KÜNNETH FORMULAS FOR COTOR.

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Abstract. We investigate the question of how to compute the cotensor product, and more generally the derived cotensor (i.e., Cotor) groups, of a tensor product of comodules. In particular, we determine the conditions under which there is a Künneth formula for Cotor. We show that there is a simple Künneth theorem for Cotor groups if and only if an appropriate coefficient comodule has trivial coaction. This result is an application of a spectral sequence we construct for computing Cotor of a tensor product of comodules. Finally, for certain families of nontrivial comodules which are especially topologically natural, we work out necessary and sufficient conditions for the existence of a Künneth formula for the 0th Cotor group, i.e., the cotensor product. We give topological applications in the form of consequences for the E_2-term of the Adams spectral sequence of a smash product of spectra, and the Hurewicz image of a smash product of spectra.

Contents

1. Introduction.

The classical, well-known Künneth formula expresses

- the homology of a tensor product of chain complexes of abelian groups[1] in terms of the (derived) tensor product of the homology of each chain complex,
- and, closely related, the homology of a Cartesian product of topological spaces in terms of the (derived) tensor product of the homology of each space.
- The stable-homotopical version of the previous example: the homology of a smash product of spectra is described in terms of the (derived) tensor product of the homology of each spectrum.

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[1] Or, more generally, of chain complexes of modules over any commutative ring R, if one is willing to consider the Künneth spectral sequence.

More generally, in any setting in which we have some notion of cohomology, and some kind of tensor product, one expects that a "Künneth formula" in that setting ought to be some expression for the cohomology of a tensor product in terms of the (derived) tensor product of the cohomology of each factor.

Now suppose that Γ is a bialgebra, or more generally, a bialgebroid, over some ground ring A. In the setting of Γ-comodules, the usual notion of cohomology is given by $\mathrm{Cotor}^*_\Gamma(A,-)$, the derived functors of the cotensor product $\mathrm{Cotor}^0_\Gamma(A,-) \cong A\square_\Gamma -$. Using the multiplication on Γ, we get a tensor product $\otimes_A$ on the category of Γ-comodules. Consequently we would like to know if there exists a well-behaved Künneth formula for the cotensor product, or more generally for Cotor. There is apparently no place in the existing literature where such a Künneth formula is considered. The purpose of this paper is to fill that "hole" in the literature.

Before explaining the main results of this paper, we pause to explain why it is both reasonable and unreasonable to expect some kind of Künneth formula in Cotor. Given an algebraic group G over some field k, the representations of G (taken in the broad sense, with no requirement of finite-dimensionality) are equivalent to the comodules over the representing Hopf algebra kG. The Cotor-groups of the comodules recover the cohomology groups of the representations. Of course it is unrealistic to expect a simple relationship between $H^*(G;\rho_1), H^*(G;\rho_2)$, and $H^*(G;\rho_1\otimes_k\rho_2)$ for arbitrary representations ρ_1 and ρ_2! From this class of examples, it is clear that there cannot be a straightforward Künneth formula in Cotor without some kind of restrictive hypotheses on the bialgebroid or on the comodules involved.

On the other hand, consider the occurrences of Cotor in algebraic topology. This is a paper in pure algebra, but motivated by topological questions, so we hope the reader will forgive a digression into topology. Given a generalized homology theory E_* satisfying standard conditions[2], the ring E_*E of stable cooperations on E_*-homology forms a bialgebroid (E_*, E_*E). For any spectrum X we get the *generalized Adams spectral sequence*

$$E_2^{*,*} \cong \mathrm{Cotor}^{*,*}_{E_*E}\left(E_*, E_*(X)\right) \Rightarrow \pi_*\left(\hat{X}_E\right).$$

In the particular case when E is classical mod p homology, we have $E_* \cong \mathbb{F}_p$ concentrated in degree zero, and we have $E_*E \cong A_*$, the mod p dual Steenrod algebra, a commutative bialgebra over $\mathbb{F}_p$. When X is a CW-complex with finitely many cells in each dimension, the spectral sequence then takes the form

$$E_2^{*,*} \cong \mathrm{Cotor}^{*,*}_{A_*}\left(\mathbb{F}_p, H_*(X;\mathbb{F}_p)\right) \Rightarrow \left(\pi_*^{st}(X)\right)\hat{}_p,$$

where $\left(\pi_*^{st}(X)\right)\hat{}_p$ is the p-adic completion of the stable homotopy groups of X, i.e., $\left(\pi_*^{st}(X)\right)\hat{}_p \cong \lim_{n\to\infty} \pi_*^{st}(X)/p^n\pi_*^{st}(X)$.

Now suppose X and Y are spectra, and write $\pi_*^{st}(X\wedge Y)\hat{}_p$ for the p-adically completed stable homotopy groups of the smash product $X\wedge Y$. The image of the Hurewicz homomorphism

$$\pi_*^{st}(X\wedge Y)\hat{}_p \to H_*(X\wedge Y;\mathbb{F}_p) \cong H_*(X;\mathbb{F}_p)\otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)$$

is precisely the 0-line in the Adams E_∞-page, i.e., those elements in

$$\mathbb{F}_p\square_{A_*}\left(H_*(X;\mathbb{F}_p)\otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)\right)$$

which survive the Adams spectral sequence.

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[2] Standard references here are [1] and appendix 1 of [17].

Mod p homology *does* satisfy a Künneth formula. The question of whether the *Hurewicz image in mod p homology* satisfies a Künneth formula is nearly[3] the same question as asking whether Cotor^0, i.e., the cotensor product, satisfies the Künneth formula

$$\mathbb{F}_p\square_{A_*}\left(H_*(X;\mathbb{F}_p)\otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)\right)\cong\left(\mathbb{F}_p\square_{A_*}H_*(X;\mathbb{F}_p)\right)\otimes_{\mathbb{F}_p}\left(\mathbb{F}_p\square_{A_*}H_*(Y;\mathbb{F}_p)\right).$$

An unbridled optimist might even hope for some kind of Künneth theorem in higher Cotor-groups as well, yielding a way to express the entire Adams E_2-page

$$\mathrm{Cotor}^{*,*}_{A_*}\left(\mathbb{F}_p, H_*(X;\mathbb{F}_p)\otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)\right)$$

for $X\wedge Y$ in terms of the Adams E_2-pages for X and for Y.

The topological considerations make it seems plausible that a Künneth formula could exist in Cotor^0, or perhaps Cotor^*, at least under appropriate hypotheses. On the other hand, the representation-theoretic considerations make it seem implausible. We hope the reader is now interested in knowing how the story turns out, which is the subject of this paper.

Our findings are outlined below:

- Section 2 reviews basic ideas about categories of comodules. The main result is a simple observation, Proposition 2.1, which establishes that we cannot possibly have a Künneth formula for comodules over a bialgebroid Γ unless the left and right unit maps η_L,η_R of Γ are equal to one another, i.e., the bialgebroid Γ is in fact a bialgebr*a*. Consequently we restrict our attention to comodules over bialgebras for the rest of the paper.
- Section 3 reviews the case where one comodule is trivial, i.e., all of its elements are primitive. This is the easiest case, and the results in section 3 essentially all follow from a lemma that, as far as the author knows, first appeared in a 2002 paper of Al-Takhman, [3]. The main result in section 3 is Corollary 3.4, which establishes that, when N is a *trivial* Γ-comodule which is flat over the ground ring A, we have isomorphisms

$$\mathrm{Cotor}^n_\Gamma(L, M\otimes_A N)\cong\mathrm{Cotor}^n_\Gamma(L,M)\otimes_A N\cong\mathrm{Cotor}^n_\Gamma(L,M)\otimes_A(A\square_\Gamma N)\tag{1.1}$$

  for all n and L.
- Given that isomorphism (1.1) is a reasonable Künneth-like theorem whenever N is a trivial comodule, we can approach the situation when M, N are each nontrivial by filtering N so that its filtration quotients are trivial. In section 4 we carry out that idea. Specifically, in Definition-Proposition 4.1, we define a canonical such filtration, the "primitive filtration," and if that filtration of a given comodule N is exhaustive, then we get a canonical grading on N. In Proposition 4.2 we show that, when Γ is a connected graded bialgebra, all bounded-below graded Γ-comodules have exhaustive primitive filtration; since the dual Steenrod algebra is connected and graded, the primitive filtration is exhaustive in the topological examples which motivate the investigations in this paper.

  The result is a spectral sequence

$$E_1^{s,t}\cong\mathrm{Cotor}^s_\Gamma(L,M)\otimes_A N^t\Rightarrow\mathrm{Cotor}^s_\Gamma(L,M\otimes_A N),\tag{1.2}$$

  constructed in Theorem 4.6. See that theorem for the necessary conditions for the existence of this spectral sequence, as well as the definition of the

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[3] "Nearly" here means "up to Adams spectral sequence differentials originating on the 0-line."

relevant grading on N in the description of the E_1-page. Among several consequences of the existence of this spectral sequence, the most important is Corollary 4.10: if the canonical map

$$(L\Box_\Gamma M) \otimes_A (A\Box_\Gamma N) \to L\Box_\Gamma (M \otimes_A N)$$

is an isomorphism, and if the canonical map

$$\mathrm{Cotor}^1_\Gamma(L, M) \otimes_A (A\Box_\Gamma N) \to \mathrm{Cotor}^1_\Gamma(L, M \otimes_A N)$$

is also injective, then N has trivial coaction. This is a negative result: it tells us that, unless one of the tensor factors is a comodule with trivial coaction, *we cannot expect any reasonable kind of Künneth formula for both* Cotor^0 *and* Cotor^1. See Corollary 4.10 for the mild hypotheses necessary for this result.

