COMMUTING UNBOUNDED HOMOTOPY LIMITS WITH MORAVA K-THEORY

GABRIEL ANGELINI-KNOLL* AND ANDREW SALCH†

*Université Sorbonne Paris Nord, LAGA, 93430, Villetaneuse, France.**

Department of Mathematics, Wayne State University
Detroit, Michigan, U.S.A.†

Abstract. This paper provides conditions for Morava K-theory to commute with certain homotopy limits. These conditions extend previous work on this question by allowing for homotopy limits of sequences of spectra that are not uniformly bounded below. As an application, we prove the $K(n)$-local triviality (for sufficiently large n) of the algebraic K-theory of algebras over truncated Brown–Peterson spectra, building on work of Bruner–Rognes and extending a classical theorem of Mitchell on $K(n)$-local triviality of the algebraic K-theory spectrum of the integers for large enough n.

Contents

1. Introduction

Given a generalized homology theory E_* and a sequence

$$\cdots \to X_2 \to X_1 \to X_0$$

E-mail address: `gak@math.fu-berlin.de,asalch@wayne.edu`.

of spectra, one often needs to know whether there is an isomorphism

$$\lim_i E_*(X_i) \cong E_*(\operatorname{holim}_i X_i).$$

This cannot be true in full generality. For example, the limit of the sequence

$$\cdots \to S/p^2 \to S/p$$

is the p-complete sphere $S_p^\wedge$. Consequently,

$$H_*(\operatorname{holim}_i S/p^i; \mathbb{Q}) \cong H_*(S_p^\wedge; \mathbb{Q}) \cong \mathbb{Q}_p.$$

On the other hand, $H_*(S/p^i; \mathbb{Q}) \cong 0$ for $i \geqslant 1$, and therefore

$$\lim_i H_*(S/p^i; \mathbb{Q}) \cong 0.$$

Additionally, $R^1 \lim_i H_*(S/p^i; \mathbb{Q})$ vanishes, so we cannot even recover the groups $H_*(\operatorname{holim}_i S/p^i; \mathbb{Q})$ from a "Milnor sequence." This example can be generalized to Morava K-theory $K(n)_*$ for $n > 0$ by letting V be a type n, finite spectrum with v_n-power self map $v \colon \Sigma^d V \to V$ and considering the limit of the sequence

$$\cdots \to \operatorname{cof}(v^2) \to \operatorname{cof}(v)$$

where $\operatorname{cof}(v^i)$ denotes the cofiber of the iterated composite $v^i \colon \Sigma^{di} V \to V$.

This motivates the question: what conditions on E_* and the sequence

$$\cdots \to X_2 \to X_1 \to X_0$$

allow us to commute the homotopy limit with E_*? There are known results along these lines, most famously a commonly-used result of Adams from [1], but the usual hypotheses are that

- the spectra X_i are uniformly bounded below, and
- the homology theory E_* is connective.

In this paper, we remove each of these assumptions, under some reasonable additional hypotheses. Our particular focus is on the case where E_* is a Morava K-theory $K(n)_*$.

There is previous unpublished work of Sadofsky [44, 43] on this question in the case when each X_i is $K(n)$-local. Given a seqeunce

$$\cdots \to X_2 \to X_1 \to X_0$$

of $K(n)$-local spectra, Sadofsky constructs a spectral sequence computing the groups $K(n)_*(\lim_i X_i)$ whose E_2-term is the right derived functors of lim in the category of $K(n)_*K(n)$-comodules (cf. [40, Theorem A.0.1]). One may therefore view the higher right derived functors in $K(n)_*K(n)$-comodules as an obstruction to commuting the limit with Morava K-theory. Our results are, in a sense, orthogonal to the work of Sadofsky since we consider cases where the Sadofsky spectral sequence does not provide useful information because it does not convergence.

This paper is written with a view towards filtered spectra that arise when studying topological periodic cyclic homology,

$$TP(R) := THH(R)^{t\mathbb{T}}.$$

In particular, the Greenlees filtration [19] on topological periodic cyclic homology is not uniformly bounded below. Nevertheless, these filtered spectra often have nice enough homological properties to apply the main result of this paper.

Following the red-shift program of Ausoni–Rognes [9], we are most interested in the chromatic complexity of topological periodic cyclic homology and related invariants. Therefore, a generalized homology theory of primary interest is Morava K-theory $K(n)_*$. Calculating Morava K-theory of topological periodic cyclic homology using the Greenlees filtration requires that one be able to commute a non-bounded-below generalized homology theory (Morava K-theory) with a non-uniformly-bounded-below homotopy limit, so existing results on generalized homology of limits, like Adams' theorem from [1] reproduced as Theorem 2.5 below, do not suffice.

Our main result may then be summarized as follows.

Theorem 1.1 (Theorem 2.23)**.** *Fix an integer M and a prime p. Let*

$$\dots \to X_2 \to X_1 \to X_0 \to X_{-1} \to \dots$$

be a sequence of spectra satisfying:

1. *each X_i is bounded below and p-complete,*
2. *for each i, the graded $\mathbb{F}_p$-vector space $H_*(X_i;\mathbb{F}_p)$ is finite type,*
3. *the sequence of graded $\mathbb{F}_p$-vector spaces $\{H(H_*(X_i;\mathbb{F}_p),Q_n)\}$ is pro-isomorphic to zero,*
4. *there exists an integer L such that for each $s \geqslant L$ the groups $\pi_s(\lim_i X_i)/p$ are finite, and*
5. *for each i, the A_*-comodule primitives of $H_*(X_i;\mathbb{F}_p)$ are trivial in grading degrees $\geqslant N$.*

Then $\lim_i X_i$ is $K(n)$-acyclic.

As the main application in the present paper, we prove a higher chromatic height analogue of Mitchell's theorem for algebraic K-theory of truncated Brown–Peterson spectra, building on work of Bruner–Rognes [15]. In particular, Mitchell proves in [38] that $K(m)_*(K(\mathbb{Z}))$ vanishes for $m \geqslant 2$, and consequently the algebraic K-theory of every $H\mathbb{Z}$-algebra is $K(m)$-acyclic. First, we fix conventions.

Definition 1.2 (cf. [32, Definition 4.1])**.** *Let n be an integer, let $m \in \{1,2,\dots\} \cup \{\infty\}$, and let p be a prime number. By a* p-primary E_m form *of $BP\langle n\rangle$ we mean a p-local E_m ring spectrum R equipped with a complex orientation $MU_{(p)} \to R$ such that the composite map*

$$\mathbb{Z}_{(p)}[v_1,\dots,v_n] \hookrightarrow \pi_*BP \hookrightarrow \pi_*(MU_{(p)}) \to \pi_*R$$

is an isomorphism.

We also note that, for every m, any two p-primary E_m forms of $BP\langle n\rangle$ become homotopy-equivalent (as spectra) after p-completion. This is a consequence of the main theorem of [5].

Our proof builds on a result of Bruner and Rognes [15, Prop. 6.1], in which it is proven that the E_∞-term of the homological homotopy fixed-point spectral sequence

$$H^*(\mathbb{T}, H_*(THH(R);\mathbb{F}_p)) \Rightarrow H_*^c(TC^-(R);\mathbb{F}_p) \tag{1}$$

is isomorphic to $(P(t)\otimes M_1 \otimes M_2)\oplus T$ in the case R is a p-primary E_∞-form of $BP\langle n\rangle$. Here $P(t)$ and M_1 and M_2 and T are certain comodules over the dual Steenrod algebra; see Section 3.2 for an explicit description of these comodules.

In this paper, we offer some applications of our main theorem to Morava K-theory of algebraic K-theory spectra of various forms of $BP\langle n\rangle$.

