

Iwasawa invariants of finite spectra

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Abstract

We calculate the classical Iwasawa invariants of the Iwasawa modules associated to the p -adic topological K -theory of finite spectra. We show that the graded average of the orders of n consecutive $K(1)$ -local homotopy groups of a finite spectrum X grows asymptotically like $\frac{-\log_p(n)}{2}$ times the total Iwasawa λ -invariant of X . We show that the Iwasawa μ -invariants of finite spectra are all zero. Finally, we prove a spectral analogue of a weak form of the Iwasawa Main Conjecture, describing the orders of the $K(1)$ -local homotopy groups of a certain “torsion-free replacement” of X in terms of the characteristic polynomials of the Iwasawa modules associated to X .

1 Introduction

1.1 The point of this paper

Throughout this document, we will let p be an odd prime.

In classical Iwasawa theory, an *Iwasawa module* is a module over the group algebra $\hat{\mathbb{Z}}_p[\hat{\mathbb{Z}}_p]$. Iwasawa modules arise naturally in the study of towers of Galois extensions of number fields

$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\infty = \bigcup_n F_n \quad (1)$$

such that, for each n , the Galois group $\text{Gal}(F_n/F)$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Iwasawa’s structure theory for Iwasawa modules defines three important algebraic invariants of an Iwasawa module: the λ -invariant, the μ -invariant, and the characteristic polynomial. The λ - and μ -invariants are integers, while the characteristic polynomial is an element of the polynomial ring $\hat{\mathbb{Z}}_p[T]$.

Iwasawa showed that the class groups of the number fields in (1) naturally fit together to form an Iwasawa module whose λ - and μ -invariants satisfy the equality

$$h_{F_n}^{(p)} = p^{n\lambda + p^n\mu + \nu} \quad (2)$$

for some integer ν and for all sufficiently large n . Here $h_{F_n}^{(p)}$ is the p -part of the class number of F_n . The point is that, if we consider how the p -part of the class group grows as we move up the tower (1), the asymptotic rate of growth is controlled by the λ - and μ -invariants. A fuller introduction to the λ - and μ -invariants is given in section 2.1.

Meanwhile, the characteristic polynomial of the Iwasawa module is used to define an *algebraic p -adic L -function* of the tower (1). In section 2 we give a fuller exposition of the number-theoretic background, including the *analytic p -adic L -function* and the Iwasawa Main Conjecture, i.e., the relationship between the algebraic and analytic p -adic L -functions.

Iwasawa modules also arise in stable homotopy theory, although they have not received nearly as much study in stable homotopy theory as in number theory. Via the stable Adams operations, the profinite group $\hat{\mathbb{Z}}_p^\times$ acts on the p -adic K -theory of any spectrum¹. Using the isomorphism of profinite groups $\hat{\mathbb{Z}}_p^\times \cong \hat{\mathbb{Z}}_p \times \mathbb{F}_p^\times$, one can canonically split the p -adic K -theory of any spectrum X into $2(p-1)$ summands

$$\epsilon_0(KU_p^\wedge)^0(X), \epsilon_1(KU_p^\wedge)^0(X), \dots, \epsilon_{p-2}(KU_p^\wedge)^0(X), \epsilon_0(KU_p^\wedge)^{-1}(X), \epsilon_1(KU_p^\wedge)^{-1}(X), \dots, \epsilon_{p-2}(KU_p^\wedge)^{-1}(X),$$

each of which is an Iwasawa module. This splitting was already given in 1969 by Adams in [1], although to our knowledge the first published mention of Iwasawa theory in connection with this splitting was not until Ravenel’s paper [17] in 1984.

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¹For readers with a background in number theory but not in stable homotopy theory, we remark that the main theorems in this paper concern *finite* spectra. The suspension spectrum of every finite pointed CW-complex is a finite spectrum, and conversely, every finite spectrum is a finite desuspension of the suspension spectrum of a finite CW-complex. Our point is that, for a reader who is unfamiliar with spectra, it is harmless to think of finite spectra as simply *the stable homotopy types of finite pointed CW-complexes*, as well as formal desuspensions thereof. In particular, all the theorems in this paper about $K(1)$ -local homotopy groups of finite spectra are also theorems about $K(1)$ -local homotopy groups of finite pointed CW-complexes.

Steve Mitchell wrote several papers [15],[16] about how the Iwasawa-theoretic arguments used by number theorists could be seen in terms of algebraic K -theory spectra, but the only published paper about Iwasawa theory *of spectra themselves*—that is, the only paper which applies Iwasawa theory to draw conclusions about topology, not about number theory—is the 2007 paper [11] of Rebekah Hahn and Steve Mitchell. Hahn and Mitchell focused on category-theoretic results, e.g. a classification of the thick subcategories of the homotopy category of weakly dualizable $L_{K(1)}S^0$ -module spectra, given in terms of subsets of $\text{Spec } \hat{\mathbb{Z}}_p[\hat{\mathbb{Z}}_p^\times]$. We review some of Hahn–Mitchell’s work in section 2.3.

By contrast, the aim of this paper is computational, rather than category-theoretic: we calculate the Iwasawa invariants of all finite spectra, express them in topological terms, and we prove topological analogues of well-known Iwasawa-theoretic results in number theory. We will now sketch the three main results in this paper—one for each of the classical Iwasawa invariants (λ -invariant, μ -invariant, and characteristic polynomial)—together with their number-theoretic precedents.

1.2 The λ -invariant

The formula (2), above, shows the precise sense in which the λ -invariant (together with the μ -invariant, but this vanishes, as we explain below) controls the asymptotic rate of growth of the p -part of the class number in an appropriate tower of number fields. Our topological analogue, stated below in Theorem A, is that *graded average of the orders of n consecutive $K(1)$ -local homotopy groups of a spectrum grows asymptotically like $\frac{-\log_p(n)}{2}$ times the total λ -invariant of the spectrum*. To be clear, throughout this paper, \log_p denotes the ordinary base p logarithm, not any kind of p -adic logarithm. “Graded averages” are explained in section 4.1, but the idea is very simple: instead of $\frac{1}{n} \sum_{j=1}^n |\pi_j(L_{K(1)}X)|$, the average of the orders of the first n $K(1)$ -local homotopy groups of X , we count the odd-degree homotopy groups as *negative* and the even-degree homotopy groups as *positive*. This yields the alternating sum $\frac{1}{n} \sum_{j=1}^n (-1)^j |\pi_j(L_{K(1)}X)|$. In our asymptotic analysis, we also allow skipping finitely many initial homotopy groups, i.e., we consider the asymptotic properties of the alternating sum $\frac{1}{n} \sum_{j=1+m}^{n+m} (-1)^j |\pi_j(L_{K(1)}X)|$ as both m and n go to infinity.

Theorem A (Theorem 4.6). For all finite spectra X , the limit

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \sum_{j=1+m}^{n+m} (-1)^j |\pi_j(L_{K(1)}X)|}{-\lambda(X) \cdot \log_p(n)/2}$$

is equal to 1.

Here $\lambda(X)$ is the *total λ -invariant of X* , defined in Definition 4.4 as the alternating sum of the λ -invariants of the Iwasawa modules of X :

$$\lambda(X) = \sum_{j=0}^{p-2} \left(\lambda(\epsilon_j(KU_p^\wedge)^0(X)) - \lambda(\epsilon_j(KU_p^\wedge)^{-1}(X)) \right).$$

We remark that our proof of Theorem A involves a bit of algebraic K -theory. In Proposition 4.2, we prove that, in a general triangulated category \mathcal{C} satisfying appropriate hypotheses, the asymptotic growth rate of the graded average of the orders of n consecutive homotopy groups is an *additive invariant*. Hence this asymptotic growth rate depends only on an object’s equivalence class in the Grothendieck group $K_0(\mathcal{C})$. Since K_0 of the category of finite spectra is isomorphic to \mathbb{Z} and generated by $[S^0]$, we are able to prove Theorem A by reducing to the case where X is a sphere. Another consequence of the algebraic- K -theoretic approach is that it is easy to see that the total λ -invariant of a finite spectrum is also equal to its Euler characteristic (see Lemma 4.5).

As far as we know, the idea of treating asymptotic growth of orders of homotopy groups as an additive invariant, and using algebraic K -theoretic methods to study it, is new. The method is general and flexible enough to be applied more broadly, e.g. at higher heights using the noncommutative Iwasawa theory of Venjakob [21]. The authors are pursuing this approach at height 2 at primes $p > 3$, but this goes beyond the scope of this paper. More generally, throughout this paper we have attempted to use methods and ideas which appear to be generalizable to higher heights, at sufficiently large primes, using Venjakob’s theory.

Aside from the total λ -invariant of a finite spectrum X , one can also consider the λ -invariants of each of the individual Iwasawa modules $\epsilon_j(KU_p^\wedge)^0(X)$ and $\epsilon_j(KU_p^\wedge)^{-1}(X)$. These are not as exciting as the total λ -invariant: in the proof of Lemma 4.5, we show that the individual λ -invariants are described topologically by the equalities

$$\begin{aligned} \lambda(\epsilon_j(KU_p^\wedge)^0(X)) &= \sum_{n \equiv j \pmod{p-1}} \dim_{\mathbb{Q}} H_{2n}(X), \text{ and} \\ \lambda(\epsilon_j(KU_p^\wedge)^{-1}(X)) &= \sum_{n \equiv j \pmod{p-1}} \dim_{\mathbb{Q}} H_{2n-1}(X). \end{aligned}$$

1.3 The μ -invariant

Iwasawa conjectured that, in the case of a tower (1) obtained from a number field F by iteratively adjoining a primitive p th root of unity, p^2 th root of unity, p^3 th root of unity, etc., the μ -invariant of the resulting Iwasawa module is zero. The celebrated

theorem of Ferrero–Washington [10] proved this conjecture in the case where F is an abelian extension of \mathbb{Q} . Our point is that, in number theory, one often expects the μ -invariant to vanish.

In Theorem B, we prove that this expectation is borne out for finite spectra:

Theorem B (Theorem 3.4). Let X be a finite spectrum. Then, for every i, j , the μ -invariant of each of the Iwasawa modules $\epsilon_j(KU_p^\wedge)^i(X)$ is zero.