- In section 5 we run spectral sequence (1.2) very explicitly in a family of examples where there are nontrivial differentials of several lengths. In particular, the spectral sequence calculations in section 5 demonstrate that spectral sequence (1.2) does not necessarily collapse at E_1 or E_2, and is indeed capable of having arbitrarily long nonzero differentials.
- Section 6 takes up the question of when we have a Künneth theorem for the cotensor product, i.e., Cotor^0. In light of the negative result of Corollary 4.10, a Künneth theorem for Cotor^0 is as much as can be hoped for, unless one of the comodules in question is trivial. Theorem 6.2 is the main result in section 6: it establishes that, when M is a subcomodule of Γ, we have a Künneth isomorphism $(A\Box_\Gamma M) \otimes_A (A\Box_\Gamma N) \to A\Box_\Gamma(M \otimes_A N)$ if and only if, whenever $n \in N$ satisfies $\psi_N(n) \in M \otimes_A N \subseteq \Gamma \otimes_A N$, then n is primitive. Here $\psi_N : N \to \Gamma \otimes_A N$ is the coaction map of the comodule N. This result is generalized by Theorem 6.4 and its corollaries, which allow M to be a subcomodule of a finite (or, in the graded case, finite-type) direct sum of copies of Γ.
- Finally, in section 7, we give some topological consequences. In Corollaries 7.2 and 7.4, we give necessary and sufficient conditions on a spectrum Y for the 0-line in the Adams E_2-page for $X \wedge Y$ to decompose as a tensor product of the 0-line in the Adams E_2-page for X with the 0-line in the Adams E_2-page for Y, with the criteria being especially explicit in the cases $X = BP, BP\langle n\rangle, ku, ko$, and tmf.

2. Basic ideas, and the restriction to bialgebras.

We recall a few basic facts about comodules; excellent references in the coalgebra case include the book-length treatment in [9], and in the bialgebroid case, Appendix 1 of [17]. Given a coalgebra or a bialgebroid (A, Γ), a *left* Γ*-comodule* is a left A-module M equipped with a coassociative, counital left A-module map $\psi : M \to \Gamma \otimes_A M$. If Γ is a bialgebroid[4] and flat over A, then we have an abelian category

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[4] We discuss comodules over a bialgebroid, not the more general case of comodules over a coalgebroid, because one needs a multiplication on Γ in order to define a tensor product of Γ-comodules over A. The argument is dual to the familiar argument that, if R is a k-algebra and M, N are R-modules, then we need to have a k-linear comultiplication on R in order to get a natural R-module structure on $M \otimes_k N$. To be clear, the material on tensor product of comodules over a coalgebra in section 3.8 of [9] is about the tensor product of a Γ-comodule with an A-module, rather than a tensor product of two Γ-comodules.

Comod(Γ) of left Γ-comodules equipped with a monoidal product given by tensor product over A, and the *relatively injective left* Γ*-comodules* are the retracts of those of the form $\Gamma \otimes_A M$ for some A-module M, which are called *extended comodules*. The extended comodule functor $E : \mathrm{Mod}(A) \to \mathrm{Comod}(\Gamma)$ is *right* adjoint to the forgetful functor $G : \mathrm{Comod}(\Gamma) \to \mathrm{Mod}(A)$. All derived functors in this paper are *relative* derived functors with respect to the allowable class whose injectives are the relative injectives. See chapter IX of [13] for an excellent textbook treatment of relative homological algebra in general, or Appendix 1 of [17] (oriented toward generalized Adams spectral sequences) or section 10.11 of [15] (oriented toward Eilenberg-Moore spectral sequences) for an introductory treatment in the present setting, that is, relative derived functors with respect to this particular allowable class on comodules[5].

Here is the basic question: under what conditions do we have a Künneth formula for comodule primitives? A first attempt at making such a question precise is as follows: given left Γ-comodules M and N, we ask under what conditions we might have an isomorphism

$$A\Box_\Gamma (M \otimes_A N) \cong (A\Box_\Gamma M) \otimes_A (A\Box_\Gamma N)\,. \tag{2.3}$$

The most obvious requirement is that $(A\Box_\Gamma M) \otimes_A (A\Box_\Gamma N)$ must actually be *defined*. So $A\Box_\Gamma M$ and $A\Box_\Gamma N$ must be A-modules. Here is the relevant observation:

Proposition 2.1. *Let (A, Γ) be a bialgebroid. Then the following conditions are equivalent:*

1. *The subgroup inclusion*
$$M\Box_\Gamma N \hookrightarrow M \otimes_A N \tag{2.4}$$
is A-linear for every right Γ-comodule M and left Γ-comodule N.
2. *The subgroup inclusion*
$$A\Box_\Gamma M \hookrightarrow M \tag{2.5}$$
is A-linear for all left Γ-comodules M.
3. *The bialgebroid (A, Γ) is a bialgebra. That is, the left unit and right unit maps η_L, η_R of (A, Γ) coincide, i.e., $\eta_L = \eta_R$.*

Proof.

1 implies 2: Trivial.

2 implies 3: Let $M = A$. Then $1 \in A\Box_\Gamma A$, so (2.5) implies in particular that

$$\begin{aligned}\eta_R(a) \otimes 1 &= 1 \otimes a\\ &= \psi(a \cdot 1)\\ &= a\psi(1)\\ &= a(1 \otimes 1)\\ &= \eta_L(a) \otimes 1 \in \Gamma \otimes_A A,\end{aligned}$$

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[5] The reader who prefers not to deal with relative homological algebra may be relieved to know that, when the ground ring A of the bialgebroid (A, Γ) is a field, these relative derived functors agree with ordinary derived functors. So relative homological algebra need not be mentioned at all unless the ground ring A fails to be a field.

so $\eta_R(a) = \eta_L(a)$ for all $a \in A$. Here ψ is the coaction map on M.

3 implies 1: Recall that the structure map $\psi : M \to \Gamma \otimes_A M$ of a left Γ-comodule M is required to be a *left* A-module morphism. If $\eta_L = \eta_R$, then ψ is a left A-module morphism if and only if it is a right A-module morphism. Consequently the maps $\psi_M \otimes N$ and $M \otimes \psi_N$ in the equalizer sequence

$$\begin{tikzcd} M\Box_\Gamma N \arrow[r] & M \otimes_A N \arrow[r, shift left, "\psi_M\otimes N"] \arrow[r, shift right, "M\otimes\psi_N"'] & M \otimes_A \Gamma \otimes_A N \end{tikzcd}$$

are A-module morphisms, and so the inclusion (2.4) is A-linear.

□

Remark 2.2. As a consequence of Proposition 2.1, we could have a Künneth isomorphism (2.3) for all Γ-comodules M, N only if (A, Γ) is a bialgebr*a*, not only a bialgebr*oid*. But even restricting to comodules over a bialgebra, the formula (2.3) still often fails to hold. For example, if we let $A = k$ for some field k, let $\Gamma = k[x]$ with x primitive, and let $M = N$ be the sub-Γ-comodule of Γ which is k-linearly spanned by 1 and x, then $A\Box_\Gamma M \cong A\Box_\Gamma N \cong k$ spanned by 1, while $A\Box_\Gamma(M \otimes_A N) \cong k \oplus k$ with basis $\{1 \otimes 1, x \otimes 1 - 1 \otimes x\}$. Since the k-vector-space dimensions differ, there is no way we could have an isomorphism exactly of the form (2.3). But the "moral" of Proposition 2.1 remains true: any Künneth formula for Cotor—i.e., any way of describing Cotor of a tensor product over A in terms of a tensor product over A of Cotor-modules—first requires that A be a bialgebra, so that the tensor product over A of Cotor-modules is defined at all.

In light of Proposition 2.1, we restrict our attention to comodules over bialgebras for the rest of this paper.

3. The case where one comodule is trivial.

We have a tensor-cotensor relation given by the result[6]

Proposition 3.1. *If Γ is a coalgebra*[7] *over a commutative ring A, M is a right Γ-comodule, and N is a left Γ-comodule, and W is an A-module, then we have natural A-linear maps*

$$W \otimes_A (M\Box_\Gamma N) \to (W \otimes_A M)\Box_\Gamma N, \tag{3.6}$$

$$(M\Box_\Gamma N) \otimes_A W \to M\Box_\Gamma (N \otimes_A W), \tag{3.7}$$

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[6] Proposition 3.1 is very easy to prove—indeed, it appears as a lemma with a five-line proof in the earliest published paper where I am aware that it appears, [3]—but since Proposition 3.1 is the closest thing to a Künneth theorem for Cotor which is already in the literature, we hope the reader will forgive us to trying to give a bit of history of this lemma. After appearing as Lemma 3.8 in [3], it shows up as Lemma 2.3 in [4], the published form of Al-Takhman's Düsseldorf thesis [2], which is easier to locate than [3], and finally it appears as 10.6 in the book [9].

[7] The isomorphisms (3.6) and (3.7) appear to involve tensor products, over A, of Γ-comodules, and yet Γ is only a coalgebra, not a bialgebra. The reason these tensor products make sense, as Γ-comodules, despite the lack of a multiplication on Γ is that in each case, one of the tensor factors is a *trivial* comodule.

where $W \otimes_A M$ and $N \otimes_A W$ are each given the trivial *Γ-comodule structure, i.e., the coaction maps are*

$$\begin{aligned} W \otimes_A M &\to W \otimes_A M \otimes_A \Gamma \\ w \otimes m &\to w \otimes \psi_M(m), \quad \textit{and} \\ N \otimes_A W &\to \Gamma \otimes_A N \otimes_A W \\ n \otimes w &\to \psi_N(n) \otimes w, \end{aligned}$$

where $\psi_M : M \to M \otimes_A \Gamma$ and $\psi_N : N \to \Gamma \otimes_A N$ are the coaction maps of M and N, respectively.