We do not really need the full strength of an E_∞ form of $BP\langle n\rangle$: for our arguments, it is enough to have an E_2-form R of $BP\langle n\rangle$ *such that the E_∞-page of spectral sequence* (1) *is isomorphic to that computed by Bruner and Rognes as a comodule over the p-primary dual Steenrod algebra*[1]. This leads us to the following assumption:

Running Assumption 1.3. Let $B\langle n\rangle$ be a p-primary E_2-form of $BP\langle n\rangle$ such that the conclusion of [15, Prop. 6.1] holds.

By the previous discusssion, we know that Running Assumption 1.3 holds for and p-primary E_∞-form of $BP\langle n\rangle$, which exist at all primes when $n = 1$ by [10], when $p = n = 2$ by [32] and when $n = 2$ and $p = 3$ by [26]. Let $K(R)$ denote the algebraic K-theory spectrum associated to an E_1 ring spectrum R.

Theorem 1.4 (Theorem 3.10)**.** *Fix a prime $p \geqslant 3$ and let $B\langle n\rangle$ be a p-primary E_2 form of $BP\langle n\rangle$ satisfying Running Assumption 1.3. For example, let $B\langle n\rangle$ be any p-primary E_∞ form of $BP\langle n\rangle$. Let m be an integer, $m \geqslant n+2$. Then there is a weak equivalence*

$$L_{K(m)}K(B\langle n\rangle) \simeq 0.$$

Consequently, the algebraic K-theory spectrum $K(A)$ is $K(m)$-acyclic for every $B\langle n\rangle$-algebra A.

This result recovers the vanishing of $K(m)_*(K(\mathbb{Z}_{(p)}))$ for $m \geqslant 2$ by Mitchell [38], since $H\mathbb{Z}_{(p)}$ is an E_∞ form of $BP\langle 0\rangle$. Our result also proves that $K(m)_*(K(\ell)) = 0$ for $m \geqslant 3$ and $p \geqslant 3$, since the Adams summand ℓ is an p-primary E_∞ form of $BP\langle n\rangle$ by [10]. This recovers the vanishing of $K(m)_*(K(\ell))$ for $m \geqslant 3$ and $p \geqslant 5$ proven by Ausoni–Rognes [8], although our proof uses entirely different methods. At $p = 3$, there is an E_∞-ring spectrum taf^D, which is an E_∞ form of $BP\langle 2\rangle$ constructed by Hill–Lawson [26] using a Shimura variety $\mathcal{X}^D$ associated to a quaternion algebra of discriminant 14 . Our result therefore also proves that $K(m)_*K(\mathrm{taf}^D)$ vanishes for $m \geqslant 4$.

We say a ring spectrum R has height n if $K(m)_*R = 0$ for all $m > n$ and $K(n)_*R \neq 0$. Our result proves, in particular, that $K(BP\langle 2\rangle)$ has height $\leqslant 3$. To prove that $K(BP\langle 2\rangle)$ has height exactly 3, one needs to prove the nonvanishing result $K(3)_*K(BP\langle 2\rangle) \neq 0$. Since the first draft of this paper, Hahn–Wilson [22] have shown that in fact $K(n+1)_*K(BP\langle n\rangle) \neq 0$ for any p-primary E_3-form of $BP\langle n\rangle$. In fact, Hahn–Wilson [22] prove the stronger statement that $K(BP\langle n\rangle)$ has fp-type $n+1$ in the sense of [34, p.5].

After work of Mitchell in 1990 [38] and work of Ausoni–Rognes in 2002 [8], there has been renewed interest and progress on questions of this nature. In particular, after the first draft of this paper was posted in pre-print form, work of Clausen–Mathew–Naumann–Noel [16] and Land–Mathew–Meier–Tamme [30] proved that if R is an E_1-ring spectrum and $T(n-1)_*R = 0$ and $T(n)_*R = 0$, then $T(n)_*K(R) = 0$, where $T(n)$ is the v_n-telescope $v_n^{-1}V$ of a type n finite complex V. We note that our Theorem 3.10 is proven using entirely different techniques to these other

[1] We suspect, but not made serious efforts to prove, that a significantly weaker form of $BP\langle n\rangle$ suffices for this. See Remark 3.2, below, for further discussion.

authors and a feature of our approach is that it applies to topological periodic cyclic homology. We therefore regard our methods as complementary to those of [16, 30].

Our work in this paper was motivated initially by a conjecture in an early draft of a paper by the first author and Quigley [4]. Let $y(n)$ be the Thom spectrum of the 1-fold loop map

$$\Omega J_{2^n-1}(S^2) \to \Omega J(S^2) \simeq \Omega^2 S^3 \to BGL_1 S$$

where $\Omega^2 S^3 \to BGL_1 S$ is the unique (up to homotopy) 2-fold loop map, with target $BGL_1 S$ is (a model for) the classifying space of stable spherical bundles, and $J(S^2)$ is the James construction [28] equipped with its usual filtration

$$J_{2^n-1}(S^2) \hookrightarrow J(S^2)$$

for all $n \geqslant 0$. Here we write $TP(y(n))[k]$ for the Greenlees filtration of topological periodic cyclic homology of $y(n)$ (see [19] for details about this filtration).

In [4], the first author and Quigley conjectured that the pro-vanishing of Margolis homology of $H_*(TP(R)[k]; \mathbb{F}_p)$ should be sufficient to show that $K(n)_*TP(R)$ vanishes, in the case of $R = y(n)$. This conjecture is resolved by Theorem 1.1 together with calculations of the first author and Quigley which verify that the remaining hypotheses of Theorem 1.1 are satisfied. This is used by the first author and Quigley to prove that $L_{K(m)}y(n) = 0$ implies $L_{K(m+1)}TP(y(n)) = 0$ for $0 \leqslant m < n$ and each $n \geqslant 0$ in [4].

1.1. **Conventions.** Our conventions are all relatively standard, but we prefer to state them explicitly to avoid any possible confusion.

When κ is a cardinal number and A an object in some category, we will write A^κ for the κ-fold categorical product and $A^{\coprod \kappa}$ for the κ-fold categorical coproduct. If our category is additive and κ is finite, then we write $A^{\oplus \kappa}$ for the κ-fold categorical biproduct of A with itself.

Let R be a commutative ring. In this paper, the term "finite type" is used in the sense common in algebraic topology: a graded R-module V is *finite type* if, for each integer n, the grading degree n summand V^n is a finitely generated R-module. In particular, if R is a field, then we say V is finite type if, for each integer n, the grading degree n summand V^n is a finite dimensional vector space.

We will write $P(x_1, \dots, x_n)$ for a polynomial algebra over $\mathbb{F}_p$ with generators $x_1, \dots, x_n$. We write $E(x_1, \dots, x_n)$ for an exterior algebra over $\mathbb{F}_p$ with generators $x_1, \dots, x_n$. We write $P_k(x_1, \dots, x_n)$ for the truncated polynomial algebra over $\mathbb{F}_p$ with generators $x_1, \dots, x_n$ and relations $x_1^k, ..., x_n^k$. Note that $P_k(x)$ is often also denoted $\mathbb{F}_p[x]/x^k$ in the literature.

Suppose we have a ring R, a non-zero-divisor $r \in R$, and a left R-module M.

- An element $m \in M$ is *r-power-torsion* if there exists an integer n such that $r^n m = 0$.
- The module M is *r-power-torsion* if, for each $m \in M$, there exists some integer n such that $r^n m = 0$.
- An element $m \in M$ is *simple r-torsion* if $rm = 0$.
- The module M is *simple r-torsion* if $rm = 0$ for all $m \in M$.

Given categories $\mathcal{C}$ and $\mathcal{D}$, we will write $\mathcal{D}^{\mathcal{C}}$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$. By a *sequence* of objects in a category $\mathcal{C}$, we mean an object in $\mathcal{C}^{(\mathbb{Z}^{\mathrm{op}})}$, where $\mathbb{Z}$ is regarded as a partially-ordered set and hence as a small category. Having adopted this convention, whenever we write $\lim_k$, it always indicates a limit as $k \to \infty$.

