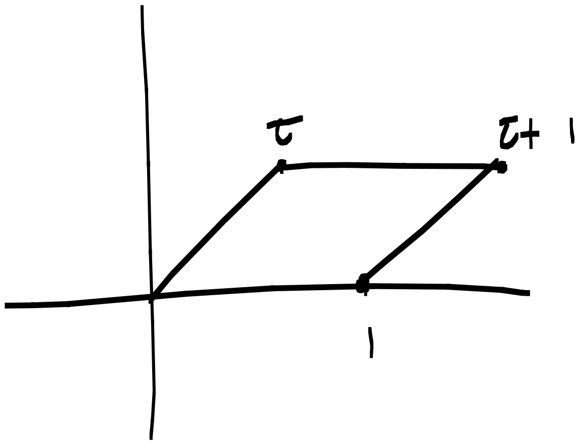
# Lecture 5 - Modular forms seminar

Note Title

Let $\tau \in \mathbb{H}$ and consider the lattice $\Lambda_\tau = \langle \tau, 1 \rangle$




Define the Weierstrass $\wp$-function (fix $\tau$)

$$\wp(\tau; z) : \mathbb{C} \longrightarrow \mathbb{C}$$

$$z \longmapsto \frac{1}{z^2} + \sum_{\lambda \in \Lambda_\tau - \{0,0\}} \frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2}$$

Then $\wp$ satisfies:

$$\wp(\tau; z+\lambda) = \wp(\tau; z) \qquad \lambda \in \Lambda$$

(is "doubly - periodic")

so $\wp$ can be viewed as a meromorphic function on the quotient $\mathbb{C}/\langle\tau,1\rangle$, a complex torus of dimension one. This is the prototypical example of an "elliptic" function.

- Let $\wp'(\tau; z) = \frac{d}{dz}\wp(\tau; z)$. Then $\wp$ satisfies the following differential equation:

$$(\wp')^2 = 4\wp^3 - g_2(\tau)\wp - g_3(\tau)$$

where
$$g_2(\tau) = 60 \cdot \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4}$$

$$= 120\, \zeta(4)\, E_4(\tau) \dashrightarrow \text{wt 4 Eisenstein Series}$$

$$= \frac{120}{90}\pi^4 E_4(\tau)$$

$$g_3 = 120 \cdot \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^6}$$

$$= 240\, \zeta(6)\, E_6(\tau) \cdots\to \text{wt } 6 \text{ Eisenstein series}$$

$$= \frac{240}{945} \pi^6 E_6(\tau)$$

- So we get a map, for each $\tau$

$$\mathbb{C}/\Lambda_\tau - \{0\} \longrightarrow \mathbb{C}^2$$

$$z \longmapsto (\wp(\tau;z), \wp'(\tau;z))$$

whose image is the algebraic curve

$$Y^2 = 4X^3 - g_2(\tau)X - g_3(\tau)$$

- Extend it to a map

$$\mathbb{C}/\Lambda_\tau \longrightarrow \mathbb{P}^2$$

$$0 \longmapsto [0:1:0]$$

so that $\mathbb{C}/\Lambda_\tau \simeq E_\tau = \{y^2 = 4x^3 - g_2 x - g_3\} \subseteq \mathbb{P}^2$

what kind of curve is $E_\tau$?

① $E_\tau$ is smooth: the discriminant is

$$\Delta = g_2^3 - 27 g_3^2 \qquad \text{(divided by 16)}$$

$$= (2\pi)^{12} \eta(\tau)^{24}$$

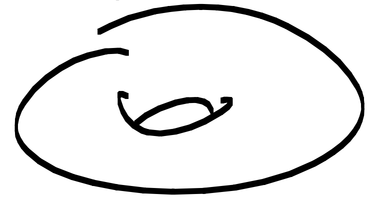
Now $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$, $q = e^{2\pi i \tau}$, is Dedekind's $\eta$-function, a m.f. of wt $1/2$ on $SL_2(\mathbb{Z})$

$$\Rightarrow \Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

is a m.F. of wt 12 on $SL_2(\mathbb{Z})$ and $\neq 0$ for any value of $\tau$

$$\Rightarrow E_\tau \text{ is smooth}$$

② $E_\tau \simeq \mathbb{C}/\Lambda_\tau$ has genus one

$\simeq$ 

(alternatively, use Riemann-Hurwitz on $x : E_\tau \longrightarrow \mathbb{P}^1$)

$\Rightarrow$ $E_\tau$ is an elliptic curve

(smooth, complete algebraic curve of genus one)

- Conversely, given any elliptic curve $E$ over $\mathbb{C}$, it is possible to find $\tau \in \mathbb{H}$ such that $E = E_\tau \simeq \mathbb{C}/\langle \tau, 1 \rangle$ : let

$\omega \in H^0(E, \Omega^1)$ ⇢ 1-dim $/\mathbb{C}$

$\gamma_1, \gamma_2 \in H_1(E, \mathbb{Z})$ ⇢ rk $2/\mathbb{Z}$

oriented: $\langle \gamma_1, \gamma_2 \rangle = 1$

then

$$\tau := \frac{\int_{\gamma_2} \omega}{\int_{\gamma_1} \omega} \in \mathbb{C}$$

is the required "period ratio".

