

It turns out that the graded Σ-injectivity of dA is precisely the key to answering that question:

Theorem 3.2. *Let A be the mod p Steenrod algebra for any prime p. Then the covariant embedding ι of graded A_*-comodules into graded A-modules does* not *send gr-injectives to gr-injectives.*

However, if M is a bounded above *graded A_*-comodule, then $\iota(M)$* is *gr-injective.*

Proof. The gr-injective right A_*-comodules are the retracts of the extended graded right A_*-comodules, i.e., the summands of those of the form $V \otimes_{\mathbb{F}_p} A_*$ for a graded $\mathbb{F}_p$-vector space V. Equivalently, the gr-injective right A_*-comodules are the summands of coproducts of suspensions of A_*. The functor ι sends A_* to dA, and ι preserves coproducts since it is a left adjoint[11]. Hence Proposition 3.1 yields immediately that $\iota(M)$ is gr-injective if M is a bounded-above gr-injective comodule, while $\iota(M)$ must fail to be gr-injective for some non-bounded-above gr-injective comodules. □

To close, here is a related question that the author is curious about, but does not know an answer to:

Question 3.3. Let A be the mod p Steenrod algebra for any prime p. Suppose that M is a graded right A_*-comodule which is gr-injective. Is the gr-injective dimension of the A-module $\iota(M)$ at most 1?

While Question 3.3 is motivated by the desire to understand the homological algebra of A_*-comodules because of their applications in algebraic topology, the question is also quite close to one that has been considered for other reasons, as follows. Suppose that R is a *left Noetherian* nonnegatively-graded ring. A theorem of van den Bergh ([16], see also [17] for Yekutieli's elegant proof) establishes that, given a gr-injective left R-module M, the injective dimension of the underlying ungraded R-module of M is at most one.

Of course the Steenrod algebra is not left Noetherian! But Yekutieli remarks, in [17], that "[w]e do not know if the noetherian condition in Theorem 1 is necessary." If the theorem of van den Bergh and Yekutieli can be proven without the Noetherian hypothesis, then the injective dimension of a coproduct of copies of dA is at most 1, and consequently, for any gr-injective A_*-comodule M, the gr-injective dimension of $\iota(M)$ could not exceed 1, yielding an affirmative answer to Question 3.3.

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