When we write colim_k, it always indicates a limit as $k \to -\infty$. We use the notation $\{X_i\}$ as shorthand for a sequence of objects $\cdots \to X_1 \to X_0 \to X_{-1} \to \dots$ in $\mathcal{C}$.

Let p be an odd prime. We write A for the p-primary dual Steenrod algebra and A_* for its dual. By Milnor [36], there is an isomorphism

$$A_* \cong P(\bar{\xi}_i : i \geqslant 1) \otimes E(\bar{\tau}_i : i \geqslant 0)$$

where $\bar{\xi}_i$ and $\bar{\tau}_i$ are the conjugates of Milnor's generators ξ_i and τ_i. That is, if we write χ for the antipode map of the Hopf algebra A_*, then $\bar{\xi}_i = \chi(\xi_i)$ and $\bar{\tau}_i = \chi(\tau_i)$. The coproduct is defined by the formulas

$$\Delta(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i} \text{ and} \tag{2}$$

$$\Delta(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i} \tag{3}$$

where $\bar{\xi}_0 = 1$ by convention. Given A_*-comodules M and n, we write $\mathrm{Hom}_{A_*}(M, N)$ for the $\mathbb{F}_p$-vector space of A_*-comodule maps from M to N and $\mathrm{Ext}_{A_*}^{*,*}(\mathbb{F}_p, N)$ the right derived functors of the functor $\mathrm{Hom}_{A_*}(\mathbb{F}_p, -)$.

1.2. **Organization.** In Section 2.1, we give sufficient conditions for the canonical map $X \to X^{<N}$ to induce an injection on mod p homology. In Section 2.4, we prove the main theorem. In Section 3, we give our main application, which is a proof of a higher chromatic height analogue of Mitchell's theorem. We also provide an appendix (Appendix A), including results on Margolis homology.

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2. Morava K-theory of homotopy limits

Let X be a spectrum, and let N be an integer. Let $X^{<N}$ denote the spectrum obtained by attaching cells to X to kill the homotopy groups in degrees greater than or equal to N. In section 2.3, we will find ourselves needing an answer to the following natural question: when does the natural map

$$X \to X^{<N}$$

induce a *one-to-one* map in mod p homology? Consequently our first task, in Section 2.1, is to answer that question.

2.1. **When does killing homotopy induce an injection in homology?** Let $P(M) = \mathrm{Hom}_{A_*}(\mathbb{F}_p, M)$ denote the graded $\mathbb{F}_p$-vector space of comodule primitives in a graded left A_*-comodule M, and let $Q(Z) = \mathbb{F}_p \otimes_A Z$ denote the A-module indecomposables of Z.

Definition 2.1. *Let p be a prime number and let N be an integer. We will say X* satisfies condition $H(N)$ *if X is bounded below, and the A_*-comodule primitives of $H_*(X; \mathbb{F}_p)$ are trivial in grading degrees $\geqslant N$.*

We suppress the prime p from the notation $H(N)$, because it will always be clear from the context.

Proposition 2.2. *Let p be a prime number, let N be an integer, and let X satisfy condition $H(N)$. Let $X^{<N}$ be X with cells attached to kill all the homotopy groups of X in degrees $\geqslant N$. Then the map $H_*(X;\mathbb{F}_p) \to H_*(X^{<N};\mathbb{F}_p)$, induced by the canonical map $X \to X^{<N}$, is injective.*

Proof. It is classical (e.g. see Proposition 3.9 of [37]) that a map $M \to M'$ of *coalgebras* is injective if and only if it is injective on the primitives. We need to show that the same is true for a map $M \to M'$ of bounded-below *comodules* instead. Dually, we want to know that a map of graded A-modules $f\colon Z \to Z'$ is surjective if Z' is bounded below and the induced map on indecomposables $\mathbb{F}_p \otimes_A f\colon Q(Z) \to Q(Z')$ is surjective. This is elementary: every homogeneous element x of Z' is an A-linear combination $\sum_{i=1}^n a_i q_i$ of homogeneous elements $q_i \in Q(Z')$. Lift each q_i to an element $\tilde{q}_i \in Q(Z)$, and observe that $f(\sum_{i=1}^n a_i \tilde{q}_i) = x$. □

Example 2.3. A simple example where the hypotheses, and therefore the result, holds, is when $X = S^0$ and $N = 1$, so that $\mathbb{F}_p \hookrightarrow (A//E(0))_*$ is an inclusion. A simple example where the hypotheses are *not* satisfied is the case where $X = S^0$ and $N = 0$, where of course $X^{<N}$ is contractible and the result cannot hold.

2.2. **Recollections.** Recall that, for each prime number p and positive integer n, we have the homotopy fiber sequence

$$\Sigma^{2(p^n-1)} k(n) \to k(n) \to H\mathbb{F}_p,$$

and the composite map

$$H\mathbb{F}_p \to \Sigma^{2p^n-1} k(n) \to \Sigma^{2p^n-1} H\mathbb{F}_p \tag{4}$$

is the cohomology operation Q_n, which satisfies $Q_n^2 = 0$. This implies a useful relationship between Morava K-theory and Margolis homology of $E(Q_n)$-modules, which we briefly summarize in an appendix to this paper, Appendix A.

Definition 2.4. *We say that an $E(Q_n)$-module is Q_n-*acyclic *if the Q_n-Margolis homology $H(M;Q_n)$ vanishes. We say that a morphism of $E(Q_n)$-modules is a Q_n-*equivalence *if it induces an isomorphism in Q_n-Margolis homology.*

We recall the following useful result of Adams [1].

Theorem 2.5 (Theorem III.15.2 [1])**.** *Suppose that R is a subring of $\mathbb{Q}$, E is a bounded-below spectrum such that $H_r(E;R)$ is a finitely generated R-module for all r, and $\{X_i\}_{i\in I}$ is a set of spectra such that $\pi_r(X_i)$ is an R-module for all r. Suppose that there exists a uniform lower bound for $\pi_*(X_i)$, i.e., there exists an integer N such that $\pi_n(X_i) \cong 0$ for all $n < N$. Then the canonical map of spectra*

$$E \wedge \prod_{i\in I} X_i \to \prod_{i\in I} (E \wedge X_i) \tag{5}$$

is a weak equivalence.

The following corollary is a straightforward consequence of Theorem 2.5.

Corollary 2.6. *Let $\{X_i\}$ be a sequence of spectra. Suppose that there exists a uniform lower bound on $\pi_*(X_i)$. Then the canonical maps of spectra*

$$H\mathbb{F}_p \wedge \operatorname{holim}_i X_i \to \operatorname{holim}_i (H\mathbb{F}_p \wedge X_i) \quad \text{and}$$
$$k(n) \wedge \operatorname{holim}_i X_i \to \operatorname{holim}_i (k(n) \wedge X_i)$$

are weak equivalences.

2.3. **$K(n)$-acyclicity of products.**

Definition 2.7. *Given a spectrum X, by the* Whitehead filtration of X *we mean the functor* $\mathrm{Wh}(X) : \mathbb{Z}^{\mathrm{op}} \to Sp$ *given by letting* $\mathrm{Wh}^n(X) = \mathrm{Wh}(X)(n)$ *be the fiber* $X^{\geqslant n}$ *of the canonical map* $X \to X^{<n}$*. The map* $\mathrm{Wh}^{n+1}(X) \to \mathrm{Wh}^n(X)$ *is the natural map* $X^{\geqslant n+1} \to X^{\geqslant n}$*.*

One agreeable property of Whitehead filtrations is that they are compatible with products:

Lemma 2.8. *Let I be a set, and for each $i \in I$, let X_i be a spectrum. Then the natural map* $\mathrm{Wh}^n(\prod_{i\in I} X_i) \to \prod_{i\in I} \mathrm{Wh}^n(X_i)$ *is a weak equivalence for each integer n. Moreover, these weak equivalences are compatible with the maps in the Whitehead filtrations, so that*

$$\mathrm{Wh}\left(\prod_{i\in I} X_i\right) \to \prod_{i\in I} \mathrm{Wh}(X_i)$$

is a levelwise weak equivalence of sequences in SHC.