1.4 The characteristic polynomial

Hahn–Mitchell define a *pseudo-equivalence* to be a map of spectra $X \rightarrow Y$ whose induced map in p -adic K -theory $(KU_p^\wedge)^i(Y) \rightarrow (KU_p^\wedge)^i(X)$ has finite kernel and finite cokernel for each i . In Theorem 3.6 we prove that every finite spectrum X is pseudo-equivalent to a canonical-up-to-homotopy wedge product $\overset{\circ}{X} = (X_{\text{even}})_{\text{torsfree}} \vee (X_{\text{odd}})_{\text{torsfree}}$ such that

- $(X_{\text{even}})_{\text{torsfree}}$ is a finite spectrum such that $H_*((X_{\text{even}})_{\text{torsfree}}; \mathbb{Z}_{(p)})$ is torsion-free and concentrated in even degrees, and
- $(X_{\text{odd}})_{\text{torsfree}}$ is a finite spectrum such that $H_*((X_{\text{odd}})_{\text{torsfree}}; \mathbb{Z}_{(p)})$ is torsion-free and concentrated in odd degrees.

Theorem 3.6 refines a theorem of Hahn and Mitchell which we recall below as Theorem 2.7. The spectrum $\overset{\circ}{X}$ appears in the statement of Theorem C, below, which establishes that the characteristic polynomials of the Iwasawa modules of a finite spectrum carry the information of the orders of the $K(1)$ -local homotopy groups of $\overset{\circ}{X}$.

Before stating Theorem C, we state its number-theoretic precedent. One consequence of the classical Iwasawa Main Conjecture, proven by Mazur and Wiles [14], is that *the p -adic valuations of special values of the characteristic polynomial of the Iwasawa module of (1) agree with the p -adic valuations of special values of a classical Dirichlet L -function*. We call this fact—stated more precisely in section 2.1—the *weak form of the Iwasawa Main Conjecture*.

Theorem C is a spectral analogue of the weak form of the Iwasawa Main Conjecture. Rather than the characteristic polynomial of the Iwasawa module of a tower of number fields (1), Theorem C concerns values of the characteristic polynomial $f^{\epsilon_j(KU_p^\wedge)^j(X)}(T)$ of the Iwasawa module $\epsilon_j(KU_p^\wedge)^i(X)$ of X , for $j = 0, \dots, p-2$ and for $i \in \{-1, 0\}$. The role of the special values of a Dirichlet L -function is played by the orders of the $K(1)$ -local homotopy groups of $\overset{\circ}{X}$. It is reasonable to think that the orders of these homotopy groups are indeed analogous to special values of a Dirichlet L -function, since in [18], the orders of those homotopy groups are shown to have, up to sign, the same p -adic valuations as the special values of a certain complex-analytic “ KU -local zeta-function of X ” which is a product of Tate twists of Dirichlet L -functions.

The notation $x \sim_p y$ in the statement of Theorem C denotes that x and y have the same p -adic valuation.

Theorem C (Theorem 3.10). Let X be a finite spectrum. Let α and β be the least and greatest integers i , respectively, such that $H_i(X; \mathbb{Q})$ is nontrivial. Then we have

$$\left| \pi_{2m-1} L_{K(1)} \overset{\circ}{DX} \right| \sim_p f^{\epsilon_{-m}(KU_p^\wedge)^0(X)} ((1+p)^{-m} - 1) \text{ and}$$

$$\left| \pi_{2m} L_{K(1)} \overset{\circ}{DX} \right| \sim_p f^{\epsilon_{-m}(KU_p^\wedge)^{-1}(X)} ((1+p)^{-m} - 1)$$

for all integers m satisfying $m < \frac{\beta}{2}$, and also for all integers m satisfying $m > \frac{\alpha}{2}$.

The subscript $-m$ of ϵ_{-m} , in the statement of Theorem C, is to be understood as being defined modulo $p-1$.

The proof of Theorem C resembles the proof of [18, Theorem 2.8], and more generally, the material on characteristic polynomials in this paper can be seen as p -adic versions of results on complex-analytic “ KU -local zeta-functions” of spectra given in [18]. These p -adic versions are, in some ways, stronger and more general: for example, Theorem C applies to all finite spectra, while [18, Theorem 2.8] required the relevant spectrum to have trivial homology in all odd degrees.

1.5 Conventions

- p will always be an *odd* prime.
- Given rational numbers a, b , we write $a \sim_p b$ to denote that a and b have the same p -adic valuation.

1.6 Acknowledgments

We thank Francesc Castella for graciously visiting Wayne State University and answering many of our questions about Iwasawa theory.

2 Background ideas from Iwasawa theory

2.1 Review of classical Iwasawa theory

In this section, we will take a brisk walk through classical Iwasawa theory, arriving at a statement of the Iwasawa Main Conjecture. Our treatment here is based primarily on [20], [22], and [19], but it is by no means an exhaustive exposition. Instead, we will simply highlight the objects and tools that will be used to applied to the Iwasawa modules of finite spectra in the subsequent sections.

Consider a tower of Galois extensions of the form (1). Write Γ_n for the Galois group $\text{Gal}(F_n/F)$, and Γ for the limit $\lim_n \Gamma_n$ of the Galois groups. Iwasawa studied such towers in the case that $\Gamma_n \cong \mathbb{Z}/p^n\mathbb{Z}$ and $\Gamma \cong \hat{\mathbb{Z}}_p$. Under such circumstances, the tower (1) is called a $\hat{\mathbb{Z}}_p$ -extension. Iwasawa was interested in determining the asymptotic growth rate of the class number $h_{F_n} = |\text{Cl}_{F_n}|$ as n increases. Iwasawa focused in particular on the growth rate of the order of the p -Sylow subgroup A_n of Cl_{F_n} . We will write $h_{F_n}^{(p)}$ for the order of this p -Sylow subgroup. Iwasawa proved the following result:

Theorem 2.1 (Iwasawa). *There exist nonnegative integers λ and μ and an integer ν such that*

$$h_{F_n}^{(p)} = p^{n\lambda + p^n\mu + \nu}$$

for all sufficiently large n .

Iwasawa conjectured that $\mu = 0$ in the case of the cyclotomic $\hat{\mathbb{Z}}_p$ -extension, i.e., the unique $\hat{\mathbb{Z}}_p$ -extension F_∞/F inside $\bigcup_n F(\zeta_{p^n})$. This conjecture was proven for F an abelian extension of \mathbb{Q} by Ferrero and Washington [10].

For each $\hat{\mathbb{Z}}_p$ -extension, let X_∞ be the limit $\lim_n A_n$ of the sequence of norm maps

$$\dots \xrightarrow{N} A_2 \xrightarrow{N} A_1 \xrightarrow{N} A_0.$$

Since each A_n is a $\hat{\mathbb{Z}}_p[\Gamma_n]$ -module, X_∞ is a module over the Iwasawa algebra $\Lambda = \hat{\mathbb{Z}}_p[[\Gamma]]$. Modules over the ring Λ are called Iwasawa modules, and there are at least two reasons why Iwasawa modules are central in classical Iwasawa theory. First, as observed by Serre, Λ is isomorphic to a power series ring with a single generator

$$\Lambda = \hat{\mathbb{Z}}_p[[\Gamma]] \cong \hat{\mathbb{Z}}_p[[T]]$$

where a choice of topological generator $\gamma \in \Gamma$ maps to $1 + T \in \hat{\mathbb{Z}}_p[[T]]$. We will fluidly use both descriptions of Λ throughout this paper. Second, finitely generated Iwasawa modules have a nice structure theorem, similar to finitely generated modules over a principal ideal domain, but the structure theory describes the module up to pseudo-isomorphism rather than up to isomorphism. A map of Iwasawa modules is called a *pseudo-isomorphism* if its kernel and cokernel are both finite. Given Iwasawa modules M, N , we write $M \simeq N$ if there exists a pseudo-isomorphism $M \rightarrow N$.²

Theorem 2.2 (Serre, Iwasawa). *For any finitely generated Iwasawa module M , we have*

$$M \simeq \Lambda^r \oplus \bigoplus_{i=1}^s \Lambda/p^{m_i} \oplus \bigoplus_{j=1}^t \Lambda/f_j(T)^{n_j}, \quad (3)$$

where each $f_j(T) \in \hat{\mathbb{Z}}_p[[T]]$ is an irreducible, distinguished polynomial, that is, an irreducible, monic polynomial such that $f(T) \equiv T^{\deg(f)} \pmod{p}$.

The invariants r, m_i, n_j, f_j of M are unique up to permutation of the sequence (m_1, \dots, m_s) and simultaneous permutation of the sequences (n_1, \dots, n_t) and $(f_1(T), \dots, f_t(T))$. The nonnegative integer r is, of course, the *rank* of M . The rank of an Iwasawa module is a pseudo-isomorphism invariant. When M is also Λ -torsion (i.e., of rank zero), three additional pseudo-isomorphism invariants of M play special roles in classical Iwasawa theory:

- The μ -invariant of M is $\mu(M) = \sum_{i=1}^s m_i$.
- The λ -invariant of M is $\lambda(M) = \sum_{j=1}^t n_j \deg(f_j)$.
- The characteristic polynomial of M is $f^M(T) = p^{\mu(M)} \prod_{j=1}^t f_j(T)^{n_j}$.

It turns out that X_∞ is Λ -torsion, and $\mu(X_\infty)$ and $\lambda(X_\infty)$ agree with the numbers μ and λ from Theorem 2.1. In other words, the structure of X_∞ determines the asymptotic growth rates of the p -part of class groups in a $\hat{\mathbb{Z}}_p$ -extension.

For the remainder of this section, we will focus our attention on the cyclotomic $\hat{\mathbb{Z}}_p$ -extension in which $F_n = \mathbb{Q}(\zeta_{p^n})$ for each n , and its corresponding Iwasawa module X_∞ . In this case we have $\Gamma = \lim_{n \rightarrow \infty} \text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\zeta_p))$. As a consequence of our running assumption that p is an odd prime, the profinite group $\hat{\mathbb{Z}}_p^\times$ of p -adic units splits as a product $\hat{\mathbb{Z}}_p \times \mathbb{F}_p^\times$. In terms of Galois groups, this splitting takes the form $\Gamma' \cong \Gamma \times \Delta$, where:

- $\Gamma' = \text{Gal}(F_\infty/\mathbb{Q}) \cong \hat{\mathbb{Z}}_p^\times$,
- and $\Delta = \text{Gal}(F/\mathbb{Q}) \cong \mathbb{F}_p^\times$.