Furthermore, the following conditions are equivalent:

- *The map* (3.6) *is an isomorphism.*
- *The map* (3.7) *is an isomorphism.*
- *The canonical inclusion $M\Box_\Gamma N \hookrightarrow M\otimes_A N$ remains injective after applying the functor $- \otimes_A W$.*

While Proposition 3.1 is the closest result to a Künneth theorem for Cotor (in this case, only Cotor^0, i.e., the cotensor product) appearing in the literature, it has a critical limitation that we need to overcome: it describes $\mathrm{Cotor}^*_\Gamma(L, M \otimes_A N)$ in terms of $\mathrm{Cotor}^*_\Gamma(L, M)$ and N only when either M or N has trivial Γ-coaction. Here a left Γ-comodule W is said to have *trivial coaction* if its coaction map $\psi_W : W \to \Gamma \otimes_A W$ is given by $\psi_W(w) = 1 \otimes w$. This limitation is far too strict for Proposition 3.1 to be useful for the topological applications of Cotor. One would like to find a description of $\mathrm{Cotor}^*_\Gamma(L, M \otimes_A N)$ in terms as close as possible to $\mathrm{Cotor}^*_\Gamma(L, M)$ and $\mathrm{Cotor}^*_\Gamma(L, N)$, *without* assuming that either M or N have trivial Γ-coaction.

Before moving on to the case where M and N each have nontrivial coaction, we at least remark that Proposition 3.1 has an easy corollary for the higher Cotor groups. We first need a couple of easy lemmas. Recall that $E : \mathrm{Mod}(A) \to \mathrm{Comod}(\Gamma)$ denotes the extended comodule functor, which is right adjoint to the forgetful functor $G : \mathrm{Comod}(\Gamma) \to \mathrm{Mod}(A)$. Then:

Lemma 3.2. *There exists an isomorphism $E(M \otimes_A GN) \cong (EM) \otimes_A N$ of Γ-comodules, natural in the variables M and N.*

Proof. See Proposition 9 in [11], where this is proven for A a bialgebra; the same proof applies when A is a bialgebroid. ☐

Lemma 3.3. *If M, N are left Γ-comodules and M is relatively injective, $M \otimes_A N$ is also relatively injective.*

Proof. Choose an A-module $\tilde{M}$ and left Γ-comodule morphisms $i : M \to E\tilde{M}$ and $\pi : E\tilde{M} \to M$ such that $\pi \circ i = \mathrm{id}_M$. Then we have the commutative diagram

$$\begin{tikzcd} M \otimes_A N \arrow[r, "i\otimes_A N"] \arrow[rr, bend right, "\mathrm{id}"'] & (E\tilde{M}) \otimes_A N \arrow[r, "\pi\otimes_A N"] & M \otimes_A N, \end{tikzcd}$$

and $(E\tilde{M}) \otimes_A N \cong E(\tilde{M} \otimes_A N)$ as left Γ-comodules, by Lemma 3.2. So $M \otimes_A N$ is indeed a retract of the extended comodule $E(\tilde{M} \otimes_A N)$. ☐

Now we have the following corollary of Proposition 3.1:

Corollary 3.4. *If Γ is a coalgebra over a commutative ring A, L is a right Γ-comodule, M is a left Γ-comodule, and N is a* trivial *left Γ-comodule which is flat over A, then we have isomorphisms of A-modules*

$$\begin{aligned}\mathrm{Cotor}^n_\Gamma(L, M\otimes_A N) &\cong \mathrm{Cotor}^n_\Gamma(L,M)\otimes_A N\\ &\cong \mathrm{Cotor}^n_\Gamma(L,M)\otimes_A (A\Box_\Gamma N)\end{aligned}$$

for each nonnegative integer n.

In particular, in the $n=0$ case we have a Künneth isomorphism

$$L\Box_\Gamma(M\otimes_A N)\cong (L\Box_\Gamma M)\otimes_A (A\Box_\Gamma N).$$

Proof. Since N has trivial coaction, we have an isomorphism of left Γ-comodules $N\cong A\Box_\Gamma N$, which we use freely throughout this proof.

Given a resolution $I^\bullet$ of M by relatively injective left Γ-comodules, the cochain complex of left A-comodules $I^\bullet\otimes_A N$ is exact by flatness of N, and it is a complex of relative injectives due to Lemma 3.3. Consequently the cohomology of the cochain complex $L\Box_\Gamma(I^\bullet\otimes_A N)$ yields the Cotor-groups $\mathrm{Cotor}^*_\Gamma(L, M\otimes_A N)$. Applying Proposition 3.1 yields the isomorphism of cochain complexes $L\Box_\Gamma(I^\bullet\otimes_A N)\cong (L\Box_\Gamma I^\bullet)\otimes_A N$, and the cohomology of the cochain complex on the right is $\mathrm{Cotor}^*_\Gamma(L,M)\otimes_A N$, again using flatness of N. □

Corollary 3.4 has a partial converse given below by Corollaries 4.10 and 4.11.

4. The primitive filtration of a comodule.

A basic idea in what follows is that, when the comodules M, N both have nontrivial coaction, the results of section 3 do not directly apply, but we could try to filter one of the comodules M or N so that each of the filtration quotients has trivial coaction, in order to get a spectral sequence whose input term could be simplified by some application of Corollary 3.4. There is a canonical and quite useful such filtration[8], defined in Definition-Proposition 4.1:

Definition-Proposition 4.1.

- *Let (A,Γ) be a bialgebroid, and let M be a left Γ-comodule. By the* primitive filtration of M *we mean the filtration*

$$0=M_{-1}\subseteq M_0\subseteq M_1\subseteq\dots\subseteq M \tag{4.8}$$

  of M by subgroups defined as follows: M_i is the kernel of the projection $M\to M(i+1)$, where

$$M=M(0)\to M(1)\to M(2)\to\dots \tag{4.9}$$

  is a sequence of surjective group homomorphisms defined inductively by letting $M(i+1)$ be the cokernel of the inclusion $A\Box_\Gamma M(i)\hookrightarrow M(i)$. Consequently we have short exact sequences of groups

$$0\to M_i\to M\to M(i+1)\to 0, \tag{4.10}$$

$$0\to A\Box_\Gamma M(i)\to M(i)\to M(i+1)\to 0,\quad \textit{and} \tag{4.11}$$

$$0\to M_i\to M_{i+1}\to A\Box_\Gamma M(i+1)\to 0. \tag{4.12}$$

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[8] This primitive filtration on a comodule is *not* the same as the primitive filtration on a bialgebra, from [14]. We have never seen our primitive filtration in the literature and have never heard it mentioned by others, but it is a very simple and effective idea, and we expect it has probably been considered by others on more than occasion.

- *We will say that M* has exhaustive primitive filtration *if $\cup_i M_i = M$. We will say that M* has finite primitive filtration *if $M_i = M$ for some M.*
- *If (A, Γ) is a bialgebra, then the primitive filtration (4.8) is a filtration by sub-Γ-comodules, not merely by subgroups, and the extensions (4.10),(4.11), and (4.12) are extensions of Γ-comodules.*
- *When the underlying A-module extensions of the left Γ-comodule extensions (4.12) are split, we say that primitive filtration on M is* split*. The primitive filtration on M is automatically split if, for example the commutative ring A is semisimple, e.g. a field.*
- *When the primitive filtration on M is split and exhaustive, then we have an isomorphism of A-modules*[9] *$M \cong \coprod_{i \geq 0} A \Box_\Gamma M(i)$. We then refer to this grading on M, whose degree i summand M^i is $A \Box_\Gamma M(i)$, as the* primitive grading *on M.*

Proof. By Proposition 2.1, if (A, Γ) is a bialgebra, then each of the inclusions $A \Box_\Gamma M(i) \hookrightarrow M(i)$ is a left Γ-comodule morphism, and consequently (4.9) is a sequence of left Γ-comodule morphisms, so (4.8) is also a sequence of left Γ-comodule morphisms, and similarly for (4.10),(4.11), and (4.12). □

Of course every left Γ-comodule whose underlying A-module is Artinian has finite primitive filtration. Another useful source of comodules with well-behaved primitive filtrations is Proposition 4.2:

Proposition 4.2. *Let (A, Γ) be a graded bialgebra which is connected, i.e., A and Γ each are trivial in negative degrees and the unit map $\eta : A \to \Gamma$ is surjective (equivalently, an isomorphism) in grading degree zero. Then every bounded-below*[10] *graded left Γ-comodule has exhaustive primitive filtration.*

Proof. If M is a bounded-below graded left Γ-comodule which is trivial in grading degrees $< n$, then $A \Box_\Gamma M \to M$ is an isomorphism in the bottommost grading degree, so its cokernel is also bounded-below and with a strictly higher lower bound on the grading degrees of its nontrivial summands. By induction, in (4.9) each $M(i)$ is trivial in grading degrees $< n + i$, and in (4.8), the inclusion $M_i \subseteq M$ is an isomorphism in grading degrees $< n + i - 1$. So every homogeneous element of M is contained in $M(i)$ for some i. □

We need a couple of easy lemmas, Lemma 4.3 and 4.4, which are certainly not new:

Lemma 4.3. *Let (A, Γ) be a bialgebra, let I be a set, and let $\{M_i : i \in I\}$ be a set of left Γ-comodules. Let L be a left Γ-comodule. Then the natural map*[11]

$$\coprod_{i \in I} (L \Box_\Gamma M_i) \to L \Box_\Gamma \left(\coprod_{i \in I} M_i \right) \tag{4.13}$$

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[9] But generally not an isomorphism of Γ-comodules.

[10] It is standard that a graded group, module, ring, etc. M is said to be *bounded below* if there exists an integer n such that M is trivial in grading degrees $< n$.