Proof. Routine consequence of π_* commuting with arbitrary products. □

Lemma 2.9. *Let p be a prime number, let n be a positive integer, let I be a set, and for each $i \in I$, let X_i be a spectrum. We have a strongly convergent spectral sequence*

$$E^1_{s,t} \cong \prod_{i\in I} k(n)_t\left(\Sigma^s H\pi_s(X_i)\right) \Rightarrow k(n)_t\left(\prod_{i\in I} X_i\right) \tag{6}$$
$$d^r : E^r_{s,t} \to E^r_{s+r,t-1} \tag{7}$$

given by the exact couple arising from applying connective Morava K-theory $k(n)$ to the Whitehead tower of $\prod_{i\in I} X_i$. The action of $v_n \in k(n)_{2(p^n-1)}$ on the abutment increases filtration by one, so that its action in the spectral sequence is zero.

Proof. Throughout, we use the terminology and methods of Boardman's paper [12]. Consider the spectral sequence of the unrolled exact couple obtained by applying $k(n)_*$ to the Whitehead filtration of $\prod_{i\in I} X_i$. For each integer t, the $k(n)$-homology group $k(n)_t\left(\mathrm{Wh}^s\left(\prod_{i\in I} X_i\right)\right)$ vanishes for all $s > t$, by the Hurewicz theorem. Consequently

$$\lim_s k(n)_t\left(\mathrm{Wh}^s\left(\prod_{i\in I} X_i\right)\right) \text{ and } R^1 \lim_s k(n)_t\left(\mathrm{Wh}^s\left(\prod_{i\in I} X_i\right)\right)$$

each vanish for all t. Consequently the spectral sequence converges conditionally to the colimit.

The description of the E^1-page given in (6) follows from Lemma 2.8. The spectral sequence's E^1-page vanishes in bidegrees (s,t) with $t < s$, so by (7), it is a half-plane[2] spectral sequence with exiting differentials, convergent to the colimit. By Theorem 6.1 of [12], the spectral sequence must then be strongly convergent.

The spectral sequence converges to the colimit, which is

$$\begin{aligned}\operatorname{colim}_s k(n)_*\left(\mathrm{Wh}^s\left(\prod_{i\in I} X_i\right)\right) &\cong k(n)_*\left(\operatorname{hocolim}_s\left(\mathrm{Wh}^s\left(\prod_{i\in I} X_i\right)\right)\right)\\ &\cong k(n)_*\left(\prod_{i\in I} X_i\right).\end{aligned}$$

Hence the spectral sequence converges to the $k(n)$-homology of the product $\prod_{i\in I} X_i$, as claimed. ☐

Lemma 2.10. *Let p be a prime and let n be a positive integer. Write $k(n)$ for the nth p-primary connective Morava K-theory spectrum. Let M be an integer, and let X be a bounded-below spectrum satisfying condition $H(M)$. Suppose that $k(n)_*(X)$ is simple v_n-torsion. Then the canonical map $k(n)_*(X) \to k(n)_*(X^{<M})$ is injective.*

Proof. Since X is bounded-below, the smash product $k(n) \wedge X$ is also bounded-below, and moreover has p-adically complete homotopy groups. Consequently Bousfield's convergence results in [13] yield that the $H\mathbb{F}_p$-nilpotent completion map $k(n) \wedge X \to (k(n) \wedge X)^\wedge_{H\mathbb{F}_p}$ is an equivalence. Hence the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{Comod(E(Q_n)^*)}(\mathbb{F}_p, H_*(X;\mathbb{F}_p)) \Rightarrow \pi_{t-s}\left((k(n) \wedge X)^\wedge_{H\mathbb{F}_p}\right) \tag{8}$$

converges to the $k(n)$-homology groups of X. Since $k(n)_*(X)$ is simple v_n-torsion, the E_∞-term of (8) is concentrated on the $s = 0$-line. Consequently the Hurewicz map $k(n)_*(X) \to H_*(X;\mathbb{F}_p)$ is injective. We have a commutative square

$$\begin{array}{ccc} k(n)_*X & \longrightarrow & k(n)_*(X^{<M}) \\ \big\downarrow & & \big\downarrow \\ H_*(X;\mathbb{F}_p) & \longrightarrow & H_*(X^{<M};\mathbb{F}_p) \end{array}$$

whose left-hand vertical map is injective. The square's bottom horizontal map is injective since X satisfies condition $H(M)$. Consequently the top vertical map is also injective, as claimed. ☐

Theorem 2.11. *Let p be a prime number, let M be an integer, let n be a positive integer, let I be a set, and for each $i \in I$, let X_i be a bounded-below[3] spectrum satisfying condition $H(M)$. Suppose that $k(n)_*(X_i)$ is simple v_n-torsion for all $i \in I$. Finally, suppose that the $\mathbb{F}_p$-vector space $\prod_{i\in I} k(n)_t(H\pi_s X_i)$ is finite-dimensional for each t and for each $s \geqslant M$. Then the product $\prod_i X_i$ is $K(n)$-acyclic.*

---

[2]That is, the spectral sequence is half-plane after a suitable linear re-indexing of bidegrees to move the line $t - s = 0$ to a coordinate axis. This re-indexing does not affect the fact that the spectral sequence has exiting differentials, in Boardman's sense.

[3]We emphasize that we are not assuming a *uniform* lower bound on the spectra X_i.

Proof. Since each X_i satisfies condition $H(M)$, for each $i \in I$ the map $\mathrm{Wh}^M(X_i) \to X_i$ is zero in $k(n)$-homology, by Lemma 2.10. Consequently the map

$$\prod_{i\in I} k(n)_* \left(\mathrm{Wh}^M(X_i)\right) \to k(n)_* \left(\prod_{i\in I} (\mathrm{Wh}^q(X_i))\right),$$

defined for all $q \leqslant M$, is trivial on the factor $k(n)_*(\mathrm{Wh}^m(X_i))$ corresponding to any element $i \in I$ such that X_i is q-connective. For each integer s, the image of the map

$$k(n)_t \left(\prod_{i\in I} (\mathrm{Wh}^s(X_i))\right) \to \operatorname{colim}_q k(n)_t \left(\prod_{i\in I} (\mathrm{Wh}^q(X_i))\right) \tag{9}$$

is the sth filtration layer in the abutment of the spectral sequence (6). If $s > M$, then the map (9) factors through the map

$$\prod_{i\in I} k(n)_t \left(\mathrm{Wh}^M(X_i)\right) \to \operatorname{colim}_q k(n)_t \left(\prod_{i\in I} \mathrm{Wh}^q(X_i)\right), \tag{10}$$

so we aim to show that the map (10) is trivial, yielding $E^\infty_{s,t} = 0$ for all $s > M$.

A homogeneous element of $\prod_{i\in I} k(n)_* \left(\mathrm{Wh}^M(X_i)\right)$ of degree t is given by specifying, for each $i \in I$, an element $x_i \in k(n)_t \left(\mathrm{Wh}^M(X_i)\right)$. In the sequence of abelian groups
(11)

$$\prod_{i\in I} k(n)_t \left(\mathrm{Wh}^M(X_i)\right) \to \prod_{i\in I} k(n)_t \left(\mathrm{Wh}^{M-1}(X_i)\right) \to \prod_{i\in I} k(n)_t \left(\mathrm{Wh}^{M-2}(X_i)\right) \to \dots,$$

$x_j \in k(n)_t(\mathrm{Wh}^M(X_j))$ is trivialized as soon as we reach $\prod_{i\in I} k(n)_t (\mathrm{Wh}^u(X_i))$, where u is the connectivity of X_j.