# Moduli spaces

The assignment

$$\tau \in \mathbb{H} \longrightarrow \mathbb{C}/\langle \tau, 1 \rangle$$

gives

$$\mathbb{H} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{complex 1-dim tori} \\ + \text{ a choice of oriented} \\ \text{basis } \langle \tau, 1 \rangle \subseteq H_1(\mathbb{Z}) \end{array} \right\}$$

We say that $\mathbb{H}$ is a "moduli space" for complex tori + oriented choice of homology basis.

Let now $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ act on the lattice $\Lambda_\tau = \langle \tau, 1 \rangle$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}$$

so that $SL_2(\mathbb{Z}) = \mathrm{Aut}(\Lambda_\tau)$. These automonphisms lift uniquely to autos. of the torus $\mathbb{C}/\langle \tau, 1 \rangle$. We get

$$\left\{ \begin{array}{c} \text{iso. classes of} \\ \text{1-dimensional} \\ \text{complex tori} \end{array} \right\} \xleftrightarrow{1:1} SL_2(\mathbb{Z}) \backslash \mathbb{H} = Y(1)$$

$$= \mathcal{M}_{1,1}/\mathbb{C}$$

$$\Big\| \qquad \begin{array}{c} \mathbb{C}/\Lambda_\tau \\ \rotatebox{90}{$\mapsto$} \\ E_\tau \end{array}$$

$$\left\{ \begin{array}{c} \text{iso. classes of} \\ \text{elliptic curves} \\ \text{over } \mathbb{C} \end{array} \right\}$$

- Therefore $Y(1)$ is a <u>moduli space for elliptic curves over $\mathbb{C}$.</u>

## Line bundles on moduli spaces

For any elliptic curve $E$ over $\mathbb{C}$, the vector space

$$H^0(E, \Omega^1_E) = \text{hold. 1-forms}$$

is 1-dimensional (dim = genus). The assignment

$$E \longmapsto H^0(E, \Omega^1_E)$$

Gives a complex line bundle $\mathcal{L}_1 = \omega$ on $Y(1)$, since $\Omega^1_E$ is <u>functorial</u>:

(e.g. given an automorphism of tori, $\phi : E_\tau \xrightarrow{\sim} E_{\tau}$, then $\phi^* \Omega^1_{E_\tau} \simeq \Omega^1_{E_\tau}$)

Prop: $\mathcal{L}_1 \simeq SL_2(\mathbb{Z}) \backslash \mathbb{H} \times \mathbb{C}$ with action

given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z) = \left( \frac{a\tau + b}{c\tau + d}, (c\tau + d) z \right)$

- More generally,

$\mathcal{L}_k := \mathcal{L}_1^{\otimes k}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z) = \left( \frac{a\tau + b}{c\tau + d}, (c\tau + d)^k z \right)$$

$$\Longrightarrow H^0(Y(1), \mathcal{L}_k) = M_k^! (SL_2(\mathbb{Z}))$$

= (weakly holomorphic) modular forms of weight $k$, level $SL_2(\mathbb{Z})$.

# Algebraic version

Suppose now $R$ is any commutative ring. There is a moduli space $\mathcal{M}_{1,1}/R$ of <u>elliptic curves over $R$</u>, a Deligne-Mumford stack, smooth of relative dimension 1 over $R$. The assignment

$$E/R \longmapsto H^0(E, \Omega^1_{E/R}) \quad \left(\begin{array}{l}\text{Rk } 1 \\ \text{Projective} \\ R\text{-module}\end{array}\right)$$

gives a line bundle $\omega$ over $\mathcal{M}_{1,1}, R$.

<u>Definition</u> | A (weakly holomorphic) algebraic modular form of weight $k \in \mathbb{Z}$, level 1, over $R$, is a global section of $\omega^{\otimes k}$ over $\mathcal{M}_{1,1}/R$.

Notation: $H^0(\mathcal{M}_{1,1}/R, \omega^{\otimes k}) = M^!_k(1, R)$