[11] Since the forgetful functor from Γ-comodules to A-modules is a left adjoint, it preserves colimits, and so the coproduct in the domain of (4.13) can equally well be regarded as a coproduct in A-modules or a coproduct in Γ-comodules. Of course the coproduct in the codomain of (4.13) must be regarded as a coproduct in Γ-comodules, since otherwise it would not make sense to apply the cotensor product to it.

is an isomorphism.

Proof. We have a commutative diagram of A-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \coprod_i L\Box_\Gamma M_i \arrow[r] \arrow[d] & \coprod_i L \otimes_A M_i \arrow[r] \arrow[d, "\cong"] & \coprod_i L \otimes_A \Gamma \otimes_A M_i \arrow[d, "\cong"] \\
0 \arrow[r] & L\Box_\Gamma \coprod_i M_i \arrow[r] & L \otimes_A \coprod_i M_i \arrow[r] & L \otimes_A \Gamma \otimes_A \coprod_i M_i
\end{tikzcd} \quad (4.14)$$

and the maps indicated as isomorphisms are isomorphisms since tensor products, in A-modules, commute with coproducts. Hence the remaining vertical map in (4.14)—i.e., the map (4.13)—is also an isomorphism. ☐

Lemma 4.4. *Let (A, Γ) be a bialgebra, and let*

$$M_0 \to M_1 \to M_2 \to \dots \quad (4.15)$$

be a sequence of morphisms of left Γ-comodules. Let L be a left Γ-comodule, and let n be a nonnegative integer. Then the natural group homomorphism

$$\operatorname{colim}_i \operatorname{Cotor}^n_\Gamma(L, M_i) \to \operatorname{Cotor}^n_\Gamma(L, \operatorname{colim}_i M_i) \quad (4.16)$$

is an isomorphism.

Proof. Let $D^\bullet_\Gamma(M_i)$ denote the unreduced cobar resolution of M_i (as in Definition A1.2.11 of Appendix 1 of [17], but using Γ in place of the unit coideal $\ker \epsilon$). Since $D^n_\Gamma(M_i) \cong \Gamma^{\otimes_A (n+1)} \otimes_A M_i$ for each n, the natural map of cochain complexes of A-modules

$$\operatorname{colim}_i D^\bullet_\Gamma(M_i) \to D^\bullet_\Gamma(\operatorname{colim}_i M_i) \quad (4.17)$$

is an isomorphism. Since the forgetful functor from Γ-comodules to A-modules is faithful and a left adjoint, it preserves and reflects colimits, so (4.17) is in fact an isomorphism of cochain complexes of Γ-comodules.

In the case that the colimit is merely a coproduct (i.e., the maps (4.15) are split injections), applying Lemma 4.3 to (4.17) yields that

$$\coprod_i \operatorname{Cotor}^n_\Gamma(L, M_i) \to \operatorname{Cotor}^n_\Gamma(L, \coprod_i M_i) \quad (4.18)$$

is an isomorphism.

More generally—i.e., when the maps in (4.15) are not necessarily split injections—we have the short exact sequence of Γ-comodules

$$\coprod_i M_i \overset{\mathrm{id} - T}{\longrightarrow} \coprod_i M_i \to \operatorname{colim}_i M_i \quad (4.19)$$

and, applying $\mathrm{Cotor}^*_\Gamma(L,-)$ to (4.19), a commutative diagram with exact columns

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\coprod_i \mathrm{Cotor}^n_\Gamma(L, M_i) \arrow[r, "\cong"] \arrow[d] & \mathrm{Cotor}^n_\Gamma(L, \coprod_i M_i) \arrow[d] \\
\coprod_i \mathrm{Cotor}^n_\Gamma(L, M_i) \arrow[r, "\cong"] \arrow[d] & \mathrm{Cotor}^n_\Gamma(L, \coprod_i M_i) \arrow[d] \\
\mathrm{colim}_i \mathrm{Cotor}^n_\Gamma(L, M_i) \arrow[r] \arrow[d] & \mathrm{Cotor}^n_\Gamma(L, \mathrm{colim}_i M_i) \arrow[d] \\
\coprod_i \mathrm{Cotor}^{n+1}_\Gamma(L, M_i) \arrow[r, "\cong"] \arrow[d] & \mathrm{Cotor}^{n+1}_\Gamma(L, \coprod_i M_i) \arrow[d] \\
\coprod_i \mathrm{Cotor}^{n+1}_\Gamma(L, M_i) \arrow[r, "\cong"] \arrow[d] & \mathrm{Cotor}^{n+1}_\Gamma(L, \coprod_i M_i) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

in which the indicated maps are isomorphisms due to (4.18) being an isomorphism. Now the Five Lemma gives us that the remaining map—i.e., (4.16)—is an isomorphism. □

Proposition 4.5. *Let (A,Γ) be a bialgebra, and let M be a left Γ-comodule with exhaustive primitive filtration $M_0 \subseteq M_1 \subseteq \dots$. Then we have a first-quadrant spectral sequence*

$$\begin{aligned} E^1_{s,t} \cong \mathrm{Tor}^A_s(A\square_\Gamma M(t), N) &\Rightarrow \mathrm{Tor}^A_s(M,N) \\ d^r : E^r_{s,t} &\to E^r_{s-1,t+r}. \end{aligned}$$

Proof. This is simply the spectral sequence of the exact couple obtained by applying $\mathrm{Tor}^A_*(-,N)$ to the tower of extensions of A-modules

$$\begin{tikzcd}
M_{-1} \arrow[r] & M_0 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & \dots \\
 & A\square_\Gamma M(0) & A\square_\Gamma M(1) & A\square_\Gamma M(2). &
\end{tikzcd}$$

□

Of course, in most motivating examples of bialgebras (A,Γ), A is a field and consequently the spectral sequence of Proposition 4.5 collapses to the $s=0$ line already at the E^1-page, yielding an unsurprising isomorphism. So the spectral sequence of Proposition 4.5 is not our focus at all, and we mention it only for completeness. A much more interesting spectral sequence is given by the following:

Theorem 4.6. (The Künneth spectral sequence for Cotor.) *Let (A, Γ) be a bialgebra, and let L be a right Γ-comodule and M, N left Γ-comodules. Suppose that M is flat over A, and suppose that N has exhaustive primitive filtration.*

Then we have a first quadrant spectral sequence

$$E_1^{s,t} \cong \operatorname{Cotor}_\Gamma^s(L, M) \otimes_A (A \square_\Gamma N(t)) \Rightarrow \operatorname{Cotor}_\Gamma^s(L, M \otimes_A N) \tag{4.20}$$
$$d_r : E_r^{s,t} \to E_r^{s+1,t-r}.$$

If the primitive filtration on N is also split (for example, if A is a field), then the E_1-term of (4.20) is also given by $E_1^{s,t} \cong \operatorname{Cotor}_\Gamma^s(L, M) \otimes_A N^t$ where $N^t = A \square_\Gamma N(t)$ is the degree t summand in the primitive grading on N, as in Definition-Proposition 4.1.

Proof. Since M is flat, applying $M \otimes_A -$ to the primitive filtration on N yields a tower of extensions of left Γ-comodules

$$\begin{tikzcd}
0 = M \otimes_A N_{-1} \arrow[r] & M \otimes_A N_0 \arrow[r] \arrow[d] & M \otimes_A N_1 \arrow[r] \arrow[d] & \dots \\
& M \otimes_A (A \square_\Gamma N(0)) & M \otimes_A (A \square_\Gamma N(1)) &
\end{tikzcd}$$

such that $\operatorname{colim}_{n \to \infty} M \otimes_A N_n \to M \otimes_A N$ is an isomorphism, and consequently a spectral sequence

$$E_1^{s,t} \cong \operatorname{Cotor}_\Gamma^s (L, M \otimes_A (A \square_\Gamma N(t))) \Rightarrow \operatorname{colim}_i \operatorname{Cotor}_\Gamma^s(L, M \otimes_A N_i) \tag{4.21}$$
$$d_r : E_r^{s,t} \to E_r^{s+1,t-r}.$$

Recall that $A \square_\Gamma N(t)$ has a natural left Γ-coaction given by Proposition 2.1, since (A, Γ) is a bialgebra, but that left Γ-coaction is clearly the trivial one, since every element of $A \square_\Gamma N(t)$ is primitive. Triviality of the coaction is what allows us to apply Corollary 3.4, to get that $\operatorname{Cotor}_\Gamma^s (L, M \otimes_A (A \square_\Gamma N(t))) \cong \operatorname{Cotor}_\Gamma^s(L, M) \otimes_A (A \square_\Gamma N(t))$ for each s, t. Finally, the abutment $\operatorname{colim}_i \operatorname{Cotor}_\Gamma^*(L, M \otimes_A N_i)$ of spectral sequence (4.21) is isomorphic to

$$\operatorname{Cotor}_\Gamma^* (L, \operatorname{colim}_i(M \otimes_A N_i)) \cong \operatorname{Cotor}_\Gamma^*(L, M \otimes_A N)$$

by Lemma 4.4. $\square$

In case it helps the reader to visualize the spectral sequence, we provide a picture of a portion of the E_1-page of spectral sequence of Theorem 4.6, with s as the horizontal coordinate, t as the vertical coordinate, the d_1-differentials colored in red, the d_2-differentials colored in orange, and the d_3-differentials colored in blue:
(4.22)