Consequently, for each element x of $\prod_{i\in I} k(n)_* \left(\mathrm{Wh}^M(X_i)\right)$, the image of x in the colimit $\operatorname{colim}_s \prod_{i\in I} k(n)_t (\mathrm{Wh}^s(X_i))$ is zero upon projection to the quotient

$$\operatorname{colim}_s \prod_{i\in I_u} k(n)_t (\mathrm{Wh}^s(X_i)),$$

where I_u is the subset of I consisting of those $i \in I$ such that the spectrum X_i has connectivity precisely equal to u. For each integer u, we have a spectral sequence

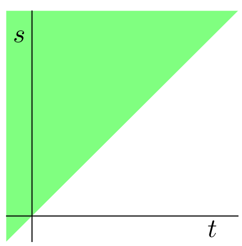
$$\begin{aligned} {}^uE^1_{s,t} \cong \prod_{i\in I_u} k(n)_t (\Sigma^s H\pi_s(X_i)) &\Rightarrow k(n)_t \left(\prod_{i\in I_u} X_i\right) \\ d^r : {}^uE^r_{s,t} &\to {}^uE^r_{s+r,t-1} \end{aligned} \tag{12}$$

obtained by applying $k(n)_*$ to the product of the Whitehead filtrations of the spectra X_i with $i \in I_u$. In spectral sequence (12), the bidegrees ${}^uE^1_{s,t}$ with $s < u$ are trivial. Consequently there is an upper bound on the lengths of nonzero differentials in spectral sequence (12) which can hit bidegree ${}^uE^r_{s,t}$.

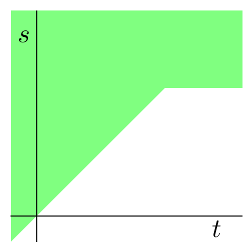
The unrolled exact couple which yielded the original spectral sequence (6) is the product, over all integers u, of the unrolled exact couple which yields the spectral sequence ${}^uE^*_{*,*}$ of (12). Since each given element x of $\prod_{i\in I} k(n)_* \left(\mathrm{Wh}^M(X_i)\right)$ maps to zero in $\operatorname{colim}_s \prod_{i\in I_u} k(n)_t (\mathrm{Wh}^s(X_i))$, the image of x in $\prod_{i\in I_u} k(n)_* \left(\mathrm{Wh}^M(X_i)\right)$ must represent a class in the spectral sequence ${}^uE^*_{*,*}$ of (12) which is hit by a differential. There is an upper bound, depending on u, on the length of that differential; this is a consequence of a lower vanishing line in the ${}^{u}E^1$-page of (12), simply due to the u-connectivity of the spectra whose $k(n)$-homology yields that ${}^{u}E^1$-page. Consequently, if we regard $x \in \prod_{i\in I} k(n)_* \left(X_i^{\geqslant M}\right)$ as an I-tuple $(x_i)_{i\in I}$, there is some finite page of spectral sequence ${}^{u}E^*_{*,*}$ by which all the components x_i with $i \in I_u$ have been wiped out by differentials.

Hence, if $s \geqslant M$, the only chance for x to represent a nonzero element in the abutment $\operatorname{colim}_q \prod_{i\in I} k(n)_t \left(X_i^{\geqslant q}\right)$ of spectral sequence (6) is if x represents an element in $E^1_{*,*}$ in a bidegree which is hit by arbitrarily long differentials (cf. Remark 2.12). But whatever element of $E^1_{*,*}$ is represented by x, that element sits above the $s = M$-line, and consequently lives in a bidegree $E^1_{s,t} \cong \prod_{i\in I} k(n)_t \left(\Sigma^s H\pi_s(X_i)\right)$ which is, by assumption, finitely generated. Consequently only finitely many lengths of differentials can have nonzero image in that bidegree. Hence x indeed must map to zero in the abutment of (6).

This yields the desired triviality of $E^\infty_{s,t}$ in (6) for all $s > M$. Drawn with the Adams convention, its E^1-page also vanishes above the line $s = t$. In the following diagrams, the vanishing regions are colored green:



(A) Vanishing region at E^1.



(B) Vanishing region at E^∞.

The action of v_n on the spectral sequence increases stem (i.e., position along the horizontal axis) by $2(p^n - 1)$, and increases filtration (i.e., position along the vertical axis) by at least one in the E^∞-page of spectral sequence (6), by the claim about the v_n-action in Lemma 2.9. Of course, there is also the possibility of longer filtration jumps, so that the action of v_n on the abutment $k(n)_* \left(\prod_{i\in I} X_i\right)$ raises filtration by more than one. Nevertheless, the vanishing line in the E^∞-page establishes that, starting in any given bidegree, there is an upper bound on the integers j such that v_n^j times elements in that bidegree can be nonzero in $k(n)_* \left(\prod_{i\in I} X_i\right)$, even taking into account possible filtration jumps. Consequently every homogeneous element of $k(n)_* \left(\prod_{i\in I} X_i\right)$ is v_n-power-torsion. Hence $v_n^{-1} k(n)_* \left(\prod_{i\in I} X_i\right) \cong K(n)_* \left(\prod_{i\in I} X_i\right)$ is trivial, as claimed. □

Remark 2.12. In the sequence

$$\cdots \to \prod_{i\in I} k(n)_t(\mathrm{Wh}^M(X_i)) \to \prod_{i\in I} k(n)_t(\mathrm{Wh}^{M-1}(X_i)) \to \dots, \tag{13}$$

given an I-tuple $(x_i)_{i\in I} \in \prod_{i\in I} k(n)_t(\mathrm{Wh}^M(X_i))$, we know that all the components x_i with $i \in I_u$ map to zero at the stage $\prod_{i\in I} k(n)_t(\mathrm{Wh}^q(X_i))$, so every individual x_i eventually maps to zero. However, we must be careful about the possibility that $(x_i)_{i\in I}$ still does not map to zero in the colimit of (13). This is the reason why we need the assumption that the $\mathbb{F}_p$-vector space $\prod_{i\in I} k(n)_t(H\pi_s X_i)$ is finite-dimensional for each t and for each $s \geqslant M$. For a simple example of what could go wrong, consider the image of the element $(1,1,\dots) \in \prod_{n\geqslant 0}\mathbb{Z}$ in the colimit of the sequence of abelian groups

$$\prod_{n\geqslant 1}\mathbb{Z} \xrightarrow{f_1} \prod_{n\geqslant 1}\mathbb{Z} \xrightarrow{f_2} \dots$$

in which f_n sends the first n factors of $\prod_{n\geqslant 1}\mathbb{Z}$ to zero, and is the identity on the remaining components.

2.4. $K(n)$-acyclicity of sequential homotopy limits.

Lemma 2.13. *Let $\{X_i\}$ be a sequence in the stable homotopy category. Then we have a commutative diagram of spectra*
(14)

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Wh}^{q+1}(\mathrm{holim}_i X_i) & \longrightarrow & \mathrm{holim}_i \left(\mathrm{Wh}^{q+1}(X_i)\right) & \longrightarrow & \Sigma^q H\left(R^1 \lim_i \pi_{q+1} X_i\right) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Wh}^{q}(\mathrm{holim}_i X_i) & \longrightarrow & \mathrm{holim}_i \left(\mathrm{Wh}^{q}(X_i)\right) & \longrightarrow & \Sigma^{q-1} H\left(R^1 \lim_i \pi_{q} X_i\right) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Wh}^{q-1}(\mathrm{holim}_i X_i) & \longrightarrow & \mathrm{holim}_i \left(\mathrm{Wh}^{q-1}(X_i)\right) & \longrightarrow & \Sigma^{q-2} H\left(R^1 \lim_i \pi_{q-1} X_i\right) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

whose rows are each homotopy fiber sequences.

