Modular forms Seminar Lecture 3

Note Title 9/17/2018

We have already encountered

$$\Theta : \mathbb{R}_{>0} \longrightarrow \mathbb{C}$$

$$\Theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$$

which satisfies

$$\Theta\left(\frac{1}{t}\right) = \sqrt{t}\,\Theta(t) \qquad (\forall\ t > 0)$$

- We now extend this function to all of $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{im}(\tau) > 0\}$

$$\Theta : \mathbb{H} \longrightarrow \mathbb{C}$$

$$\tau \longmapsto \Theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$$

[Recover previous Θ by setting $\tau = it$)

- Then Θ is <u>holomorphic</u> and it satisfies:

1) $\Theta(\tau+2) = \Theta(\tau)$ (obvious)

2) $\Theta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\Theta(\tau)$

For #2, note that the property is true for $\tau = it$, so it must be true on all of $\mathbb{H}$.

- To organize these functional equations, note that

$$SL_2(\mathbb{R}) \longrightarrow \mathrm{Aut}(\mathbb{H})$$
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{a\tau+b}{c\tau+d} = \gamma\tau$$

(in fact, $PSL_2(\mathbb{R}) \simeq \mathrm{Aut}(\mathbb{H})$)

so that we can rewrite

$$\tau+2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\tau$$
$$-\frac{1}{\tau} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\tau$$

Now the matrices $\gamma_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

generate a discrete group, the "Theta group"

$$\Gamma(1,2) \subseteq SL_2(\mathbb{Z}) \subseteq SL_2(\mathbb{R})$$

and one can show that

$$\left(\Theta\left(\frac{a\tau+b}{c\tau+d}\right) = \right) \Theta(\gamma\tau) = \sqrt{\lambda(\gamma)(c\tau+d)}\ \Theta(\tau)$$

where $\lambda : \Gamma(1,2) \longrightarrow \mu_4 = \{\pm 1, \pm i\}$

is a certain character...

- In general:

Definition | Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a finite-index subgroup. A "modular form" of weight k and level Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

Character

$$f(\gamma\tau) = \chi(\gamma)(c\tau+d)^k f(\tau) \quad , \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Note: in general, $k \in \mathbb{Z}$ or $k \in \frac{1}{2}\mathbb{Z}$

- So $\Theta(\tau)$ is a modular form of weight $1/2$ and level $\Gamma(1,2)$.

Examples - Θ^k is a m.f. of weight $k/2$ and level $\Gamma(1,2)$

- $$\Theta(2\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 \tau} \qquad q = e^{2\pi i \tau}$$

$$= 1 + 2q + 2q^4 + \dots$$

is a m.f. weight $1/2$, level $\Gamma_0(2) \subseteq SL_2(\mathbb{Z})$

$$\left(\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\} \right)$$

. Consider the partition function:

$$p(\tau) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

$$= \sum_{n=1}^{\infty} p(n) q^n \qquad q = e^{2\pi i \tau}$$

where $p(n) = \#$ ways of writing n as a sum of integers $0 < k \leq n$ (unordered)

E.g.

$$\left.\begin{aligned} 4 &= 4+0 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}\right\} \text{5 ways}$$

asy. estimates, circle method, ...

Amazing Fact: $q^{-1/24} p(\tau)$ is a modular form of weight $k = -1/2$ and level $SL_2(\mathbb{Z})$. (level 1, "full" level)

- Note: $q^{1/24} p(\tau)^{-1} = q^{1/24} \prod_{n=1}^{\infty} (1-q)^n =: \eta(\tau)$ is a m.f. weight $1/2$, level 1, called the Dedekind η-function.

Generalization: VOAs, monstrous moonshine...

E.g. The coefficients of Θ also have a combinatorial interpretation:

$$\Theta(\tau) = 1 + 2q + 2q^4 + 2q^9 + \dots \qquad \underline{\underline{q = e^{\pi i \tau}}}$$

$$\Theta^2(\tau) = \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)\left(\sum_{m\in\mathbb{Z}} q^{m^2}\right)$$

$$= \sum_{n\in\mathbb{Z}} r_2(n)\, q^n$$

$r_k(n) =$ # ways of representing an integer as a sum of k squares

In general,

$$\Theta^k(\tau) = \sum_{n\in\mathbb{Z}} r_k(n)\, q^n$$

<u>Waring</u> <u>problem</u>: Find a "formula" for $r_k(n)$.