²It is worth noting that $M \simeq N$ does not imply $N \simeq M$ unless M, N are both finitely-generated Λ -torsion Iwasawa modules.

We furthermore define Λ' as $\hat{\mathbb{Z}}_p[[\Gamma']]$. We have an isomorphism of topological $\hat{\mathbb{Z}}_p$ -algebras $\Lambda' \cong \Lambda[\Delta]$. Using the product decomposition $\Gamma' \cong \Gamma \times \Delta$, it is natural to think of Γ as the group of principal units $1 + p\hat{\mathbb{Z}}_p \cong \hat{\mathbb{Z}}_p^\times$, and of Δ as the $(p-1)$ st roots of unity, both sitting inside $\hat{\mathbb{Z}}_p^\times$.

The theory of Iwasawa modules can be extended very simply to Λ' -modules in the following way. There exist idempotent elements $\epsilon_0, \epsilon_1, \dots, \epsilon_{p-2} \in \hat{\mathbb{Z}}_p[\Delta]$ of the form

$$\epsilon_j = \frac{1}{p-1} \sum_{a=1}^{p-1} \omega^{-j}(\sigma_a) \sigma_a.$$

For $a = 1, 2, \dots, p-1$, $\sigma_a \in \Delta$ sends ζ_p to ζ_p^a , and $\omega : \Delta \rightarrow \hat{\mathbb{Z}}_p^\times$ denotes the *Teichmüller character*, given by letting $\omega(\sigma_a)$ be the unique $(p-1)$ st root of unity in $\hat{\mathbb{Z}}_p$ such that $\omega(\sigma_a) \equiv a \pmod{p}$.

The idempotents $\epsilon_0, \dots, \epsilon_{p-2}$ yield a splitting $\Lambda' \cong \epsilon_0 \Lambda' \times \dots \times \epsilon_{p-2} \Lambda'$, in which each factor $\epsilon_j \Lambda'$ is isomorphic, as a topological ring, to Λ . Hence, for any Λ' -module M , we have a canonical splitting $M \cong \bigoplus_{j=0}^{p-2} \epsilon_j M$ where each eigenspace $\epsilon_j M$ is an Iwasawa module. Since X_∞ is, in fact, a Λ' -module, we can consider its eigenspaces $\epsilon_j X_\infty$ and the associated characteristic polynomials $f^{\epsilon_j X_\infty}(T)$.

In [13], Kubota and Leopoldt analytically constructed p -adic L -functions which interpolate the special values of Dirichlet L -functions. Specifically, if χ is a Dirichlet character, then there exists a p -adic L -function $L_p(s, \chi)$, with s taking values in the domain $\hat{\mathbb{Z}}_p$, such that

$$L_p(1-n, \chi) = (1 - \chi \omega^{-n}(p) p^{n-1}) \cdot L(1-n, \chi \omega^{-n})$$

for all $n \geq 1$. The essential point here is that the Kubota–Leopoldt p -adic L -function is *constructed by means of p -adic analysis*, and it is constructed in such a way that its special values are known in advance to be meaningful.

Iwasawa [12] constructed p -adic L -functions of Dirichlet characters by an entirely different method, exploiting the isomorphism $\hat{\mathbb{Z}}_p[[\Gamma]] \cong \hat{\mathbb{Z}}_p[[T]]$ to great effect. Iwasawa constructed a power series $f(T, \omega^{1-j}) \in \hat{\mathbb{Z}}_p[[T]]$ such that, after a change of variable,

$$f((1+p)^s - 1, \omega^{1-j}) = L_p(s, \omega^{1-j})$$

and he famously made the following conjecture which was later proved by Mazur and Wiles [14].

Theorem 2.3 (Iwasawa Main Conjecture). *For $j = 3, 5, \dots, p-2$, the following is an equality of ideals in Λ .*

$$(f^{\epsilon_j X_\infty}(T)) = (f(T, \omega^{1-j})) \tag{4}$$

The left-hand side of (4) is an *algebraic p -adic L -function*, i.e., a p -adic L -function extracted in a simple way from an Iwasawa module, but whose special values at integers are *a priori* of uncertain significance. The right-hand side of (4) is an *analytic p -adic L -function* (like those constructed by Kubota–Leopoldt), with *a priori* meaningful special values at negative integers, but of completely uncertain connection to algebraic constructions like cohomology or Iwasawa modules. The Iwasawa Main Conjecture asserts that, up to multiplication by a unit in the power series ring $\hat{\mathbb{Z}}_p[[T]]$, the algebraic p -adic L -function coincides with the analytic p -adic L -function.

The equality (4) yields that

$$f^{\epsilon_j X_\infty}((1+p)^{1-n} - 1) \sim_p L(1-n, \omega^{1-j-n})$$

for all $n \geq 1$. The moral is that, as a consequence of the Iwasawa Main Conjecture,

The p -adic valuations of the special values of the characteristic polynomial $f^{\epsilon_j X_\infty}$ agree with the p -adic valuations of the special values of a classical Dirichlet L -function. \tag{5}

We will refer to statement (5) as the *weak form of the Main Conjecture*. It is this weak form for which we will be able to formulate a spectral analogue, in Theorem 3.10.

2.2 The p -adic complex K -theory of a spectrum yields a $2(p-1)$ -tuple of Iwasawa modules

Let KU denote the periodic complex K -theory spectrum, and KU_p^\wedge its p -adic completion. For a spectrum X , the splitting of the Λ' -module $(KU_p^\wedge)^0(X)$ as a direct sum $\epsilon_0(KU_p^\wedge)^0(X) \oplus \dots \oplus \epsilon_{p-2}(KU_p^\wedge)^0(X)$ is well-known in topology, dating back to Adams; our understanding is that [1] is the original reference. This splitting is described in explicitly Iwasawa-theoretic terms in [15] and in [11]. Similarly, in odd degrees, we have the $(p-1)$ -tuple of Iwasawa modules $\epsilon_0(KU_p^\wedge)^{-1}(X) \oplus \dots \oplus \epsilon_{p-2}(KU_p^\wedge)^{-1}(X)$. By Bott periodicity, we do not need to work with the p -adic K -groups in all degrees: it suffices to consider the $2(p-1)$ -tuple of Iwasawa modules coming from any single even degree and any single odd degree. In this paper we choose to work with $(KU_p^\wedge)^0$ and $(KU_p^\wedge)^{-1}$.

2.3 Review of Hahn–Mitchell’s results

In [11], Rebekah Hahn and Steve Mitchell developed some basic ideas and results in spectral Iwasawa theory. We review the most relevant ideas from their paper.

Definition 2.4. A map of $K(1)$ -local spectra $X \rightarrow Y$ is a pseudo-equivalence if the induced maps in p -adic K -theory $(KU_p^\wedge)^0(Y) \rightarrow (KU_p^\wedge)^0(X)$ and $(KU_p^\wedge)^{-1}(Y) \rightarrow (KU_p^\wedge)^{-1}(X)$ each have finite kernel and finite cokernel.

Hence a map of $K(1)$ -local spectra $X \rightarrow Y$ is a pseudo-equivalence if and only if its induced map of Iwasawa modules $\epsilon_j(KU_p^\wedge)^j(Y) \rightarrow \epsilon_j(KU_p^\wedge)^j(X)$ is a pseudo-isomorphism for all i, j —or equivalently, a pseudo-isomorphism for all $i \in \{0, -1\}$ and all $j \in \{0, 1, \dots, p-2\}$.

Definition 2.5. An Iwasawa module M is elementary cyclic if M is isomorphic to either Λ , or Λ/p^i for some positive integer i , or Λ/f^i for some positive integer i and some irreducible distinguished polynomial f .

An Iwasawa module M is elementary if it is a direct sum of elementary cyclic Iwasawa modules.

A $K(1)$ -local spectrum X is elementary if the Iwasawa module $\epsilon_j(KU_p^\wedge)^j(X)$ is elementary for all i, j .

Recall that we have a single fixed odd prime p throughout this paper. Consequently it is safe to use the terms defined in Definitions 2.4 and 2.5 more broadly, by saying that:

- a spectrum X (not necessarily $K(1)$ -local) is *elementary* if its $K(1)$ -localization is elementary.
- A map of (not necessarily $K(1)$ -local) spectra $X \rightarrow Y$ is a *pseudo-equivalence* if the induced maps $(KU_p^\wedge)^0(Y) \rightarrow (KU_p^\wedge)^0(X)$ and $(KU_p^\wedge)^{-1}(Y) \rightarrow (KU_p^\wedge)^{-1}(X)$ each have finite kernel and finite cokernel.
- Spectra X, Y are *pseudo-equivalent* if there is a zigzag of pseudo-equivalences connecting X and Y .

Theorem 2.6. ([11, Theorem 9.3]) Let X be a $K(1)$ -local spectrum such that the Λ' -modules $(KU_p^\wedge)^0(X)$ and $(KU_p^\wedge)^{-1}(X)$ are each finitely generated. Then there exist $K(1)$ -local spectra Y and Z , and pseudo-equivalences

$$X \xrightarrow{\sim} Y \xleftarrow{\sim} Z,$$

with Z elementary.

One has to be a bit careful when writing something like “ X and Y are pseudo-equivalent,” since if there exists a pseudo-equivalence $X \rightarrow Y$, there does not necessarily exist a pseudo-equivalence $Y \rightarrow X$.

Under slightly stronger hypotheses, there is a convenient particular case of Theorem 2.6 in which Hahn–Mitchell get a slightly stronger result:

Theorem 2.7. ([11, Theorem 9.6]) Let X be a $K(1)$ -local spectrum such that the Λ' -modules $(KU_p^\wedge)^0(X)$ and $(KU_p^\wedge)^{-1}(X)$ are each finitely generated, and such that each of the Λ -modules $\epsilon_j(KU_p^\wedge)^j(X)$ is Λ -torsion for all i, j . Then there exists an elementary $K(1)$ -local spectrum \tilde{X} and pseudo-equivalences

$$X \xrightarrow{\sim} \tilde{X} \quad \text{and} \quad \tilde{X} \xrightarrow{\sim} X.$$

We will refer to \tilde{X} as the *Hahn–Mitchell replacement* for X .