Proof. To get that each row in (14) is a fiber sequence, compare the Milnor sequence for the homotopy groups of $\mathrm{holim}_i X_i$ to the Milnor sequence for the homotopy groups of $\mathrm{holim}_i \mathrm{Wh}^q(X_i)$. The compatibility with the structure maps of the sequences of spectra follows from functoriality of Wh^q and the universality of the comparison map from $\mathrm{Wh}^q \circ \mathrm{holim}_i$ to $\mathrm{holim}_i \circ \mathrm{Wh}^q$. □

Lemma 2.14. *Let $\{X_i\}$ be a sequence of morphisms of spectra. Then the natural map*

$$\underset{i}{\mathrm{holim}} X_i \to \underset{q}{\mathrm{hocolim}}\ \underset{i}{\mathrm{holim}} \left(\mathrm{Wh}^q(X_i)\right)$$

is an equivalence.

Proof. Take the homotopy colimit of each column in (14). □

Proposition 2.15. *Let $\{X_i\}$ be a sequence of morphisms of spectra. Suppose that $R^1 \lim_i \pi_*(X_i)$ vanishes. Then there exists a conditionally convergent spectral sequence*

$$E^1_{s,t} \cong k(n)_t \left(\Sigma^s H \lim_i \pi_s X_i\right) \Rightarrow k(n)_t \operatorname{holim}_i X_i \qquad (15)$$
$$d_r : E^r_{s,t} \to E^r_{s+r,t-1}.$$

Proof. For each $i \in \mathbb{N}$, we have the Whitehead filtration $\mathrm{Wh}(X_i)$, a sequence of spectra. Consider the homotopy limit $\operatorname{holim}_i \mathrm{Wh}(X_i)$ of the Whitehead filtrations, and regard $\operatorname{holim}_i \mathrm{Wh}(X_i)$ as a tower of spectra. Applying $k(n)_*$ to that tower yields an exact couple. The claimed spectral sequence (15) is the spectral sequence of that exact couple.

By a connectivity argument exactly like the one used in the proof of Lemma 2.9, for each fixed integer t the group $k(n)_t\left(\operatorname{holim}_i \mathrm{Wh}^s(X_i)\right)$ is trivial for sufficiently large s, so the spectral sequence converges conditionally to the colimit.

A priori, that colimit is $\operatorname{colim}_q k(n)_t\left(\operatorname{holim}_i \mathrm{Wh}^q(X_i)\right)$. We must show that this colimit agrees with the claimed abutment $k(n)_t \operatorname{holim}_i X_i$. This agreement follows from the isomorphisms:

$$\begin{aligned} \operatorname{colim}_q k(n)_t\left(\operatorname{holim}_i \mathrm{Wh}^q(X_i)\right) &\cong k(n)_t\left(\operatorname{hocolim}_q \operatorname{holim}_i \left(\mathrm{Wh}^q(X_i)\right)\right) \\ &\cong k(n)_t(\operatorname{holim}_i X_i) \qquad (16) \end{aligned}$$

where isomorphism (16) is due to Lemma 2.14.

We also must show that the spectral sequence's input is as claimed. We have the fiber sequence

$$\operatorname{holim}_i(\mathrm{Wh}^{s+1}(X_i)) \to \operatorname{holim}_i(\mathrm{Wh}^s(X_i)) \to \operatorname{holim}_i \Sigma^s H\pi_s X_i, \qquad (17)$$

and using the Milnor exact sequence, the right-hand term in (17) is easily seen to be a two-stage Postnikov system with the same homotopy groups as

$$\Sigma^s H \lim_i \pi_s X_i \vee \Sigma^{s-1} H R^1 \lim_i \pi_s X_i. \qquad (18)$$

The vanishing assumption on $R^1 \lim{}_i \pi_*(X_i)$ now ensures that the E^1-term is as claimed. □

The following useful lemma is proven by Lunøe-Nielsen and Rognes, as part of the proof of Proposition 2.2 in their paper [33]:

Lemma 2.16. *Suppose that $\{X_i\}$ is a sequence in the stable homotopy category. Suppose that each X_i is a bounded below spectrum and each graded $\mathbb{F}_p$-module $H_*(X_i; \mathbb{F}_p)$ is finite type. Then $R^1 \lim_i \pi_*((X_i)^\wedge_p) = 0$.*

Given a prime number p, recall that an abelian group is said to be *p-reduced* if it has no nonzero infinitely p-divisible elements.

Lemma 2.17. *Let p be a prime number, let L be an integer, and let $\{X_i\}$ be a sequence of p-complete bounded-below spectra. Make the following assumptions:*

1. *The derived limit $R^1 \lim_i \pi_s(X_i)$ is trivial for all integers s.*
2. *The mod p reduction of the abelian group $\pi_s(\operatorname{holim}_i X_i)$ is finite for all $s \geqslant L$. x*

Then the abelian group $\lim_i k(n)_t(H\pi_s X_i)$ is finite for all integers $s \geqslant L$ and all integers t.

Proof. Let $\tilde{k}(n)$ be any spectrum equipped with a map $\tilde{k}(n) \to k(n)$ such that $\pi_*(\tilde{k}(n)) = \mathbb{Z}_{(p)}[v_n]$ as a graded abelian group, and the map $\pi_*(\tilde{k}(n)) \to \pi_*(k(n))$ is the reduction modulo p map. For example, $\tilde{k}(n)$ can be the spectrum obtained from BP by using BP-module cells to cone off everything in the ideal in BP_* generated by the elements v_i with $i \neq n$.

By the first hypothesis and by the Milnor sequence for the homotopy groups of a sequential homotopy limit, we have isomorphisms

$$\begin{aligned} k(n)_t(\mathrm{holim}_i\, H\pi_s X_i) &\cong k(n)_t(H \lim_i \pi_s X_i) \\ &\cong k(n)_t(H\pi_s \,\mathrm{holim}_i\, X_i) \\ &\cong \tilde{k}(n)_t\left(S/p \wedge H\pi_s \,\mathrm{holim}_i\, X_i\right). \end{aligned}$$

The spectrum $S/p \wedge H\pi_s \,\mathrm{holim}_i\, X_i$ is a two-stage Postnikov system, with

$$\pi_j\left(S/p \wedge H\pi_s \,\mathrm{holim}_i\, X_i\right) \cong \begin{cases} \mathbb{F}_p \otimes_{\mathbb{Z}} \pi_s \,\mathrm{holim}_i\, X_i & \text{if } j = 0 \\ \left(\pi_s \,\mathrm{holim}_i\, X_i\right)[p] & \text{if } j = 1 \\ 0 & \text{if } j \neq 0, 1, \end{cases}$$

where the notation $G[p]$ denotes the p-torsion subgroup of an abelian group G.

Since the mod p reduction of $\pi_s \,\mathrm{holim}_i\, X_i$ was assumed finite, we know that $\pi_0\left(S/p \wedge H\pi_s \,\mathrm{holim}_i\, X_i\right)$ is finite. Since $\pi_s \,\mathrm{holim}_i\, X_i$ is p-reduced[4], the group $\pi_1\left(S/p \wedge H\pi_s \,\mathrm{holim}_i\, X_i\right)$ is also finite. Hence $k(n)_t(\mathrm{holim}_i\, H\pi_s X_i)$ is $\tilde{k}(n)_t$ of a finite Postnikov system whose layers are each Eilenberg-Mac Lane spectra of finite abelian groups. Consequently $k(n)_t(\mathrm{holim}_i\, H\pi_s X_i)$ is finite.