$$\begin{tikzcd}[column sep=small]
& \vdots & \vdots & \vdots & \vdots & \\
t=3 & (L \square_\Gamma M) \otimes_A N^3 \arrow[dr, red] \arrow[ddr, orange] \arrow[dddr, blue] & \operatorname{Cotor}_\Gamma^1(L, M) \otimes_A N^3 \arrow[dr, red] \arrow[ddr, orange] \arrow[dddr, blue] & \operatorname{Cotor}_\Gamma^2(L, M) \otimes_A N^3 \arrow[dr, red] \arrow[ddr, orange] \arrow[dddr, blue] & \operatorname{Cotor}_\Gamma^3(L, M) \otimes_A N^3 & \cdots \\
t=2 & (L \square_\Gamma M) \otimes_A N^2 \arrow[dr, red] \arrow[ddr, orange] & \operatorname{Cotor}_\Gamma^1(L, M) \otimes_A N^2 \arrow[dr, red] \arrow[ddr, orange] & \operatorname{Cotor}_\Gamma^2(L, M) \otimes_A N^2 \arrow[dr, red] \arrow[ddr, orange] & \operatorname{Cotor}_\Gamma^3(L, M) \otimes_A N^2 & \cdots \\
t=1 & (L \square_\Gamma M) \otimes_A N^1 \arrow[dr, red] & \operatorname{Cotor}_\Gamma^1(L, M) \otimes_A N^1 \arrow[dr, red] & \operatorname{Cotor}_\Gamma^2(L, M) \otimes_A N^1 \arrow[dr, red] & \operatorname{Cotor}_\Gamma^3(L, M) \otimes_A N^1 & \cdots \\
t=0 & (L \square_\Gamma M) \otimes_A N^0 & \operatorname{Cotor}_\Gamma^1(L, M) \otimes_A N^0 & \operatorname{Cotor}_\Gamma^2(L, M) \otimes_A N^0 & \operatorname{Cotor}_\Gamma^3(L, M) \otimes_A N^0 & \cdots \\
& s=0 & s=1 & s=2 & s=3 &
\end{tikzcd}$$

In (4.22) we write N^i rather than $A\square_\Gamma N(i)$ as though the primitive filtration on N is split, but this is only for notational convenience, to make the illustration of the spectral sequence more readable. If the primitive filtration on N is not split, then replace all the instances of N^i in diagram (4.22) with $A\square_\Gamma N(i)$, and the resulting diagram remains a correct picture of the spectral sequence.

This convention for drawing the spectral sequence in (4.22), in particular the choice of horizontal and vertical coordinates, is convenient because the bidegrees in the s-column are precisely those bidegrees which contribute, in the E_∞-page, to $\mathrm{Cotor}^s_\Gamma(L, M\otimes_A N)$.

Theorem 4.6 has corollaries:

Corollary 4.7. *Let (A,Γ) be a bialgebra, and let L be a right Γ-comodule and M, N left Γ-comodules. Suppose that M is flat over A, and suppose that N has split exhaustive primitive filtration. Then the A-module $L\square_\Gamma(M\otimes_A N)$ is isomorphic to the sub-A-module of $(L\square_\Gamma M)\otimes_A N$ consisting of the elements in the $s=0$-column in the kernel of the d_r differential for every $r\geq 1$.*

Note that, unlike the Adams spectral sequence (whose convergence properties are discussed in [6]), there is no issue of conditional convergence in this spectral sequence which could cause $N=\oplus_{n\geq 0}N^n$ in the E_∞-page to become $\prod_{n\geq 0}N^n$ in the abutment. This is because the spectral sequence of Theorem 4.6 converges to the colimit, and so the extension problems are organized into a colimit sequence rather than a limit sequence.

Corollary 4.8. *Let (A,Γ) be a bialgebra, and let L be a right Γ-comodule and M, N left Γ-comodules. Suppose that M is flat over A and that N has primitive filtration of length 2, i.e., the quotient Γ-comodule $N/\left(A\square_\Gamma N\right)$ has trivial Γ-coaction. Then the spectral sequence of Theorem 4.6 degenerates to a long exact sequence*

$$\begin{tikzcd}
 & & 0 \arrow[dll] \\
(L\square_\Gamma M)\otimes_A (A\square_\Gamma N) \arrow[r] & L\square_\Gamma(M\otimes_A N) \arrow[r] & (L\square_\Gamma M)\otimes_A (N/(A\square_\Gamma N)) \arrow[dll] \\
\mathrm{Cotor}^1_\Gamma(L,M)\otimes_A (A\square_\Gamma N) \arrow[r] & \mathrm{Cotor}^1_\Gamma(L,M\otimes_A N) \arrow[r] & \mathrm{Cotor}^1_\Gamma(L,M)\otimes_A (N/(A\square_\Gamma N)) \arrow[dll] \\
\mathrm{Cotor}^2_\Gamma(L,M)\otimes_A (A\square_\Gamma N) \arrow[r] & \mathrm{Cotor}^2_\Gamma(L,M\otimes_A N) \arrow[r] & \dots .
\end{tikzcd}$$

Definition 4.9. *Given a commutative ring A, we will say that an A-module M* detects nontriviality *if, whenever N is an A-module such that $M\otimes_A N\cong 0$, we have $N\cong 0$.*

For example, over any commutative ring A, nonzero free A-modules detect nontriviality. If A is a field, every nonzero A-module detects nontriviality. If A is a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K, then $K\oplus A/\mathfrak{m}A$ detects nontriviality.

One of the most important consequences of Theorem 4.6 is Corollary 4.10:

Corollary 4.10. *Let (A,Γ) be a bialgebra, and let L be a right Γ-comodule and M, N left Γ-comodules. Suppose that the A-module $L\square_\Gamma M$ detects nontriviality,* *suppose that N has split exhaustive primitive filtration, suppose that the canonical map*

$$(L\square_\Gamma M)\otimes_A (A\square_\Gamma N)\to L\square_\Gamma (M\otimes_A N) \tag{4.23}$$

is an isomorphism, and suppose that the canonical map

$$\mathrm{Cotor}^1_\Gamma(L,M)\otimes_A (A\square_\Gamma N)\to \mathrm{Cotor}^1_\Gamma(L,M\otimes_A N) \tag{4.24}$$

is injective. Then N has trivial coaction.

Proof. The map (4.23) is an isomorphism if and only if all elements in the leftmost column above the bottom row in spectral sequence (4.20) support differentials, and consequently fail to survive to the E_∞-page. In particular, if $(L\square_\Gamma M)\otimes_A N^1$ were nonzero, it would have to support a nonzero d_1-differential hitting $\mathrm{Cotor}^1_\Gamma(L,M)\otimes_A N^0$, and consequently not all elements of $\mathrm{Cotor}^1_\Gamma(L,M)\otimes_A N^0$ would survive to the E_2-term, much less the E_∞-term. Consequently the map (4.24) would not be able to be injective. So $(L\square_\Gamma M)\otimes_A N^1$ must be trivial. Since $L\square_\Gamma M$ detects nontriviality, N^1 vanishes. Since $N^1 = A\square_\Gamma(N/(A\square_\Gamma N))$, vanishing of N^1 ensures that $N(2)\cong N/(A\square_\Gamma N)\cong N(1)$ and consequently N^2 also vanishes; by induction, the primitive filtration on N is constant starting at the second term. Exhaustivity of the primitive filtration on N consequently gives us that $A\square_\Gamma N = N$. □

Using Proposition 4.2 and the fact that triviality of $L\square_\Gamma M$ implies triviality of M when A is a field, (A,Γ) is a connected graded bialgebra, and L,M are bounded-below graded Γ-comodules, we have:

Corollary 4.11. *Let A be a field, and let Γ be a connected graded bialgebra over A. Let L be a graded right Γ-comodule, let M,N be graded left Γ-comodules such that that the canonical map*

$$(L\square_\Gamma M)\otimes_A (A\square_\Gamma N)\to L\square_\Gamma (M\otimes_A N) \tag{4.25}$$

is an isomorphism, suppose that L,M, and N are nonzero and bounded below, and suppose that the canonical map

$$\mathrm{Cotor}^1_\Gamma(L,M)\otimes_A (A\square_\Gamma N)\to \mathrm{Cotor}^1_\Gamma(L,M\otimes_A N) \tag{4.26}$$

is injective. Then N has trivial coaction.

Corollaries 4.10 and 4.11 are negative results, and they provide a kind of converse to Corollary 3.4: they tell us that, when we have the expected Künneth formula (4.25) for Cotor^0 (i.e., the cotensor product), then we cannot possibly have any reasonable Künneth formula describing Cotor^1, since the failure of the canonical map (4.26) to be injective means that $\mathrm{Cotor}^1_\Gamma(A,M\otimes_A N)$ cannot decompose as any kind of extension of $\mathrm{Cotor}^1_\Gamma(A,M)\otimes_A \mathrm{Cotor}^0_\Gamma(A,N)$ by $\mathrm{Cotor}^0_\Gamma(A,M)\otimes_A \mathrm{Cotor}^1_\Gamma(A,N)$ unless N has trivial coaction. Instead, the best general statement one can make is that one has the spectral sequence of Theorem 4.6.

One reasonable response to these negative results is to ask under what circumstances we at least have a Künneth formula for Cotor^0. We take up this question in section 6.

5. Example calculations of the Künneth spectral sequence for Cotor.

Let k be a field of characteristic p and let Γ be the bialgebra $k[\xi]/\xi^p$ with x primitive. Consider the case $L = k = M$ and $N = \Gamma$ of Theorem 4.6: the primitive filtration on N is finite, with N^n the k-vector space spanned by ξ^n for $0 \leq n \leq p-1$ and trivial otherwise. We adopt the notation $k\{x\}$ for the k-vector space with basis $\{x\}$, so that we can make a reasonable drawing of the E_1-page of the Künneth spectral sequence, here pictured with $p = 5$:

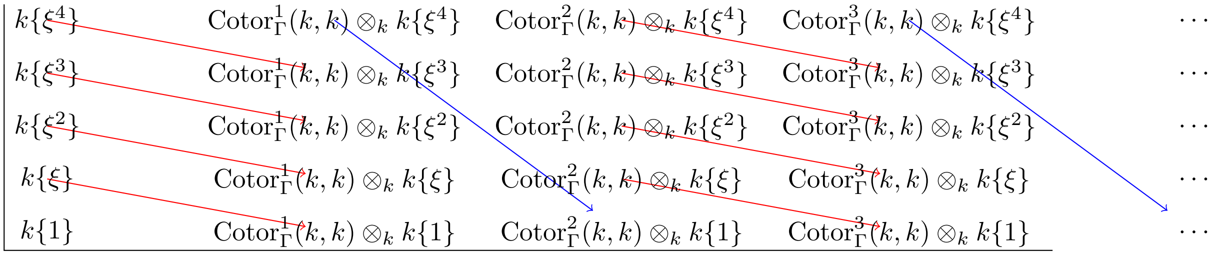

Figure 1. $E_1^{*,*} \cong \mathrm{Cotor}^*_{k[\xi]/\xi^p}(k,k) \otimes k[\xi]/\xi^p$
$\Rightarrow \mathrm{Cotor}^*_{k[\xi]/\xi^p}(k, k[\xi]/\xi^p)$.