In Theorem 2.7, the phrase “for all i, j ” can be safely replaced by “for all $i \in \{0, -1\}$ and all $j \in \{0, 1, \dots, p-2\}$ ”.

3 The characteristic polynomials and Iwasawa μ -invariants of finite spectra

3.1 KU_p^\wedge of the Spheres

As a warm-up exercise, we let X be an even-dimensional sphere S^{2i} , and we consider the Iwasawa-theoretic properties of the Iwasawa modules $\epsilon_0(KU_p^\wedge)^*(X), \dots, \epsilon_{p-2}(KU_p^\wedge)^*(X)$. We have the isomorphism $(KU_p^\wedge)^0(S^{2i}) \cong \hat{\mathbb{Z}}_p$ for all $i \in \mathbb{Z}$, and $\Gamma' \cong \hat{\mathbb{Z}}_p^\times$ acts on $(KU_p^\wedge)^0(S^{2i})$ via the stable Adams operations, i.e. for all $u \in \hat{\mathbb{Z}}_p^\times$ and all $x \in (KU_p^\wedge)^0(S^{2i})$,

$$u \cdot x = u^i x.$$

Among the $p-1$ eigenspaces $\epsilon_j(KU_p^\wedge)^0(S^{2i})$ of $(KU_p^\wedge)^0(S^{2i})$, it turns out that only one is nontrivial:

Proposition 3.1. $\epsilon_j(KU_p^\wedge)^0(S^{2i}) \cong \begin{cases} (KU_p^\wedge)^0(S^{2i}) & \text{if } i \equiv j \pmod{p-1} \\ 0 & \text{otherwise} \end{cases}$

Proof. We can think of Δ as sitting inside $\hat{\mathbb{Z}}^\times$ via the Teichmüller character which means that $\sigma_a \in \Delta$ acts on $(KU_p^\wedge)^0(S^{2i})$ by multiplication by $\omega^i(\sigma_a)$. So, for $1 \in (KU_p^\wedge)^0(S^{2i})$, we have

$$\begin{aligned} \epsilon_j \cdot 1 &= \frac{1}{p-1} \sum_{a=1}^{p-1} \omega^{-j}(\sigma_a) \sigma_a \cdot 1 \\ &= \frac{1}{p-1} \sum_{a=1}^{p-1} \omega^{i-j}(\sigma_a) \\ &= \begin{cases} 1 & \text{if } i \equiv j \pmod{p-1} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

This mean that $\epsilon_j(KU_p^\wedge)^0(S^{2i})$ is a finitely generated Λ -torsion Iwasawa module such that $\mu(\epsilon_j(KU_p^\wedge)^0(S^{2i})) = 0$ for all j , and it is nontrivial when $j \equiv i \pmod{p-1}$. We now restrict our attention to the nontrivial case $j \equiv i \pmod{p-1}$. The characteristic polynomial of an Iwasawa module essentially records the action of a topological generator in $\Gamma \cong 1 + p\hat{\mathbb{Z}}_p$, so we can construct the characteristic polynomial of $\epsilon_j(KU_p^\wedge)^0(S^{2i})$, which we will denote as $f_{i,j}(T)$ for ease of notation, in the following way. Recall that there exists an isomorphism $\hat{\mathbb{Z}}_p[[1 + p\hat{\mathbb{Z}}_p]] \cong \hat{\mathbb{Z}}_p[[T]]$ which is defined by sending the topological generator $1 + p \in 1 + p\hat{\mathbb{Z}}_p$ to $1 + T \in \hat{\mathbb{Z}}_p[[T]]$. Hence, in $\hat{\mathbb{Z}}_p[[T]]$, we identify $1 + T$ and $(1 + p)^i$ and conclude that the characteristic polynomial of $\epsilon_j(KU_p^\wedge)^0(S^{2i})$ is

$$f_{i,j}(T) = \begin{cases} T - (1 + p)^j + 1 & \text{if } i \equiv j \pmod{p-1} \\ 1 & \text{otherwise} \end{cases}$$

Note that this implies that $\epsilon_j(KU_p^\wedge)^0(S^{2i}) \simeq \Lambda / f_{i,j}(T)$ and $\lambda(\epsilon_j(KU_p^\wedge)^0(S^{2i})) = 1$ in the nontrivial case. Otherwise, the λ -invariant is 0.

In the same vein as the weak form of the Main Conjecture, (5), the p -adic valuations of the special values of the characteristic polynomials $f_{i,j}(T)$ agree with the orders of the stable homotopy groups of the $K(1)$ -local sphere. Specifically, we get the following relationship.

Proposition 3.2. *For the characteristic polynomial $f_{i,j}$ of the Iwasawa module $\epsilon_j(KU_p^\wedge)^0(S^{2i})$, where $j = 0, 1, \dots, p-2$, we have*

$$f_{i,j}((1 + p)^{1-n} - 1) \sim_p |\pi_{2(n+i-1)-1} L_{K(1)} S^0|$$

for all $n \equiv 1 - j \pmod{p-1}$.

We do not write out a proof of Proposition 3.2, since it merely amounts to a rephrasing, in Iwasawa-theoretic terms, of the classical calculation (essentially [17, Theorem 8.10])

$$\pi_r L_{K(1)} S^0 = \begin{cases} \hat{\mathbb{Z}}_p & \text{if } t = -1, 0 \\ \mathbb{Z}/p^{k+1}\mathbb{Z} & \text{if } t = 2(p-1)p^k r - 1, r \not\equiv 0 \pmod{p} \\ 0 & \text{otherwise} \end{cases}$$

The stable homotopy groups of $L_{K(1)} S^0$ are trivial in degrees $\neq -1$ modulo $2(p-1)$, with the one exception of $\pi_0(L_{K(1)} S^0)$, which is isomorphic to $\hat{\mathbb{Z}}_p$. Hence Proposition 3.2 recover the orders of all the nontrivial stable homotopy groups of $L_{K(1)} S^0$ other than $\pi_0 L_{K(1)} S^0$.

There is one special case of Proposition 3.2 which requires a bit of explanation: if $n = 1 - i$, then 3.2 makes a claim about the order of $\pi_{-1}(L_{K(1)} S^0)$, which is the infinite group $\hat{\mathbb{Z}}_p$. Since $\hat{\mathbb{Z}}_p = \lim_m \mathbb{Z}/p^m \mathbb{Z}$, we formally treat the p -adic valuation of $|\hat{\mathbb{Z}}_p|$ as the limit

$$v_p(|\hat{\mathbb{Z}}_p|) = \lim_m v_p(|\mathbb{Z}/p^m \mathbb{Z}|) = \infty,$$

which is consistent with the claim $0 \sim_p |\hat{\mathbb{Z}}_p|$ made by Proposition 3.2.

Remark 3.3. The Λ' -module $(KU_p^\wedge)^0(S^{2i})$ also arises number-theoretically. Consider the cyclotomic $\hat{\mathbb{Z}}_p$ -extension F_∞/F where $F_n = \mathbb{Q}(\zeta_{p^n})$. If we let R_n denote the group of roots of unity sitting inside the p -adic completion of $\mathcal{O}_{F_n}^\times$, then $R_n = \langle \zeta_{p^n} \rangle \cong \mathbb{Z}/p^n \mathbb{Z}$. We can arrange these groups into a tower by considering the norm maps $N : R_{n+1} \rightarrow R_n$. The reader can check that $N(\zeta_{p^{n+1}}) = \zeta_{p^n}$. Hence we have that $R = \lim_n R_n \cong \hat{\mathbb{Z}}_p$. Each $(\mathbb{Z}/p^n \mathbb{Z})^\times \cong \text{Gal}(F_n/\mathbb{Q})$ acts on $\mathbb{Z}/p^n \mathbb{Z} \cong R_n$ by left multiplication, and, in the limit, $\hat{\mathbb{Z}}_p^\times \cong \lim_n \text{Gal}(F_n/\mathbb{Q})$ acts on $\hat{\mathbb{Z}}_p \cong R$ by left multiplication. Therefore, we get the Λ' -module isomorphism $R \cong (KU_p^\wedge)^0(S^2)$.

Suppose we “twist” the Galois action of $(\mathbb{Z}/p^n \mathbb{Z})^\times$ on $\mathbb{Z}/p^n \mathbb{Z} \cong R_n$ so that, for $a \in (\mathbb{Z}/p^n \mathbb{Z})^\times$ and $x \in \mathbb{Z}/p^n \mathbb{Z}$, $a \cdot x = a^i x$. Denote the resulting Λ' -module as $R(i)$. Then $R(i) \cong (KU_p^\wedge)^0(S^{2i})$.

3.2 The μ -invariants of finite spectra.

There is little to say about μ -invariants of finite spectra: they are all zero, for easy reasons.

Theorem 3.4. *Let X be a finite spectrum. Then, for every i, j , the μ -invariant of each of the Iwasawa modules $\epsilon_j(KU_p^\wedge)^i(X)$ is zero.*

Proof. If the μ -invariant of $\epsilon_j(KU_p^\wedge)^i(X)$ were nonzero, then $\epsilon_j(KU_p^\wedge)^i(X)$ would be pseudo-isomorphic to an Iwasawa module with $\Lambda/p^k \cong \mathbb{Z}/p^k\mathbb{Z}[[T]]$ as a summand, for some positive integer k . Hence Λ/p^k , an Iwasawa module with uncountably many linearly independent p -torsion elements, would occur as a summand in that Iwasawa module. A pseudo-isomorphism could only eliminate finitely many of these elements, after which $\epsilon_j(KU_p^\wedge)^i(X) \subset (KU_p^\wedge)^i(X)$ would still have uncountably many linearly independent p -torsion elements. This is impossible for a finite spectrum. Hence the μ -invariant of $\epsilon_j(KU_p^\wedge)^i(X)$ must have been zero. \square

3.3 The characteristic polynomials and Iwasawa main conjecture for finite spectra

We begin with an easy lemma about pseudo-equivalences.

Lemma 3.5.

- A map of spectra $f : X \rightarrow Y$ with homotopy fiber $\text{fib } f$ is a pseudo-equivalence if and only if $(KU_p^\wedge)^i(\text{fib } f)$ is finite for all i .
- If X, Y are finite spectra and $f : X \rightarrow Y$ is a pseudo-equivalence, then f induces an isomorphism in rational homotopy.

Proof.