Theorem 2.5 now implies that $\lim_i k(n)_t(H\pi_s X_i)$ is a quotient of the finite group $k(n)_t(\mathrm{holim}_i\, H\pi_s X_i)$. Hence $\lim_i k(n)_t(H\pi_s X_i)$ is finite. □

Theorem 2.18. *Let p be a prime number, let M be an integer, and let n be a positive integer. Suppose we have a sequence $\{X_i\}$ of bounded-below p-complete spectra satisfying the following conditions:*

- *$R^1 \lim_i \pi_* X_i$ is trivial.*
- *There exists an integer L such that the mod p reduction of the abelian group $\pi_s(\mathrm{holim}_i\, X_i)$ is finite for all $s \geqslant L$.*
- *For each integer i, the spectrum X_i satisfies condition $H(M)$.*
- *$k(n)_*(X_i)$ is simple v_n-torsion for all $i \in \mathbb{N}$.*
- *For each i, the graded $\mathbb{F}_p$-module $H_*(X_i; \mathbb{F}_p)$ is finite type.*

Then the homotopy limit $\mathrm{holim}_i\, X_i$ is $K(n)$-acyclic.

Proof. Same argument as in Theorem 2.11, using spectral sequence (15) in place of spectral sequence (6). There is one point worth commenting on: in the proof of Theorem 2.11, we invoked the finiteness of the product $\prod_{i \in I} k(n)_t\left(\Sigma^s H\pi_s(X_i)\right)$ to bound the lengths of nonzero differentials hitting a given bidegree. In the present setting, we instead need to know that $\lim_i k(n)_t\left(\Sigma^s H\pi_s(X_i)\right)$ is finite. This is a consequence of Lemmas 2.16 and 2.17. We remark that this argument requires the assumption that each X_i is p-complete (because of the p-completion appearing in Lemma 2.16), which we did not need for Theorem 2.11. □

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[4]The p-reducedness of $\pi_s \,\mathrm{holim}_i\, X_i \cong \lim_i \pi_s X_i$ follows from the fact that p-reducedness is a property preserved by sequential limits, and the fact that each X_i is p-complete, hence has p-reduced homotopy groups. See Proposition 2.5 of [13] and VI.3.4(i) of [14] for this last implication.

Remark 2.19. Let $\{X_i\}$ be a sequence of spectra satisfying the hypotheses of Theorem 2.18. Then since $k(n)_*(X_i)$ is simple v_n-torsion, it is clear that $K(n)_*(X_i) = 0$ for each i, so $\lim_i K(n)_* X_i = 0$. It is therefore a consequence of Theorem 2.18 that

$$\lim_i K(n)_* X_i = K(n)_*(\lim_i X_i),$$

i.e., under these hypotheses, $K(n)$-homology commutes with a *non-uniformly-bounded-below* homotopy limit.

Fix a prime p as well as integers M and $n \geqslant 1$ for the remainder of this section.

Definition 2.20. *Let $\{X_i\}$ be a sequence in the stable homotopy category. We say that the sequence $\{X_i\}$ is* $K(n)$-amenable with parameter M *if the following conditions are all satisfied:*

(1) *each X_i is bounded below and p-complete,*
(2) *$H_*(X_i; \mathbb{F}_p)$ is finite type for all i,*
(3) *the sequence of graded $\mathbb{F}_p$-vector spaces $\{H(H_*(X_i; \mathbb{F}_p), Q_n)\}$ is pro-isomorphic to zero,*
(4) *there exists an integer L such that for each $s \geqslant L$ the groups $\pi_s(\lim_i X_i)/p$ are finite, and*
(5) *for each i, the spectrum X_i satisfies condition $H(M)$.*

If the parameter M is clear from context, then we simply say that the sequence $\{X_i\}$ is $K(n)$-amenable.

Remark 2.21. Recall that a map $V \to W$ of pro-objects in graded $\mathbb{F}_p$-modules is a monomorphism if its kernel is pro-isomorphic to the zero sequence. A sequence $\{Z_i\}$ of abelian groups is pro-isomorphic to zero if for each s there exists a $t \geqslant s$ such that $Z_t \to Z_s$ is the zero map. Hence a map of sequences $\{V_i\} \to \{W_i\}$ is a pro-monomorphism if for each s there exists a $t \geqslant s$ such that

$$\ker(V_t \to W_t) \to \ker(V_s \to W_s)$$

is the zero map.

Lemma 2.22. *If $\{X_i\}$ is a $K(n)$-amenable sequence with parameter M, then the canonical map*

$$\{k(n)_*(X_i)\} \to \{k(n)_*(X_i^{<M})\}$$

is a monomorphism in the category of pro-objects in graded $\mathbb{F}_p$-modules.

Proof. The naturality of the edge homomorphism in the Adams spectral sequence for $k(n) \wedge X_i$ provides a morphism of pro-objects in the category of graded $\mathbb{F}_p$-vector spaces

$$\{k(n)_*(X_i)\} \to \{\hom_{E(\tau_n)}(\mathbb{F}_p, H_*(X_i; \mathbb{F}_p))\}. \tag{19}$$

Each element of the kernel of the edge map $k(n)_*(X_i) \to \mathrm{Hom}_{E(\tau_n)}(\mathbb{F}_p, H_*(X_i; \mathbb{F}_p))$, can be represented by an element in the E_2-page of the Adams spectral sequence converging to $k(n)_*(X_i)$. That element is in $\mathrm{Ext}^{s,*}_{E(\tau_n)}(\mathbb{F}_p, H_*(X_i; \mathbb{F}_p))$ for some $s > 0$. Hence, by Corollary A.6, the pro-triviality of $\{H(H_*(X_i; \mathbb{F}_p), Q_n)\}$ implies that the map (19) of pro-vector-spaces is a monomorphism.

Since there is a natural canonical inclusion

$$\mathrm{inc}\colon\ \mathrm{Hom}_{E(\tau_n)}(\mathbb{F}_p, H_*(X_i; \mathbb{F}_p)) \subset H_*(X_i; \mathbb{F}_p)$$

we have a natural monomorphism of pro-objects in the category of graded $\mathbb{F}_p$-vector spaces

$$\{k(n)_*(X_i)\} \to \{H_*(X_i;\mathbb{F}_p)\}.$$

By naturality, the diagram

$$\begin{array}{ccccc} k(n)_*(X_i) & \longrightarrow & \mathrm{Hom}_{E(\tau_n)}(\mathbb{F}_p, H_*(X_i;\mathbb{F}_p)) & \xrightarrow{\text{inc}} & H_*(X_i;\mathbb{F}_p) \\ \downarrow & & \downarrow & & \downarrow \\ k(n)_*(X_i^{<M}) & \longrightarrow & \mathrm{Hom}_{E(\tau_n)}(\mathbb{F}_p, H_*(X_i^{<M};\mathbb{F}_p)) & \xrightarrow{\text{inc}} & H_*(X_i^{<M};\mathbb{F}_p) \end{array} \tag{20}$$

commutes. We have shown that the composite of the top horizontal maps in diagram (20) is a pro-monomorphism. The right-hand vertical map in (20) is a levelwise monomorphism, since each spectrum X_i satisfies condition $H(M)$. Hence the left-hand vertical map in (20) must also be a pro-monomorphism, as claimed. □

Finally, we prove the main theorem.