Solution: uses the fact that Θ^k is a m.f. of weight $k/2$

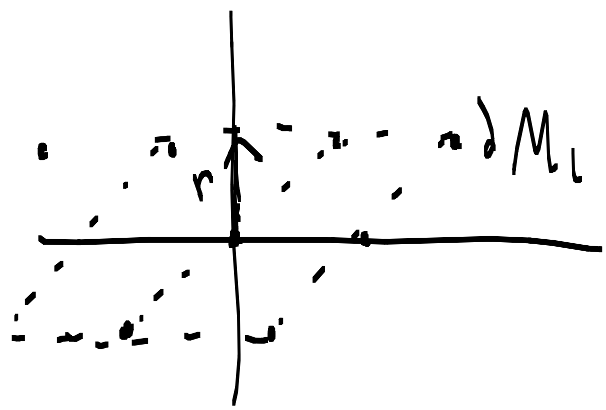
<u>Generalizations</u>: Θ-functions of lattices,
→ sphere packings, etc.

<u>Topological theta functions</u>?

Eisenstein Series

A systematic way of producing modular forms.

- let $\Lambda \subseteq \mathbb{C}^2$ be a lattice



<u>Prop:</u> The series $\sum_{\lambda \in \Lambda - \{0\}} \frac{1}{|\lambda|^k}$ converges for all $\underline{k > 2}$, $k \in \mathbb{Z}$.

<u>Proof:</u> Count how many points are on the boundary parallelogram ∂M_n, as $n \to \infty$

$\# |\partial M_n \cap \Lambda| = 8n$

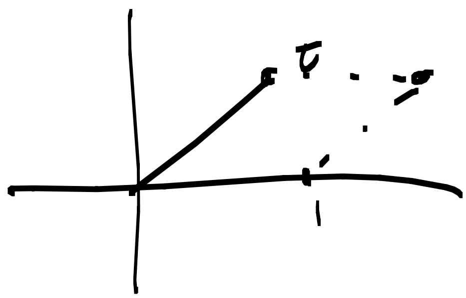
$$\Rightarrow \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{|\lambda|^k} = \sum_{n=1}^{\infty} \sum_{\lambda \in \partial M_n} \frac{1}{|\lambda|^k}$$

$$\leq \sum_{n=1}^{\infty} 8n \frac{1}{n^k r^k}$$

$$= 8 r^{-k} \zeta(k-1)$$

which converges for $k > 2$.

- for each $\tau \in \mathbb{H}$, let $\Lambda_\tau = \langle \tau, 1 \rangle$



Corollary | Let $k > 2$, $k \in \mathbb{Z}$.

1) For each $\tau \in \mathbb{H}$, the series

$$G_k(\tau) = \sum_{\Lambda_\tau - \{0\}} \frac{1}{\lambda^k} = \sum_{(n,m) \in \mathbb{Z}^2 - \{0,0\}} \frac{1}{(m\tau + n)^k}$$

converges, 2) $G_k(\tau)$ is holomorphic.

- Note: $SL_2(\mathbb{Z}) = \mathrm{Aut}(\Lambda_\tau)$ $\left(\begin{matrix}\text{orientation}\\ \text{-preserving}\end{matrix}\right)$

$$\Rightarrow \quad G_K\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^K\, G(\tau) \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

ie. G_K is a m.f. of weight K and level 1.

- <u>A few things:</u>

1) let $\overset{\gamma}{\overset{\shortparallel}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$. Then

$$G_K\left(\frac{a\tau+b}{c\tau+d}\right) = G(\tau)$$

$$\shortparallel$$

$$(-1)^K\, G(\tau)$$

so if K is <u>odd</u> $G_K = 0$.

2) let $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$. Then

$$G_K\left(\frac{a\tau+b}{c\tau+d}\right) = G(\tau+1) = G(\tau)$$

so G_K has a Fourier expansion in terms of $q = e^{2\pi i \tau}$:

$$G_K(\tau) = 2\zeta(K) + \frac{(2\pi i)^K}{(K-1)!} \sum_{n=1}^{\infty} \sigma_{K-1}(n) q^n$$

where $\sigma_K(n) = \sum_{d|n} d^K$

$$E_K(\tau) := \frac{G_K(\tau)}{2\zeta(K)} = 1 + \frac{2}{\zeta(1-K)} \sum_{n=1}^{\infty} \sigma_{K-1}(n) q^n$$

$$= 1 - \frac{2K}{B_K} \sum_{n=1}^{\infty} \sigma_{K-1}(n) q^n$$

$$\in \mathbb{Q}[[q]]$$

E.g. $E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$

$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$

Generalizations: Serre: p-adic interpolation of Dedekind ζ-function..

Heuristics Why modular forms?

Let X be a Riemann surface defined over $\overline{\mathbb{Q}}$.

Then

$$X \simeq \overline{\Gamma \backslash \mathbb{H}} \quad \text{(Belyi, late 70s)}$$

where $\Gamma \subseteq SL_2(\mathbb{Z})$ is of finite index.

Line bundles on X $\longleftrightarrow$ Γ-cocycles of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \chi(\gamma)(c\tau+d)^k$

Sections of line bundles on X $\longleftrightarrow$ modular forms!