The nonzero differentials are as pictured, but that claim deserves some justification, which we now give, below.

An alternative construction of the spectral sequence of Theorem 4.6 is given by filtering the cobar complex of Γ with coefficients of N by the primitive filtration on N. That is, the spectral sequence of Theorem 4.6 is the spectral sequence of the filtered cochain complex

$$\Gamma^{\otimes_A \bullet} \otimes_A N_0 \subseteq \Gamma^{\otimes_A \bullet} \otimes_A N_1 \subseteq \Gamma^{\otimes_A \bullet} \otimes_A N_2 \subseteq \cdots \subseteq \Gamma^{\otimes_A \bullet} \otimes_A N.$$

In the case $\Gamma = k[\xi]/\xi^p$, a cobar complex 1-cocycle which represents a generator for $\mathrm{Cotor}^1_\Gamma(k,k)$ is the primitive $\xi \in k[\xi]/\xi^p$, while a cobar complex 2-cocycle which represents a generator for $\mathrm{Cotor}^2_\Gamma(k,k)$ is the "transpotent"

$$T\xi := \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} \xi^{p-i} \otimes \xi^i \in k[\xi]/\xi^p \otimes_k k[\xi]/\xi^p$$

of ξ, and the graded k-algebra $\mathrm{Cotor}^*_\Gamma(k,k)$ is isomorphic to $\Lambda(h) \otimes_k k[b]$, where $h \in \mathrm{Cotor}^1_\Gamma(k,k)$ is represented by ξ, and $b \in \mathrm{Cotor}^2_\Gamma(k,k)$ is represented by $T\xi$. The comodule algebra structure on Γ, and the fact that the primitive filtration on Γ is a filtration by comodule ideals, yields that the spectral sequence of Theorem 4.6 is a spectral sequence of k-algebras, and so we need only calculate the differentials on k-algebra generators for each page. The E_1-page is isomorphic to $k[\xi]/\xi^p \otimes_k \Lambda(h) \otimes_k k[b]$, and since $\psi(\xi) = \xi \otimes 1 + 1 \otimes \xi$, we have $\delta(\xi) = \xi \otimes 1$ in the cobar complex, and hence the d_1-differential $d_1(\xi) = h$. For degree reasons, $d_1(h) = 0$ and $d_1(b) = 0$, so the Leibniz rule gives us that $d_1(\xi^i b^j) = i\xi^{i-1}hb^j$ for all $i < p$ and $d_1(\xi^i hb^j) = 0$, yielding the E_2-page $E_2^{*,*} \cong \Lambda(h\xi^{p-1}) \otimes_k k[b]$. The class $h\xi^{p-1}$ is represented by the cobar complex 1-cochain $\xi \otimes \xi^{p-1}$, and we have $\delta(\xi \otimes \xi^{p-1}) = -\sum_{i=1}^{p-1} \binom{p-1}{i} \xi^i \otimes \xi^{p-1-i}$ in the cobar complex. When we reach the E_{p-1}-page of the spectral sequence, we have that $d_{p-1}(h\xi^{p-1})$ is the sum of the terms of $\delta(\xi \otimes \xi^{p-1})$

of primitive filtration[12] $p-1$ less than that of $\xi \otimes \xi^{p-1}$. Consequently we have that $d_{p-1}(h\xi^{p-1})$ is the class in the E_{p-1}-page represented by the cocycle $-\xi \otimes \xi^{p-1} \otimes 1$, i.e., $d_{p-1}(h\xi^{p-1}) = -(T\xi) \otimes 1$. From the Leibniz rule we get that all that remains on the E_p-page is the copy of k in bidegree $(0,0)$, so the spectral sequence collapses at that page. This yields the long differentials pictured in Figure 1.

One consequence is that, by taking p to be a large prime, we can get arbitrarily long nonzero differentials in spectral sequence (4.20).

6. When do we have a Künneth formula for Cotor^0?

The main result in this section is Theorem 6.2, which give a criterion for the differentials supported on the $s = 0$ column of spectral sequence (4.20) to wipe out everything above the $t = 0$ row[13], at least in the situation of greatest interest for topological applications, i.e., the situation of Theorem 4.6 when $L = A$ and M is a subcomodule of Γ. The most obvious cases of those topological applications are described later, in section 7. In this section we also give Theorem 6.4, a generalization of Theorem 6.2 which weakens the hypothesis that M is a subcomodule of Γ.

Definition 6.1. *Let A be a commutative ring, let Γ be a bialgebra over A, and let M be a subcomodule of the left Γ-comodule Γ. Let N be a left Γ-comodule. By the* Künneth quotient of N relative to M *we mean the A-module $\text{Kü}(N; M)$ given by the cokernel of the natural A-module map*

$$\left(M \otimes_A N^0\right) \times_{\Gamma \otimes_A N} N \hookrightarrow (M \otimes_A N) \times_{\Gamma \otimes_A N} N$$

arising from the commutative diagram

$$\begin{tikzcd}
M \otimes_A N^0 \arrow[r, hook] & M \otimes_A N \arrow[r, hook] & \Gamma \otimes_A N \\
\left(M \otimes_A N^0\right) \times_{\Gamma \otimes_A N} N \arrow[r, hook] \arrow[u, hook] & (M \otimes_A N) \times_{\Gamma \otimes_A N} N \arrow[r, hook] \arrow[u, hook] & N \arrow[u, hook, "\psi_N"']
\end{tikzcd} \tag{6.27}$$

in which each square is a pullback square of A-modules.

The maps marked with hooked arrows in (6.27) are injective if M is flat over A.

Of the equivalent conditions given in Theorem 6.2, the fourth condition is an especially checkable (in practical situations) necessary and sufficient condition for the Künneth formula for the cotensor product, (6.28), to hold.

Theorem 6.2. *Let A be a commutative ring, let Γ be a bialgebra over A, and let M be a subcomodule of the left Γ-comodule Γ. Let N be a left Γ-comodule which is flat over A, and suppose that M is also flat over A. Then the following conditions are equivalent:*

(1) *The canonical map*

$$(A\square_\Gamma M) \otimes_A (A\square_\Gamma N) \to A\square_\Gamma (M \otimes_A N) \tag{6.28}$$

is an isomorphism.

---

[12] Remember that we have filtered the cobar complex of Γ, with coefficients in Γ, by the primitive filtration on the coefficients.

[13] Corollary 4.7 established that this behavior of the differentials in the spectral sequence is equivalent to the natural map $(L\square_\Gamma M) \otimes_A (A\square_\Gamma N) \to L\square_\Gamma (M \otimes_A N)$ being an isomorphism.

(2) *The intersection of* $M \otimes_A N \subseteq \Gamma \otimes_A N$ *with the image of the coaction map* $\psi_N : N \to \Gamma \otimes_A N$ *lands in the submodule* $M \otimes_A (A \square_\Gamma N)$ *of* $M \otimes_A N$*.*

(3) *The Künneth quotient* $\mathrm{Kü}(N; M)$ *vanishes.*

(4) *If* $n \in N$ *satisfies* $\psi_N(n) \in M \otimes_A N$*, then* n *is primitive.*

Proof.

1 is equivalent to 2: Since $M \subseteq \Gamma$ and M is flat over A, we have monomorphisms $M \otimes_A N \hookrightarrow \Gamma \otimes_A N$ and $A\square_\Gamma(M \otimes_A N) \hookrightarrow A\square_\Gamma(\Gamma \otimes_A N)$, and consequently a commutative diagram of A-modules

$$\begin{tikzcd}
A\square_\Gamma (M \otimes_A N) \arrow[r, hook] & A\square_\Gamma (\Gamma \otimes_A N) & N \arrow[l, "sh"', "\cong"] \\
A\square_\Gamma (M \otimes_A (A\square_\Gamma N)) \arrow[r, hook] \arrow[u, hook] & A\square_\Gamma (\Gamma \otimes_A (A\square_\Gamma N)) \arrow[u, hook] & A\square_\Gamma N \arrow[l, "sh|_{A\square_\Gamma N}"', "\cong"] \arrow[u, hook]
\end{tikzcd}$$

where $sh : N \to A\square_\Gamma(\Gamma \otimes_A N)$ is the "shearing isomorphism"given on elements simply by the coaction map ψ_N. Expressed more carefully, the diagram

$$\begin{tikzcd}
A\square_\Gamma (\Gamma \otimes_A N) \arrow[d, hook, "\iota"] & N \arrow[l, "sh"', "\cong"] \arrow[dl, "\psi_N"] \\
\Gamma \otimes_A N &
\end{tikzcd} \tag{6.29}$$

commutes, where ι is the canonical inclusion of the comodule primitives of $\Gamma \otimes_A N$ into $\Gamma \otimes_A N$.