- Elementary from the long exact sequence induced in p -adic K -theory by the homotopy fiber sequence $\text{fib } f \rightarrow X \xrightarrow{f} Y$.
- Suppose, by contrapositive, that $H\mathbb{Q} \wedge f : H\mathbb{Q} \wedge X \rightarrow H\mathbb{Q} \wedge Y$ is not an equivalence. Then the Spanier-Whitehead dual of f , Df , would also have the property that $H\mathbb{Q} \wedge Df$ is not an equivalence. Hence $KU_p^\wedge \wedge H\mathbb{Q} \wedge Df$ would also not be an equivalence, since $KU_p^\wedge \wedge H\mathbb{Q}$ splits as a wedge of copies of suspensions of $H\mathbb{Q}$. Now by the chain of isomorphisms

$$\begin{aligned}
(KU_p^\wedge)^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong (KU_p^\wedge)^i(X)[p^{-1}] & (6) \\
&\cong \text{colim}_p (KU_p^\wedge)^i(X) \\
&\cong \pi_{-i}(\text{hocolim}_p F(X, KU_p^\wedge)) \\
&\cong \pi_{-i}(F(X, \text{hocolim}_p KU_p^\wedge)) \\
&\cong \pi_{-i}(F(X, H\mathbb{Q} \wedge KU_p^\wedge)) \\
&\cong \pi_{-i}(F(X, S^0) \wedge H\mathbb{Q} \wedge KU_p^\wedge) \\
&\cong (KU_p^\wedge)_{-i}(DX \wedge H\mathbb{Q}), & (7)
\end{aligned}$$

the map f does not induce an isomorphism in rationalized p -adic K -theory. Hence the map induced by f in p -adic K -theory must have infinite kernel or infinite cokernel in at least one degree, i.e., f is not a pseudo-equivalence. \square

Theorem 3.6. *Let X be a finite spectrum. Then X is pseudo-equivalent to a canonical-up-to-homotopy wedge sum*

$$(X_{\text{even}})_{\text{torsfree}} \vee (X_{\text{odd}})_{\text{torsfree}}$$

of finite spectra, such that $H_((X_{\text{even}})_{\text{torsfree}}; \mathbb{Z}_{(p)})$ is torsion-free and concentrated in even degrees, and $H_*((X_{\text{odd}})_{\text{torsfree}}; \mathbb{Z}_{(p)})$ is torsion-free and concentrated in odd degrees.*

Proof. A size argument analogous to that used to prove Theorem 3.4 shows that, for all i, j , the Iwasawa module $\epsilon_j(KU_p^\wedge)^i(X)$ is finitely generated, and must have rank zero, hence must also be torsion. Theorem 2.7 then tells us that X is pseudo-equivalent to its Hahn–Mitchell replacement \tilde{X} , whose p -adic K -theory admits an isomorphism

$$(KU_p^\wedge)^i(\tilde{X}) \cong \bigoplus_{j=0}^{p-2} \epsilon_j(KU_p^\wedge)^i(\tilde{X})$$

for each i , such that $\epsilon_j(KU_p^\wedge)^i(\tilde{X})$ is a direct sum of finitely many Λ -modules of the form $\Lambda/f(T)^m$ with $f(T)$ an irreducible distinguished polynomial, and m a positive integer. Our point is simply that $(KU_p^\wedge)^*(\tilde{X})$ is p -torsion-free.

Hahn–Mitchell use Bousfield’s algebraicity results from [3] to construct \tilde{X} . Among other results in [3], Bousfield shows that every graded $(KU_{(p)})_*(KU_{(p)})$ -comodule M splits into the direct sum of a summand M_{even} concentrated in even degrees, and a summand M_{odd} concentrated in odd degrees, and for each of these summands, if its injective dimension in the category of $(KU_{(p)})_*(KU_{(p)})$ -comodules is ≤ 1 , then that summand is realizable as the $(KU_{(p)})$ -homology of a spectrum, and this spectrum is unique up to $KU_{(p)}$ -local equivalence. Bousfield also proves a finiteness result [3, Theorem 9.7]: if M_{even} is finitely generated in each degree as a $\mathbb{Z}_{(p)}$ -module, then one can find a finite spectrum whose $KU_{(p)}$ -homology is M_{even} ; and similarly for M_{odd} .

Analogous results hold for KU_p^\wedge , and are used by Hahn–Mitchell to prove the existence of \tilde{X} . There is no difficulty with switching between p -adic K -homology and p -adic K -cohomology in this argument, since X is finite, so we may take Spanier–Whitehead duals to exchange the K -homology and K -cohomology whenever convenient. Consequently \tilde{X} splits as a wedge $\tilde{X}_{\text{even}} \vee \tilde{X}_{\text{odd}}$, with \tilde{X}_{even} and \tilde{X}_{odd} each finite spectra, with the p -adic K -theory of \tilde{X} concentrated in even degrees, and with the p -adic K -theory of \tilde{X} concentrated in odd degrees.

By Theorem 2.7, \tilde{X} has torsion-free p -adic K -theory, and hence \tilde{X}_{even} and \tilde{X}_{odd} each have torsion-free p -adic K -theory. We are not quite done, since we do not know that \tilde{X}_{even} and \tilde{X}_{odd} have torsion-free p -local homology. Torsion-free homology implies torsion-free K -theory, but the converse is not true: the Smith–Toda complex $V(1)$ is a counterexample.

We will now modify \tilde{X}_{even} and \tilde{X}_{odd} each by a pseudo-equivalence to produce finite spectra whose p -local homology is torsion-free. Here is our construction, which uses only very classical tools. Let Y be a finite spectrum. Then there exists a finite sequence of maps of spectra

$$\text{pt.} \longrightarrow Y^a \longrightarrow Y^{a+1} \longrightarrow \dots \longrightarrow Y^b \quad (8)$$

such that $Y^b = Y$ and such that the cofiber of the map $Y^{j-1} \rightarrow Y^j$ is a p -local Moore spectrum³ of type $(H_j(Y; \mathbb{Z}_{(p)}), j)$. Such a sequence is called a “homology decomposition,” and in the unstable case, dates back to the 1959 papers [4] and [8]; the method used in those papers also works to produce the stable construction in spectra.

Now we begin an inductive construction. We start in the bottom degree of the homology decomposition (8): write $Y((a))$ for Y with the torsion summand $M(\text{tors } H_a(Y; \mathbb{Z}_{(p)}), a)$ pinched off. That is, $Y((a))$ is the homotopy cofiber of the composite map $M(\text{tors } H_a(Y; \mathbb{Z}_{(p)}), a) \rightarrow Y^a \rightarrow Y$. The map $Y \rightarrow Y((a))$ is a pseudo-equivalence by the first part of Lemma 3.5, and the homology $H_*(Y((a)); \mathbb{Z}_{(p)})$ is torsion-free in degrees $\leq a$.

That was the first step in an induction. Here is the inductive hypothesis: suppose that j is some integer, and suppose we have already constructed a sequence of pseudo-equivalences of finite spectra

$$Y \rightarrow Y((a)) \rightarrow Y((a+1)) \rightarrow \dots \rightarrow Y((j))$$

such that, for each $i \leq j$, the homology $H_*(Y((i)); \mathbb{Z}_{(p)})$ is torsion-free in degrees $\leq i$. Choose a homology decomposition

$$\dots \rightarrow Y((j))^{j-1} \rightarrow Y((j))^j \rightarrow Y((j))^{j+1} \rightarrow \dots \quad (9)$$

for $Y((j))$. The homology group $H_{j+1}(Y((j)); \mathbb{Z}_{(p)})$ decomposes, as a $\mathbb{Z}_{(p)}$ -module, into the direct sum of a torsion summand and a torsion-free summand. This induces a wedge splitting of the Moore spectrum $M = Y((j))^{j+1} / Y((j))^j$ as

$$M(\text{tors } H_{j+1}(Y; \mathbb{Z}_{(p)}), j+1) \vee M(\text{torsfree } H_{j+1}(Y; \mathbb{Z}_{(p)}), j+1).$$

Write f for the composite map

$$\Sigma^{-1}M(\text{tors } H_{j+1}(Y; \mathbb{Z}_{(p)}), j+1) \hookrightarrow \Sigma^{-1}(M(\text{tors } H_{j+1}(Y; \mathbb{Z}_{(p)}), j+1) \vee M(\text{torsfree } H_{j+1}(Y; \mathbb{Z}_{(p)}), j+1)) \rightarrow Y((j))^j \rightarrow Y((j)).$$

Let $Y((j+1))$ denote the homotopy cofiber of the map f . Then, from analysis of the long exact sequences induced in homology by the homotopy cofiber sequence

$$\Sigma^{-1}M(\text{tors } H_{j+1}(Y; \mathbb{Z}_{(p)}), j+1) \rightarrow Y((j)) \rightarrow Y((j+1)),$$

we get that the p -local homology groups of $Y((j+1))'$ agree with those of $Y((j))$ in degrees $\neq j+1$, hence are torsion-free in degrees $\leq j$, and furthermore $H_{j+1}(Y((j+1)); \mathbb{Z}_{(p)})$ is torsion-free. The map $Y((j)) \rightarrow Y((j+1))$ is a pseudo-equivalence by the first part of Lemma 3.5. This completes the inductive step.

Since Y is finite, its homology is trivial except in finitely many degrees, so after finitely many steps in the induction, we have killed off all the torsion in $H_*(-; \mathbb{Z}_{(p)})$. Hence the homotopy colimit $\text{hocolim}_j Y((j))$ is in fact reached after finitely many steps. We define Y_{torsfree} to be that homotopy colimit, so that $Y \rightarrow Y_{\text{torsfree}}$ is a pseudo-equivalence. At each stage in the induction, we have attached finitely many cells to Y , hence Y_{torsfree} is finite, as desired.

There is one last claim to explain: by construction, $(X_{\text{even}})_{\text{torsfree}}$ has torsion-free p -local homology, and has p -adic K -theory concentrated in even degrees. We need to show that $H_*((X_{\text{even}})_{\text{torsfree}}; \mathbb{Z}_{(p)})$ is also concentrated in even degrees. By a theorem of Dold (see 14.18 of [6] or Corollary 2.6 of [7]), the shortest nonzero differential in the Atiyah–Hirzebruch spectral sequence $H_*((X_{\text{even}})_{\text{torsfree}}; (KU_p^\wedge)_*) \Rightarrow (KU_p^\wedge)_*((X_{\text{even}})_{\text{torsfree}})$ is torsion-valued. Since $(KU_p^\wedge)_*$ is torsion-free and $(X_{\text{even}})_{\text{torsfree}}$ has torsion-free p -local homology, this Atiyah–Hirzebruch spectral sequence collapses immediately. Hence, if $(X_{\text{even}})_{\text{torsfree}}$ had any nonvanishing p -local homology in odd degrees, it would also have nonvanishing p -adic K -homology in some odd degree. This is impossible by the following argument: by the universal coefficient sequence for the p -adic K -theory of a finite CW-complex ([2], [23], [9, Theorem IV.4.5]):

$$0 \rightarrow \text{Ext}_{\mathbb{Z}_p}^1((KU_p^\wedge)_{n-1}((X_{\text{even}})_{\text{torsfree}}), \hat{\mathbb{Z}}_p) \rightarrow (KU_p^\wedge)^n((X_{\text{even}})_{\text{torsfree}}) \rightarrow \text{hom}_{\hat{\mathbb{Z}}_p}((KU_p^\wedge)_n((X_{\text{even}})_{\text{torsfree}}), \hat{\mathbb{Z}}_p) \rightarrow 0,$$

the torsion-freeness of $(KU_p^\wedge)^*((X_{\text{even}})_{\text{torsfree}})$ and the vanishing of $(KU_p^\wedge)^*((X_{\text{even}})_{\text{torsfree}})$ in odd degrees implies that $(KU_p^\wedge)_*((X_{\text{even}})_{\text{torsfree}})$ also vanishes in odd degrees. We conclude that $(X_{\text{even}})_{\text{torsfree}}$ must have had trivial p -local homology in all odd degrees. A completely similar argument shows that $(X_{\text{odd}})_{\text{torsfree}}$ must have had trivial p -local homology in all even degrees. \square

³By a p -local Moore spectrum of type (A, n) , we mean a spectrum M such that $H_j(M, \mathbb{Z}_{(p)})$ is trivial if $j \neq n$, and is isomorphic to A if $j = n$.

Definition 3.7. Given a finite spectrum X , we write $\overset{\circ}{X}$ as an abbreviation for the spectrum $(X_{\text{even}})_{\text{torsfree}} \vee (X_{\text{odd}})_{\text{torsfree}}$ constructed in Theorem 3.6. We call $\overset{\circ}{X}$ the torsion-free replacement of X .

Theorem 3.8. Let X be a finite spectrum. Then, for each i, j , the λ -invariant, μ -invariant, and characteristic polynomial of the Iwasawa module $\epsilon_j(KU_p^\wedge)^i(X)$ depend only on the rational homotopy type of X .

Proof. The characteristic polynomial of an Iwasawa module is a pseudo-isomorphism invariant, so X has the same Iwasawa modules as the wedge of finite spectra $(X_{\text{even}})_{\text{torsfree}} \vee (X_{\text{odd}})_{\text{torsfree}}$ constructed in Theorem 3.6. Since X_{even} (respectively, X_{odd}) has its p -local homology concentrated in even (respectively, odd) degrees, the Atiyah-Hirzebruch spectral sequence implies that its p -adic K -theory is also concentrated in even (respectively, odd) degrees.

Hence, for each i, j , the Iwasawa module $\epsilon_j(KU_p^\wedge)^{2i}(X)$ is pseudo-isomorphic to $\epsilon_j(KU_p^\wedge)^{2i}((X_{\text{even}})_{\text{torsfree}})$. Since $(X_{\text{even}})_{\text{torsfree}}$ has torsion-free p -local homology, it also has torsion-free p -adic K -theory by Dold's Atiyah-Hirzebruch argument (mentioned already in the proof of Theorem 3.6). Hence the p -adic K -theory of $(X_{\text{even}})_{\text{torsfree}}$ embeds into its rationalization. The Adams operations on the rationalized K -theory of a finite spectrum depend only on the rationalization of the spectrum; this is an old observation, and easily provable by the chain of isomorphisms (6) through (7) together with the fact that $H\mathbb{Q} \wedge DX$ splits as a wedge of suspensions of $H\mathbb{Q}$. Hence the characteristic polynomial of each Iwasawa module $\epsilon_j(KU_p^\wedge)^{2i}(X)$ is determined by the rational homotopy type of $(X_{\text{even}})_{\text{torsfree}}$.

A completely analogous argument shows that the characteristic polynomial of each Iwasawa module $\epsilon_j(KU_p^\wedge)^{2i+1}(X)$ is determined by the rational homotopy type of $(X_{\text{odd}})_{\text{torsfree}}$. Hence the characteristic polynomials of the Iwasawa modules of X are determined by the rational homotopy type of $(X_{\text{even}})_{\text{torsfree}} \vee (X_{\text{odd}})_{\text{torsfree}}$. The second part of Lemma 3.5 then gives us that the the characteristic polynomials of the Iwasawa modules of X are determined by the rational homotopy type of X itself. \square

Lemma 3.9. Let X be a finite spectrum such that $H_*(X; \mathbb{Z}_{(p)})$ is torsion-free and concentrated in even degrees. Let a and b be the least and greatest integers i , respectively, such that $H_{2i}(X; \mathbb{Q})$ is nontrivial. Then, for each $j = 0, 1, \dots, p-2$, we have

$$f^{\epsilon_j(KU_p^\wedge)^0(X)}((1+p)^{1-n} - 1) \sim_p |\pi_{2(n-1)-1} L_{K(1)} DX|$$

for all $n \equiv 1 - j \pmod{p-1}$ and $n \notin (1-b, 1-a]$.

Proof. To prove this, we will need two ingredients: Proposition 3.2 and the Atiyah-Hirzebruch spectral sequence

$$\begin{aligned} E_2^{s,t} &\cong H^s(X, \pi_{-t} L_{K(1)} S^0) \implies \pi_{-s-t}(DX \wedge L_{K(1)} S^0) \\ d_r : E_r^{s,t} &\rightarrow E_r^{s+r, t-r+1}. \end{aligned} \quad (10)$$

As a consequence of Proposition 3.2, we can relate the special values $f^{\epsilon_j(KU_p^\wedge)^0(X)}((1+p)^{1-n} - 1)$ to the product of the orders of certain bidegrees in the E_2 -term of (10), as follows. Since X is assumed to have torsion-free p -local homology, the universal coefficient theorem yields

$$\begin{aligned} H^{2i}(X; \pi_{-t} L_{K(1)} S^0) &\cong H_{2i}(X; \pi_{-t} L_{K(1)} S^0) \\ &\cong (\pi_{-t} L_{K(1)} S^0)^{r_i}, \end{aligned}$$

where $r_i = \text{rank}(H_{2i}(X; \mathbb{Z}_{(p)}))$. So, for each factor $f_{i,j}(T)^{r_i}$ of $f^{\epsilon_j(KU_p^\wedge)^0(X)}(T)$ and each integer n congruent to $1 - j$ modulo $p-1$, we have

$$\begin{aligned} f_{i,j}((1+p)^{1-n} - 1)^{r_i} &\sim_p |\pi_{2(n+i-1)-1} L_{K(1)} S^0|^{r_i} \\ &\sim_p |E_2^{2i, 1-2(n+i-1)}| \end{aligned}$$

Hence,

$$f^{\epsilon_j(KU_p^\wedge)^0(X)}((1+p)^{1-n} - 1) \sim_p \prod_{i \equiv j(p-1)} |E_2^{2i, 1-2(n+i-1)}| \quad (11)$$

We still need to show that the product on the right-hand side of (11) is, in fact, $|\pi_{2(n-1)-1} L_{K(1)} DX|$ when $n \notin (1-b, 1-a]$. Because $H_*(X; \mathbb{Z}_{(p)})$ is concentrated in even degrees and because $E_2^{s,t} \cong H^s(X; \pi_{-t} L_{K(1)} S^0)$, the E_2 -term of (10) is concentrated in bidegrees in which s is even and t is either 0 or of the form $1 - 2m(p-1)$ for $m \in \mathbb{Z}$. For degree reasons, this means that the domain or codomain of any nonzero differential must be on the $t = 0$ line. This, coupled with the fact that $H_*(X; \mathbb{Z}_{(p)})$ is concentrated between degrees $2a$ and $2b$, gives us that $E_2^{s,t}$ will not be hit by a nonzero differential when $s+t < 2a$ or $s+t > 2b$. Thus, when $m < 2a$ or $m > 2b$,

$$|\pi_{-m}(DX \wedge L_{K(1)} S^0)| = \prod_{s+t=m} |E_2^{s,t}|$$

A quick calculation shows that, more specifically, when $n \equiv 1 - j \pmod{p-1}$ and $n \notin (1-b, 1-a]$,

$$|\pi_{2(n-1)-1}(DX \wedge L_{K(1)} S^0)| = \prod_{i \equiv j(p-1)} |E_2^{2i, 1-2(n+i-1)}| \quad (12)$$

Finally, we need to know that $DX \wedge L_{K(1)} S^0$ is weakly equivalent to $L_{K(1)} DX$. This is, of course, not true for arbitrary spectra X , since $K(1)$ -localization is not smashing. However,

- we have a natural transformation $\xi : - \wedge L_{K(1)} S^0 \rightarrow L_{K(1)}-$,
- ξ is a weak equivalence when evaluated on a sphere,
- and both the domain and codomain of ξ are functors which send homotopy cofiber sequences to homotopy cofiber sequences.

Consequently ξ is a weak equivalence when evaluated on any *finite* spectrum. Hence, since X is assumed finite, (11) and (12) give us

$$f^{\epsilon_j(KU_p^\wedge)^0(X)}((1+p)^{1-n} - 1) \sim_p \left| \pi_{2(n-1)-1} L_{K(1)} DX \right|.$$

□

We are now in a position to prove that, for a finite spectrum X , the characteristic polynomials of the $2p - 2$ Iwasawa modules $\epsilon_0(KU_p^\wedge)^0(X), \dots, \epsilon_{p-2}(KU_p^\wedge)^0(X), \epsilon_0(KU_p^\wedge)^{-1}(X), \dots, \epsilon_{p-2}(KU_p^\wedge)^{-1}(X)$ of X determine the orders of the $K(1)$ -local homotopy groups of the torsion-free replacement $\overset{\circ}{X}$ of X . This is the content of Theorem 3.10. In particular, in the special case that X already has p -local homology which is torsion-free and concentrated in even degrees—like a complex projective space—Theorem 3.10 describes the orders of the $K(1)$ -local homotopy groups of X itself.