Consequently any given element of $A\square_\Gamma(M \otimes_A N) \subseteq \Gamma \otimes_A N$ is equal to $\psi(n)$ for some $n \in N$. So the inclusion $A\square_\Gamma(M \otimes_A N^0) \subseteq A\square_\Gamma(M \otimes_A N)$ is surjective if and only if $\psi(n)$ is contained in $M \otimes_A N^0$ whenever $\psi(n) \in M \otimes_A N$, as in the statement of the proposition.

2 is equivalent to 3, and 2 is equivalent to 4: These implications are a routine matter of unwinding the definitions.

□

Remark 6.3. It would be nice if the functor $\mathrm{Kü}(-; M) : \mathrm{Comod}(\Gamma) \to \mathrm{Mod}(A)$ were at least half-exact, as that would give us some means (by standard homological methods) of actually computing its value on various comodules. However, Kü is generally not half-exact, as one sees from the example where A is a field k, and $M = \Gamma = k[\xi]/\xi^2$ with ξ primitive. If we had any flexible and powerful tools for computing $\mathrm{Kü}(-; M)$, then the vanishing of $\mathrm{Kü}(N; M)$ could potentially be the most checkable of the four equivalent conditions listed in Theorem 6.2, instead of (as presently seems to be the case) the fourth condition being the most practically checkable. Unfortunately, the author does not know of any such tools for calculating Kü, and would be glad to learn of them.

If Γ is finitely generated as an A-module (or if (A, Γ) is graded with A concentrated in degree 0 and Γ finitely generated as an A-module in each degree) then Kü is at least a *coherent* functor in the sense of [5], but this does not seem to yield any nontrivial means of computing the values of $\mathrm{Kü}(-; M)$.

Theorem 6.2 admits a generalization to situations where M is not a subcomodule of Γ. That generalization is Theorem 6.4, which is not as clean to state as Theorem

6.2, which is why we present the result separately. The material on the Künneth quotient $\mathrm{Kü}(N;M)$ from 6.2 also has a generalization to the setting of Theorem 6.4, but since the vanishing of the Künneth quotient is the condition, among the equivalent conditions of Theorem 6.2, which we know the least applications for, we leave off the Künneth quotient from Theorem 6.4 to try to keep the statement cleaner.

Theorem 6.4. *Let (A,Γ) be a graded bialgebra with A concentrated in degree zero. Suppose that Γ is finite-type, that is, for each integer n the degree n summand of Γ is a finitely generated A-module. Let M be a graded left subcomodule of a finite-type direct sum of suspensions of Γ, that is, there exists some function $d:\mathbb{Z}\to\mathbb{N}$ and some one-to-one graded left Γ-comodule homomorphism*

$$\iota : M \hookrightarrow \coprod_{n\in\mathbb{Z}} \Sigma^n \Gamma^{\oplus d(n)}.$$

Suppose that N is a graded right Γ-comodule, and suppose that M and N are each flat over A. Then the following conditions are equivalent:

(1) *The canonical map*

$$(A\Box_\Gamma M)\otimes_A (A\Box_\Gamma N) \to A\Box_\Gamma(M\otimes_A N) \tag{6.30}$$

is an isomorphism.

(2) *The intersection of the image of*

$$\iota\otimes N : M\otimes_A N \hookrightarrow \left(\coprod_{n\in\mathbb{Z}} \Sigma^n\Gamma^{\oplus d(n)}\right)\otimes_A N$$

with the image of the direct sum of copies of the coaction map

$$\coprod_{n\in\mathbb{Z}} \Sigma^n \psi_N^{\oplus d(n)} : \coprod_{n\in\mathbb{Z}} \Sigma^n N^{\oplus d(n)} \to \coprod_{n\in\mathbb{Z}} \Sigma^n\left(\Gamma\otimes_A N\right)^{\oplus d(n)} \xrightarrow{\cong} \left(\coprod_{n\in\mathbb{Z}} \Sigma^n\Gamma\right)^{\oplus d(n)} \otimes_A N$$

lands in the submodule $M\otimes_A (A\Box_\Gamma N)$ of $M\otimes_A N$.

(3) *If $y\in\coprod_{n\in\mathbb{Z}}\Sigma^n N^{\oplus d(n)}$ satisfies*

$$\coprod_{n\in\mathbb{Z}} \Sigma^n\psi_N^{\oplus d(n)}(y) \in M\otimes_A N \subseteq \coprod_{n\in\mathbb{Z}} \Sigma^n\Gamma^{\oplus d(n)}\otimes_A N,$$

then y is a primitive element of $\coprod_{n\in\mathbb{Z}}\Sigma^n N^{\oplus d(n)}$.

Proof. The proof is essentially the same as that of Theorem 6.2. □

The most notable family of examples of graded bialgebras satisfying the hypotheses of Theorem 6.4 are the mod p Steenrod algebras and their linear duals at each prime p. The associated graded bialgebras $E^0S(n)$ of Ravenel's grading on the Morava stabilizer algebras (as in [16] and section 6.3 of [17]) form another important family of examples of finite-type, non-finite-dimensional graded bialgebras whose Cotor groups are, like Cotor over the duals of the Steenrod algebras, the input for spectral sequences which ultimately compute stable homotopy groups of various spaces and spectra.

However, the most straightforward case of Theorem 6.4 is, of course, the case where the rings and modules in question are concentrated in degree 0, i.e., the ungraded case:

Corollary 6.5. *Let A be a commutative ring, and let Γ be a bialgebra over A. Suppose that Γ is finitely generated as an A-module. Let M be a left subcomodule of a direct sum $\Gamma^{\oplus n}$ of finitely many copies of Γ, let N be a right Γ-comodule, and suppose that M, N are each flat over A. Then the following conditions are equivalent:*

1. *The canonical map*
$$(A\Box_\Gamma M) \otimes_A (A\Box_\Gamma N) \to A\Box_\Gamma (M \otimes_A N)$$
is an isomorphism.
2. *The intersection of the image of $\iota \otimes N : M \otimes_A N \hookrightarrow (\Gamma^{\oplus n}) \otimes_A N$ with the image of the direct sum of copies of the coaction map*
$$\psi_N^{\oplus n} : N^{\oplus n} \to (\Gamma \otimes_A N)^{\oplus n} \xrightarrow{\cong} \Gamma^{\oplus n} \otimes_A N$$
lands in the submodule $M \otimes_A (A\Box_\Gamma N)$ of $M \otimes_A N$.
3. *If $n \in N^{\oplus n}$ satisfies $\psi_N(n) \in M \otimes_A N \subseteq \Gamma^{\oplus n} \otimes_A N$, then n is a primitive element of $N^{\oplus n}$.*

Example 6.6.

- See the calculations of section 5 for a case in which the the isomorphism 6.30 holds, i.e., a case in which we have a Künneth formula for Cotor^0.
- On the other hand, let $A = k$ for some field k of characteristic p, and let $\Gamma = k[\xi]/\xi^p$ with ξ primitive. Then the canonical map
$$(A\Box_\Gamma \Gamma) \otimes_A (A\Box_\Gamma \Gamma) \to A\Box_\Gamma (\Gamma \otimes_A \Gamma)$$
is not surjective: its domain is isomorphic to k, while its codomain is isomorphic to Γ. (Of course, this same example works in the same way for any bialgebra over a field, but we chose Γ as above for concreteness and consistency with section 5.)

7. TOPOLOGICAL APPLICATIONS.

Now we had better explain some of the topological consequences of the results obtained in this paper. Given a pointed space or spectrum X and an integer n, it is classical that we can inductively attach cells to X to wipe out all the homotopy groups of X in degrees greater than n, yielding a pointed space or spectrum $X^{\leq n}$ and a continuous map $X \to X^{\leq n}$ such that $\pi_m(X) \to \pi_m(X^{\leq n})$ is an isomorphism for all $m \leq n$, and such that $\pi_m(X^{\leq n})$ vanishes for all $m > n$. In the unstable setting, this construction dates back to [10], solving a problem of Hurewicz's from Eilenberg's 1949 list [12] of open problems in algebraic topology.

When we take the homotopy fiber of $X \to X^{\leq n}$, we get a homotopy fiber sequence

$$X^{>n} \to X \to X^{\leq n}, \tag{7.31}$$

where $X^{>n} \to X$ induces an isomorphism in π_m for all $m > n$, and $\pi_m(X^{>n})$ vanishes for $m \leq n$. If we take this homotopy fiber in the stable homotopy category, then (7.31) is also a homotopy *co*fiber sequence, and so it induces a long exact sequence in homology groups. It is natural to ask under what conditions on X and n we can get good computational control over that long exact sequence in homology.

In particular, the homotopy cofiber sequence (7.31) induces a *short* exact sequence of graded $\mathbb{F}_p$-vector spaces

$$0 \to H_*(X;\mathbb{F}_p) \to H_*(X^{\leq n};\mathbb{F}_p) \to H_*(\Sigma X^{>n};\mathbb{F}_p) \to 0 \tag{7.32}$$

if we assume that X is bounded below, $H\mathbb{F}_p$-nilpotently complete[14], and the A_*-comodule primitives of $H^*(X;\mathbb{F}_p)$ are trivial in degrees $\geq n$. Here p is any prime, and A_* is the mod p dual Steenrod algebra.

In other words, if $\mathrm{Cotor}^0_{A_*}(\mathbb{F}_p, H_*(X;\mathbb{F}_p))$ is bounded above (i.e., vanishes in sufficiently large degrees), then (7.32) is short exact for sufficently large n.