Theorem 3.10 (Weak form of Iwasawa Main Conjecture for spectra). *Let X be a finite spectrum. Let α and β be the least and greatest integers i , respectively, such that $H_i(X; \mathbb{Q})$ is nontrivial. Then we have*

$$\begin{aligned} \left| \pi_{2m-1} L_{K(1)} D\overset{\circ}{X} \right| &\sim_p f^{\epsilon_{-m}(KU_p^\wedge)^0(X)}((1+p)^{-m} - 1) \text{ and} \\ \left| \pi_{2m} L_{K(1)} D\overset{\circ}{X} \right| &\sim_p f^{\epsilon_{-m}(KU_p^\wedge)^{-1}(X)}((1+p)^{-m} - 1) \end{aligned}$$

for all integers m satisfying $m < \frac{\beta}{2}$, and also for all integers m satisfying $m > \frac{-\alpha}{2}$.

Proof. Corollary of Theorem 3.6 and Lemma 3.9. □

To be clear, in the statement of Theorem 3.10, the subscript $-m$ in ϵ_{-m} is to be understood as being defined modulo $p - 1$.

Remark 3.11. Here is a comment on the prospects for a *strong* form of the Iwasawa Main Conjecture for spectra, i.e., a theorem to the effect that the *algebraic* p -adic L -function given simply by the characteristic polynomial of $\epsilon_j(KU_p^\wedge)^i(X)$ generates the same ideal in $\hat{\mathbb{Z}}_p[[T]]$ as some *analytic* p -adic L -function, presumably constructed by p -adically interpolating some sequence of special values of some complex-analytic L -function. The paper [18] constructed, for each finite spectrum X with torsion-free homology concentrated in even degrees, a “provisional KU -local zeta-function” $\check{\zeta}_{KU}(s, X)$. The function $\check{\zeta}_{KU}(s, X)$ is a meromorphic function on the complex plane, and the denominators of its special values in a left-hand half-plane are proven in [18] to coincide with the orders of the KU -local stable homotopy groups of X .

All the zeta-functions and L -functions considered in [18] are products of Tate twists of L -functions of primitive Dirichlet characters. Consequently, by Kubota–Leopoldt [13], after removing the Euler factors at p , the special values at negative integers can be p -adically interpolated to yield a p -adic L -function. One might hope that this p -adic L -function is the correct “analytic side” of a spectral Iwasawa Main Conjecture. We do not expect this to be true in full generality, but we find it plausible for odd *regular* primes p . This is because, given an odd regular prime p and a finite spectrum X with torsion-free homology concentrated in even degrees, the p -adic valuations of the special values of the *algebraic* p -adic L -function to X are equal to p -adic valuations of the special values of the *analytic* p -adic L -function obtained by p -adic interpolation of $\check{\zeta}_{KU}(s, X)$, as a consequence of [18, Theorem 2.8] and Theorem 3.10, above.

4 The Iwasawa λ -invariants of finite spectra.

In section 2.1, we explained the precise sense in which the λ -invariant and μ -invariant control the asymptotic growth rate of the p -part of the class number, as one moves up a suitable tower of number fields.

Consider topological analogues of these results. We have already shown in Theorem 3.4 that the μ -invariants of finite spectra are all trivial, so we focus on the question of what numerical invariant of a finite spectrum X has its asymptotic growth rate described by the λ -invariants of the $2(p - 1)$ -tuple of Iwasawa modules

$$\epsilon_0(KU_p^\wedge)^0(X), \epsilon_1(KU_p^\wedge)^0(X), \dots, \epsilon_{p-2}(KU_p^\wedge)^0(X), \epsilon_0(KU_p^\wedge)^{-1}(X), \epsilon_1(KU_p^\wedge)^{-1}(X), \dots, \epsilon_{p-2}(KU_p^\wedge)^{-1}(X)$$

associated to X in section 2.2. In this section we prove that, just as the λ -invariant of the Iwasawa module of a $\hat{\mathbb{Z}}_p$ -extension of number fields controls the asymptotic growth rate of the p -parts of the class numbers, *the λ -invariants of the Iwasawa modules of a finite spectrum X control the asymptotic growth rate of the graded average order of the $K(1)$ -local homotopy groups of X .*

Our method of proof is centered on using additive invariants and algebraic K_0 of the category of finite spectra, since this approach is one that can potentially be generalized to higher-height analogues, using noncommutative Iwasawa theory, in the sense of [21] and [5]. The authors hope to pursue this idea in later work.

4.1 Asymptotic graded averages.

In this section we will study the average values of the orders of the $K(1)$ -local homotopy groups of finite spectra. Given a finite spectrum X , we would like to consider the average of the first n terms in the sequence

$$|\pi_1 L_{K(1)} X|, |\pi_2 L_{K(1)} X|, |\pi_3 L_{K(1)} X|, \dots \quad (13)$$

However, the average $\frac{1}{n} \sum_{j=1}^n |\pi_j L_{K(1)} X|$ is not necessarily defined: the trouble is that the sequence (13) may include finitely many terms which are infinite. Since we are interested in asymptotics, it is harmless to skip the first m terms, and consider the average $\frac{1}{n} \sum_{j=1+m}^{n+m} |\pi_j L_{K(1)} X|$ for $m \gg 0$.

There is another point that needs explaining. It will be extremely convenient to arrange for our asymptotic growth rates to be *additive* invariants of spectra, i.e., if $X \rightarrow Y \rightarrow Z$ is a cofiber sequence of spectra, then the sum of the growth rates for X and Z ought to be the growth rate for Y . In the special case $Y = 0$, we see that the growth rate for ΣX must then somehow be equal to -1 times the growth rate for X . The way to arrange this is to study the asymptotics of the *alternating* sum

$$\frac{1}{n} \sum_{j=1+m}^{n+m} (-1)^j |\pi_j L_{K(1)} X|$$

as $n \rightarrow \infty$, for $m \gg 0$. The heuristic is that we are studying the asymptotic growth rate of the average order of the $K(1)$ -local stable homotopy groups of X , but ‘‘average’’ must be understood as a *graded* average, in which the even-dimensional elements are counted as positive, while the odd-dimensional elements are counted as negative.

It turns out that the graded average order of n successive $K(1)$ -local homotopy groups of a finite spectrum X grows like $\log_p(n)/2$ times a constant. Furthermore, it turns out that this constant is determined by the λ -invariants of the Iwasawa modules of X . The main result establishing these facts is Theorem 4.6.

Definition 4.1. *Let α be a nonzero real number. Given a bi-infinite sequence $\dots, k_{-1}, k_0, k_1, k_2, \dots$, we will say that its graded average grows like $\alpha \cdot \log_p(n)$ if the limit*

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \sum_{j=1+m}^{n+m} (-1)^j k_j}{\alpha \cdot \log_p(n)}$$

exists and is equal to 1.

Proposition 4.2. *Let C be a triangulated category. Fix an object S of C , and write π_i for the functor $[\Sigma^i S, -] : C \rightarrow \text{Ab}$. Make the following assumptions:*

- $X \rightarrow Y \rightarrow Z$ in C is a cofiber sequence in C .
- α_X, α_Z are real numbers such that the graded average of $\dots, |\pi_{-1}(X)|, |\pi_0(X)|, |\pi_1(X)|, \dots$ grows like $\alpha_X \log_p(n)$, and the graded average of $\dots, |\pi_{-1}(Z)|, |\pi_0(Z)|, |\pi_1(Z)|, \dots$ grows like $\alpha_Z \log_p(n)$.
- The limit $\lim_{n \rightarrow \infty} \frac{|\pi_n(X)|}{n \log_p(n)}$ exists and is equal to zero.

Then the graded average of $\dots, |\pi_{-1}(Y)|, |\pi_0(Y)|, |\pi_1(Y)|, \dots$ grows like $(\alpha_X + \alpha_Z) \log_p(n)$.

Proof. We have the long exact sequence

$$\dots \xrightarrow{g_{n+1}} \pi_{n+1}(Z) \xrightarrow{h_{n+1}} \pi_n(X) \xrightarrow{f_n} \pi_n(Y) \xrightarrow{g_n} \pi_n(Z) \xrightarrow{h_n} \pi_{n-1}(X) \xrightarrow{f_{n-1}} \dots$$

and consequently equalities

$$\begin{aligned} \frac{1}{n(\alpha_X + \alpha_Z) \log_p(n)} \sum_{j=1+m}^{n+m} (-1)^j |\pi_j(Y)| &= \frac{1}{n(\alpha_X + \alpha_Z) \log_p(n)} \sum_{j=1+m}^{n+m} (-1)^j (|\text{im } f_n| + |\text{im } g_n|) \\ &= \frac{1}{n(\alpha_X + \alpha_Z) \log_p(n)} \sum_{j=1+m}^{n+m} (-1)^j (|\pi_j(X)| + |\pi_j(Z)| - |\text{im } h_{j+1}| - |\text{im } h_j|) \\ &= \frac{1}{n(\alpha_X + \alpha_Z) \log_p(n)} \left((-1)^m |\text{im } h_{1+m}| - (-1)^{m+n} |\text{im } h_{1+m+n}| + \sum_{j=1+m}^{n+m} (-1)^j (|\pi_j(X)| + |\pi_j(Z)|) \right). \end{aligned} \quad (14)$$

$$(15)$$

Since $\text{im } h_j$ is a subgroup of $\pi_{j-1}(X)$, the assumption that $\lim_{n \rightarrow \infty} \frac{|\pi_n(X)|}{n \log_p(n)} = 0$ is enough to ensure that applying $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty}$ to (15) yields

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{(\alpha_X + \alpha_Z) \log_p(n)} \sum_{j=1+m}^{n+m} \frac{1}{n} (-1)^j (|\pi_j(X)| + |\pi_j(Z)|),$$

i.e., 1. Hence $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty}$ applied to (14) yields 1, and hence the homotopy groups of Y have graded average which grows like $(\alpha_X + \alpha_Z) \log_p(n)$. \square

As a consequence of Proposition 4.2 and a simple algebraic- K -theoretic argument, we have:

Theorem 4.3. *Let p be an odd prime, and let X be a finite spectrum with Euler characteristic $\chi(X) = \sum_n (-1)^n \dim_{\mathbb{Q}} H_n(X; \mathbb{Q})$. Then the graded average of the orders of n consecutive $K(1)$ -local homotopy groups of X grows like $\frac{-\chi(X)}{2} \cdot \log_p(n)$.*

Proof. We first prove the claim in the case that $X = S^0$. Let s_n denote the graded average $\frac{1}{2(p-1)p^n} \sum_{j=1}^{2(p-1)p^n} (-1)^j |\pi_j L_{K(1)} S^0|$. By elementary algebra and the calculation of the $K(1)$ -local homotopy groups of spheres, we have

$$\begin{aligned} s_n &= \frac{1}{2(p-1)p^n} \sum_{i=1}^{p^n} (1 - p^{v_p(i)+1}) \\ &= \frac{1}{2(p-1)p^n} \left(p^n - p \sum_{i=1}^{p^n} p^{v_p(i)} \right) \\ &= \frac{1}{2(p-1)p^n} (p^n - p((n+1)p^n - np^{n-1})) \\ &= \frac{1}{2(p-1)p^n} (n+1)(p^n - p^{n+1}) \\ &= \frac{-1-n}{2}. \end{aligned}$$

Consider the sequence of nonpositive rational numbers

$$\frac{1}{2 \log_p(2)} \sum_{j=1}^2 (-1)^j |\pi_j(L_{K(1)} S^0)|, \quad \frac{1}{3 \log_p(3)} \sum_{j=1}^3 (-1)^j |\pi_j(L_{K(1)} S^0)|, \quad \frac{1}{4 \log_p(4)} \sum_{j=1}^4 (-1)^j |\pi_j(L_{K(1)} S^0)|, \dots \quad (16)$$

The least (i.e., highest absolute value) terms in (16) are taken in the subsequence

$$\begin{aligned} &\frac{1}{(2(p-1)p^0 - 1) \log_p(2(p-1)p^0 - 1)} \sum_{j=1}^{2(p-1)p^0-1} (-1)^j |\pi_j(L_{K(1)} S^0)|, \\ &\frac{1}{(2(p-1)p^1 - 1) \log_p(2(p-1)p^1 - 1)} \sum_{j=1}^{2(p-1)p^1-1} (-1)^j |\pi_j(L_{K(1)} S^0)|, \\ &\frac{1}{(2(p-1)p^2 - 1) \log_p(2(p-1)p^2 - 1)} \sum_{j=1}^{2(p-1)p^2-1} (-1)^j |\pi_j(L_{K(1)} S^0)|, \\ &\frac{1}{(2(p-1)p^3 - 1) \log_p(2(p-1)p^3 - 1)} \sum_{j=1}^{2(p-1)p^3-1} (-1)^j |\pi_j(L_{K(1)} S^0)|, \dots \end{aligned} \quad (17)$$

Write t_n for the n th term, counting from zero, in the sequence (17). Then we have

$$t_n = \frac{2(p-1)p^n s_n - 1}{(2(p-1)p^n - 1) \log_p(2(p-1)p^n - 1)},$$

and consequently

$$\begin{aligned} \lim_{n \rightarrow \infty} t_n &= \lim_{n \rightarrow \infty} \frac{(2(p-1)p^n - 1)(-1-n)/2 + (-1-n)/2 - 1}{(2(p-1)p^n - 1) \log_p(2(p-1)p^n - 1)} \\ &= \lim_{n \rightarrow \infty} \frac{(-1-n)/2}{\log_p(2(p-1)p^n - 1)} \\ &= \frac{-1}{2}. \end{aligned}$$

Consequently the graded average of the orders of n consecutive $K(1)$ -local homotopy groups of spheres grows no slower than

$\frac{-\log_p(n)}{2}$. To bound the growth from above, consider the subsequence of (16) given by

$$\begin{aligned} & \frac{1}{(2(p-1)p^0-2)\log_p(2(p-1)p^0-2)} \sum_{j=1}^{2(p-1)p^0-2} (-1)^j |\pi_j(L_{K(1)}S^0)|, \\ & \frac{1}{(2(p-1)p^1-2)\log_p(2(p-1)p^1-2)} \sum_{j=1}^{2(p-1)p^1-2} (-1)^j |\pi_j(L_{K(1)}S^0)|, \\ & \frac{1}{(2(p-1)p^2-2)\log_p(2(p-1)p^2-2)} \sum_{j=1}^{2(p-1)p^2-2} (-1)^j |\pi_j(L_{K(1)}S^0)|, \\ & \frac{1}{(2(p-1)p^3-2)\log_p(2(p-1)p^3-2)} \sum_{j=1}^{2(p-1)p^3-2} (-1)^j |\pi_j(L_{K(1)}S^0)|, \dots \end{aligned} \tag{18}$$

The greatest (i.e., least absolute value) terms in (16) are taken in the subsequence (18). Write u_n for the n th term in (18), counting from zero. Then we have

$$u_n = \frac{2(p-1)p^n s_n - 1 + p^{n+1}}{(2(p-1)p^n - 2)\log_p(2(p-1)p^n - 2)},$$

and consequently

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \frac{(2(p-1)p^n - 2)(-1 - n)/2 - n - 2 + p^{n+1}}{(2(p-1)p^n - 2)\log_p(2(p-1)p^n - 2)} \\ &= \lim_{n \rightarrow \infty} \frac{-n/2}{\log_p(2(p-1)p^n - 2)} \\ &= \frac{-1}{2}. \end{aligned}$$

Consequently (16) converges to $-1/2$, i.e., the graded average of the orders of n consecutive $K(1)$ -local homotopy groups of spheres grows like $\frac{-\log_p(n)}{2}$.

In the sequence of positive rational numbers $|\pi_2(L_{K(1)}S^0)|/(2\log_p(2)), |\pi_3(L_{K(1)}S^0)|/(3\log_p(3)), |\pi_4(L_{K(1)}S^0)|/(4\log_p(4)), \dots$, the highest values are taken at terms of the form

$$\frac{|\pi_{2p^n(p-1)-1}(L_{K(1)}S^0)|}{(2p^n(p-1)-1)\log_p(2p^n(p-1)-1)} = \frac{p^{n+1}}{(2p^n(p-1)-1)\log_p(2p^n(p-1)-1)}.$$

As $n \rightarrow \infty$, even these highest values go to zero. Hence $\lim_{n \rightarrow \infty} |\pi_n(L_{K(1)}S^0)|/(n\log_p(n)) = 0$. By induction on cells using Proposition 4.2, we have a well-defined *additive* invariant of finite spectra, given by sending a finite spectrum X to the unique real number $\alpha(X)$ such that the graded average of the orders of n consecutive $K(1)$ -local homotopy groups of X grows like $\alpha(X) \cdot \log_p(n)$.

Since $X \mapsto \alpha(X)$ is an additive invariant of finite spectra, it depends only on the value of the *universal* additive invariant of finite spectra, i.e., the Euler characteristic. Hence we can calculate the value of α on any finite spectrum X by multiplying the Euler characteristic $\chi(X)$ by the value of α on the generator S^0 of $K_0(\text{Fin}S_p^\wedge)$. The calculation $\alpha(S^0) = -1/2$, above, yields the claim in the statement of the theorem. \square

4.2 The λ -invariants of a finite spectrum

Definition 4.4. Let X be a finite spectrum. By $\lambda(X)$, the total λ -invariant of X , we mean the sum of differences

$$\sum_{j=0}^{p-2} (\lambda(\epsilon_j(KU_p^\wedge)^0(X)) - \lambda(\epsilon_j(KU_p^\wedge)^{-1}(X)))$$

of the λ -invariants of the Iwasawa modules associated to X .

Lemma 4.5. The total λ -invariant is equal to the Euler characteristic. Consequently the total λ -invariant is an additive invariant of finite spectra.

Proof. It is probably possible to give an intrinsic proof, based on formal properties of the characteristic polynomial and not reliant on the fact, established by Theorem 3.8, that the characteristic polynomial of each Iwasawa module of a finite spectrum X depends only on the rational homotopy type of X . However, the shortest and simplest proof certainly uses that fact, in order to recognize that the λ -invariant of $\epsilon_j(KU_p^\wedge)^n(X)$ is equal to the λ -invariant of $\epsilon_j(KU_p^\wedge)^0(Y)$ where Y is a wedge of spheres with

the same rational homology as X . In section 3.1, we calculated the characteristic polynomials and λ -invariants of the Iwasawa modules $\epsilon_j(KU_p^\wedge)^0(S^{2i})$. In particular, those calculations yield that

$$\begin{aligned}\lambda(\epsilon_j(KU_p^\wedge)^0(S^{2i})) &= \lambda(\epsilon_j(KU_p^\wedge)^{-1}(S^{2i-1})) \\ &= \begin{cases} 1 & \text{if } j \equiv i \pmod{p-1} \\ 0 & \text{otherwise.} \end{cases} \\ \lambda(\epsilon_j(KU_p^\wedge)^{-1}(S^{2i})) &= \lambda(\epsilon_j(KU_p^\wedge)^0(S^{2i-1})) \\ &= 0.\end{aligned}$$

From this, and the straightforward observation that $\lambda(\epsilon_j(KU_p^\wedge)^n(X \vee X')) = \lambda(\epsilon_j(KU_p^\wedge)^n(X)) + \lambda(\epsilon_j(KU_p^\wedge)^n(X'))$ for all finite spectra X, X' , we have the formula

$$\begin{aligned}\lambda(\epsilon_j(KU_p^\wedge)^0(X)) &= \sum_{n \equiv j \pmod{p-1}} \dim_{\mathbb{Q}} H_{2n}(X) \\ \lambda(\epsilon_j(KU_p^\wedge)^{-1}(X)) &= \sum_{n \equiv j \pmod{p-1}} \dim_{\mathbb{Q}} H_{2n-1}(X),\end{aligned}$$

and consequently the total λ -invariant is equal to the Euler characteristic. □

Theorem 4.3 and Lemma 4.5 then jointly imply:

Theorem 4.6. *Let X be a finite spectrum. Then the graded average of the orders of n consecutive $K(1)$ -local homotopy groups of X grows like $\frac{-\lambda(X)}{2} \cdot \log_p(n)$.*

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