The 0-line in the Adams E_2-page for a smash product $X \wedge Y$ is

$$\mathrm{Cotor}^0_{A_*}\left(\mathbb{F}_p, H_*(X;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)\right).$$

The 0-line in the Adams spectral sequence is of particular interest, since the 0-line in the E_∞-page is precisely the image of the Hurewicz homomorphism from stable homotopy to homology. In light of the above considerations about attaching cells to kill higher homotopy, if we know that $\mathrm{Cotor}^0_{A_*}(\mathbb{F}_p, H_*(X;\mathbb{F}_p))$ and $\mathrm{Cotor}^0_{A_*}(\mathbb{F}_p, H_*(Y;\mathbb{F}_p))$ are each bounded above, we would like to know that $\mathrm{Cotor}^0_{A_*}\left(\mathbb{F}_p, H_*(X;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)\right)$ is also bounded above, so that

$$0 \to H_*\left(X \wedge Y;\mathbb{F}_p\right) \to H_*\left((X \wedge Y)^{<n};\mathbb{F}_p\right) \to H_*\left(\Sigma(X \wedge Y)^{\geq n};\mathbb{F}_p\right) \to 0$$

is short exact for some n.

Of course $\mathrm{Cotor}^0_{A_*}(\mathbb{F}_p, -)$ is simply the A_*-comodule primitives functor, so as a special case of Theorem 6.2, we have:

Corollary 7.1. *Let X, Y be spectra, and suppose that $H_*(X;\mathbb{F}_p)$ is a A_*-subcomodule of A_*. Let $e(X), e(Y), e(X \wedge Y)$ denote the 0-line in the Adams E_2-term for X, Y, and $X \wedge Y$, respectively. Then the canonical map $e(X) \otimes_{\mathbb{F}_p} e(Y) \to e(X \wedge Y)$ is an isomorphism if and only if the following condition is satisfied:*

> *For all homogeneous $n \in H_*(Y;\mathbb{F}_p)$ such that $\psi(n) \in H_*(X;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p) \subseteq A_* \otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)$, we have that n is an A_*-comodule primitive.*

In particular, if the A_*-comodule primitives in $H_*(X;\mathbb{F}_p)$ and $H_*(Y;\mathbb{F}_p)$ are each bounded above, and condition (7.1) *is satisfied, then the A_*-comodule primitives in $H_*(X;\mathbb{F}_p)$ are also bounded above.*

There are many familiar and compelling examples of spectra X such that $H_*(X;\mathbb{F}_p)$ is a A_*-subcomodule of A_*: for example, the case $X = BP$, or the case $X = BP\langle n\rangle$ for any positive integer n, or the cases $X = ko$ or $X = tmf$ when $p = 2$. In these cases, we have:

Corollary 7.2. *Let Y be a spectrum. We continue to write $e(X)$ for the 0-line in the Adams E_2-spectral sequence for a spectrum X. We write ζ_n for the conjugate $\overline{\xi}_n$ of ξ_n in the dual Steenrod algebra A_*.*

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[14] The standard reference for nilpotent completion of spectra is [7]. We offer a bit of explanation to make the idea concrete, for readers not already familar with nilpotent completion of spectra: under the assumption that a spectrum X is bounded below, Bousfield proves that X is $H\mathbb{F}_p$-nilpotently complete if and only if its homotopy groups are Ext-p-complete in the sense of [8]. More concretely, if X is bounded-below and each of its homotopy groups are finitely generated, then X is $H\mathbb{F}_p$-nilpotently complete if and only if each of its homotopy groups are p-adically complete.

- *Let* $p = 2$. *Then the canonical map* $e(BP) \otimes_{\mathbb{F}_2} e(Y) \to e(BP \wedge Y)$ *is an isomorphism if and only if, for each homogeneous element* $y \in H_*(Y;\mathbb{F}_2)$ *such that* $\psi(y) \in P(\zeta_1^2, \zeta_2^2, \dots) \otimes_{\mathbb{F}_2} H_*(Y;\mathbb{F}_2)$, *we have* $\psi(y) = 1 \otimes y$.
- *Let* $p > 2$. *Then the canonical map* $e(BP) \otimes_{\mathbb{F}_p} e(Y) \to e(BP \wedge Y)$ *is an isomorphism if and only if, for each homogeneous element* $y \in H_*(Y;\mathbb{F}_p)$ *such that* $\psi(y) \in P(\xi_1, \xi_2, \dots) \otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)$, *we have* $\psi(y) = 1 \otimes y$.
- *Let* $p = 2$. *The canonical map* $e(BP\langle n\rangle) \otimes_{\mathbb{F}_2} e(Y) \to e(BP\langle n\rangle \wedge Y)$ *is an isomorphism if and only if, for each homogeneous element* $y \in H_*(Y;\mathbb{F}_2)$ *such that* $\psi(y) \in P(\zeta_1^2, \dots, \zeta_n^2, \zeta_{n+1}, \zeta_{n+2}, \dots) \otimes_{\mathbb{F}_2} H_*(Y;\mathbb{F}_2)$, *we have* $\psi(y) = 1 \otimes y$.
- *Let* $p > 2$. *The canonical map* $e(BP\langle n\rangle) \otimes_{\mathbb{F}_p} e(Y) \to e(BP\langle n\rangle \wedge Y)$ *is an isomorphism if and only if, for each homogeneous element* $y \in H_*(Y;\mathbb{F}_p)$ *such that* $\psi(y) \in P(\xi_1, \xi_2, \dots) \otimes_{\mathbb{F}_p} E(\tau_n, \tau_{n+1}, \dots) \otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)$, *we have* $\psi(y) = 1 \otimes y$.
- *Let* $p = 2$. *The canonical map* $e(ko) \otimes_{\mathbb{F}_2} e(Y) \to e(ko \wedge Y)$ *is an isomorphism if and only if, for each homogeneous element* $y \in H_*(Y;\mathbb{F}_2)$ *such that* $\psi(y) \in P(\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \dots) \otimes_{\mathbb{F}_2} H_*(Y;\mathbb{F}_2)$, *we have* $\psi(y) = 1 \otimes y$.
- *Let* $p = 2$. *The canonical map* $e(tmf) \otimes_{\mathbb{F}_2} e(Y) \to e(tmf \wedge Y)$ *is an isomorphism if and only if, for each homogeneous element* $y \in H_*(Y;\mathbb{F}_2)$ *such that* $\psi(y) \in P(\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots) \otimes_{\mathbb{F}_2} H_*(Y;\mathbb{F}_2)$, *we have* $\psi(y) = 1 \otimes y$.

The cases of Corollary 7.2 can be handled by an alternate method, since the spectra BP and $BP\langle n\rangle$ have the property that their mod p homology is not only an A_*-subcomodule of A_* but also a sub-*bialgebra* of A_*. The same is true of ko and tmf at the prime 2. When $H_*(X;\mathbb{F}_p)$ is a sub-bialgebra of A_*, we can use a change-of-rings isomorphism

$$\begin{aligned}\mathrm{Cotor}^n_{A_*}(\mathbb{F}_p, H_*(X \wedge Y;\mathbb{F}_p)) &\cong \mathrm{Cotor}^n_{A_*}\left(\mathbb{F}_p, H_*(X;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p)\right) \\ &\cong \mathrm{Cotor}^n_{H_*(X;\mathbb{F}_p)}(\mathbb{F}_p, H_*(Y;\mathbb{F}_p))\end{aligned}$$

to arrive at the same conclusions as in Corollary 7.2. The advantage of using Corollary 7.1 rather than change-of-rings isomorphisms is flexibility and generality: Corollary 7.1 does not require $H_*(X;\mathbb{F}_p)$ to have its own comultiplication.

More broadly, the methods of this paper, particularly Theorem 6.4, also apply when $H_*(X;\mathbb{F}_p)$ is not even a subcomodule of A_*, much less a subcoalgebra. This includes some familiar classical cases: for example, $X = ku$ or $X = MU$ at odd primes. As a special case of Theorem 6.4, we have the generalization of Corollary 7.1:

Corollary 7.3. *Let* X, Y *be spectra, and suppose that* $H_*(X;\mathbb{F}_p)$ *is a graded sub-*A_**-comodule of* $\coprod_{n\in\mathbb{Z}} \Sigma^n A_*^{\oplus d(n)}$ *for some function* $d : \mathbb{Z} \to \mathbb{N}$. *Then* $e(X) \otimes_{\mathbb{F}_p} e(Y) \to e(X \wedge Y)$ *is an isomorphism if and only if, for each homogeneous element* $y \in H_*(Y;\mathbb{F}_p)$ *such that*

$$\coprod_{n\in\mathbb{Z}} \Sigma^n \psi^{\oplus d(n)}(y) \in H_*(X;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p) \subseteq \coprod_{n\in\mathbb{Z}} \Sigma^n A_*^{\oplus d(n)} \otimes_{\mathbb{F}_p} H_*(Y;\mathbb{F}_p),$$

we have $\coprod_{n\in\mathbb{Z}} \Sigma^n \psi^{\oplus d(n)}(y) = \coprod_{n\in\mathbb{Z}} \Sigma^n (1 \otimes y)^{\oplus d(n)}$, *i.e.,* y *is a* A_**-comodule primitive.*

A very familiar example of a spectrum to which Corollary 7.3 applies is the case $X = ku$:

Corollary 7.4. *Let $p > 2$. The canonical map $e(ku) \otimes_{\mathbb{F}_p} e(Y) \rightarrow e(ku \wedge Y)$ is an isomorphism if and only if, for each homogeneous element $y \in H_*(Y; \mathbb{F}_p)$ such that*

$$\coprod_{i=0}^{p-2} \Sigma^{2i}\psi(y) \in \coprod_{i=0}^{p-2} \Sigma^{2i} P(\xi_1, \xi_2, \dots) \otimes_{\mathbb{F}_p} E(\tau_2, \tau_3, \dots) \otimes_{\mathbb{F}_p} H_*(Y; \mathbb{F}_p) \subseteq \coprod_{i=0}^{p-2} \Sigma^{2i} A_* \otimes_{\mathbb{F}_p} H_*(Y; \mathbb{F}_p),$$

we have $\psi(y) = 1 \otimes y$.

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