Theorem 2.23. *Suppose $\{X_i\}$ is a $K(n)$-amenable sequence with parameter M. Then* $\mathrm{holim}_i X_i$ *is $K(n)$-acyclic.*

Proof. Essentially the same argument as that of Theorem 2.11, although some care is required to establish the horizontal vanishing line in spectral sequence (15). Our argument for that vanishing line is as follows. Recall (e.g. from Scholie 3.5 in the appendix of [7]) that a map of sequences of abelian groups $\{A_i\} \to \{B_i\}$ is pro-zero if and only if, for every integer i, there exists an integer $j(i) \geqslant i$ such that the composite map of abelian groups $A_{j(i)} \to A_i \to B_i$ is zero.

For each integer k, let $\gamma(k)$ be the minimum of the connectivities of the spectra $X_0, \dots, X_k$, so that $Wh^{\gamma(k)}(X_i) \cong X_i$ for all $i \leqslant k$. By Lemma 2.22, the map of sequences of graded abelian groups $\{k(n)_* Wh^M(X_i)\} \to \{k(n)_*(X_i)\}$ is pro-zero, i.e., zero in the pro-category of graded abelian groups[5]. Consequently, for each integer i, there exists an integer $j(i) \geqslant i$ such that $k(n)_* \mathrm{Wh}^M(X_{j(i)}) \to k(n)_*(X_i)$ is zero.

Hence, given an element $x = (x_0, x_1, \dots)$ of $\lim_i k(n)_t \mathrm{Wh}^M(X_i)$, for each given initial subsequence $(x_0, \dots, x_k)$ of x, the image of $x_{j(k)} \in k(n)_t \mathrm{Wh}^M(X_{j(k)})$ in $k(n)_t \mathrm{Wh}^{\gamma(j(k))}(X_k)$ is zero. By the compatibility of the elements of the sequence $(x_0, x_1, \dots)$, the image of x_i in $k(n)_t \mathrm{Wh}^{\gamma(j(k))}(X_k)$ is zero for all $i \leqslant k$. Consequently the map

$$\lim_i k(n)_t \mathrm{Wh}^M(X_i) \to \operatorname*{colim}_s \lim_i k(n)_t \mathrm{Wh}^s(X_i)$$

sends every initial subsequence of $(x_0, x_1, \dots)$ to zero. Hence x maps to zero in the colimit $\mathrm{colim}_s \lim_k k(n)_t \mathrm{Wh}^M(X_k)$, by the same argument (using finite-dimensionality of $\lim_{i\in I} k(n)_t(H\pi_s X_i)$ for all $s \geqslant M$) as in the proof of Theorem 2.23. □

[5] To be clear: "zero in the pro-category of graded abelian groups" is a stronger condition than "each individual degree is pro-zero in the category of abelian groups." Lemma 2.22 yields the stronger of these two conditions.

3. A higher chromatic height analogue of Mitchell's theorem

In this section, we give a sample application of the main theorem of this paper, Theorem 2.23. Our sample application is to the Morava K-theory of the topological periodic cyclic homology and the topological negative cyclic homology of $B\langle n\rangle$, where $B\langle n\rangle$ is any p-primary E_2 form of $BP\langle n\rangle$ satisfying Running Assumption 1.3. See Definition 1.2 for the definition of a "p-primary E_2 form of $BP\langle n\rangle$."

In [38], Mitchell proved that $K(m)_*(K(\mathbb{Z})) \cong 0$ for $m \geqslant 2$, and consequently $K(m)_*(K(R)) \cong 0$ for any $H\mathbb{Z}$-algebra R. We might consider the following higher chromatic height analogue of Mitchell's theorem.

Question 3.1. Suppose n is some integer, $n \in [-1, \infty)$ and $B\langle n\rangle$ is a p-primary E_2 form of $BP\langle n\rangle$. If R is a $B\langle n\rangle$-algebra spectrum, then does $K(m)_*(K(R))$ vanish for all $m \geqslant n + 2$?

This gives an upper bound on the chromatic complexity of the algebraic K-theory of a $B\langle n\rangle$-algebra. By Ravenel [42, Thm. 2.1(d),(f),(i)], we know that

$$K(m)_*(B\langle n\rangle) \cong 0$$

for $m \geqslant n + 1$, and every $B\langle n\rangle$-algebra is likewise $K(m)$-acyclic. Consequently a positive answer to Question 3.1 implies that, if there is a "red-shift" in algebraic K-theory of a $B\langle n\rangle$-algebra spectrum, then this shift is a shift of at most one.

The main goal of this section is to answer Question 3.1 for all (n, p) such that there exists a p-primary E_2 form $B\langle n\rangle$ of $BP\langle n\rangle$ satisfying Running Assumption 1.3. Question 3.1 is already known to have a positive result for $n = -1$ by Quillen [41], for $n = 0$ by Mitchell [38], and for $n = 1$ and $p \geqslant 5$ by Ausoni–Rognes [8] after p-completion. Consequently our main new contributions are the cases $n = 1, 2$ at $p = 3$, though we do not know of obstacles to working out the same calculations at $p = 2$.

Remark 3.2. Our results rely on the calculation, by Bruner and Rognes in [15], of the continuous homology $H^c_*(TC^-(BP\langle n\rangle); \mathbb{F}_p)$, for all primes p and integers n. The proof uses the fact that when $BP\langle n\rangle$ is an E_∞ ring spectrum than it admits certain Kudo–Araki–Dyer–Lashof operations.

If n is sufficiently large, then the spectrum $BP\langle n\rangle$ is known to *not* admit an E_∞ ring structure: at $p = 2$, this is a theorem of Lawson [31], and at $p > 2$, a theorem of Senger [45]. This seems to suggest that the main theorem in this section, Theorem 3.10, cannot have consequences for large integers n. On the other hand, the spectrum BP was shown to admit an E_4 ring spectrum model by Basterra–Mandell [11]. Since the first draft of this paper appeared in preprint form, Hahn–Wilson [22] have proven that there are specific forms of $BP\langle n\rangle$ built from $MU_{(p)}$, which are E_3 ring spectra, and consequently THH of such a form of $BP\langle n\rangle$ is an E_2-ring spectrum. If the calculations of Bruner–Rognes [15] can be made to work using only the E_m Dyer–Lashof–Kudo–Araki operations, for appropriate values of m, rather than the classical (E_∞) Dyer–Lashof–Kudo–Araki operations, then our Theorem 3.10 would yield a positive answer to Question 3.1 for all primes p and heights n.

Remark 3.3. Since the first draft of this paper was posted, the papers [16, 30] appeared. These papers establish that Question 3.1 has a positive answer at all primes for any p-primary E_1 form of $BP\langle n\rangle$. Their work uses entirely different methods from ours, and our work applies to approximations of algebraic K-theory such as TC^- and TP, whereas their approach works directly with algebraic K-theory.

3.1. **Trace methods.** Now we briefly recall the setup for topological periodic cyclic homology and topological negative cyclic homology. Let R be an E_1 ring spectrum. Write $\mathbb{T}$ for the circle, regarded as the compact Lie group of unit vectors in the complex numbers. It is well-known that the topological Hochschild homology of R, denoted $THH(R)$, has a canonical action of the circle group $\mathbb{T}$. Let $E\mathbb{T} = S(\mathbb{C}^\infty)$ be the unit sphere in $\mathbb{C}^\infty$ where $\mathbb{T}$ acts on $\mathbb{C}^\infty$ coordinate-wise. Therefore, $E\mathbb{T}$ is a $\mathbb{T}$-space with free $\mathbb{T}$-action whose underlying space is contractible. We also consider the homotopy cofiber sequence

$$E\mathbb{T}_+ \to S^0 \to \widetilde{E}\mathbb{T}$$

in topological spaces where the map $E\mathbb{T}_+ \to S^0$ is induced by the unique map $E\mathbb{T} \to *$ of unpointed topological spaces.

We define *topological negative cyclic homology* as the homotopy fixed-point spectrum

$$TC^-(R) := THH(R)^{h\mathbb{T}} = F(E\mathbb{T}_+, THH(R))^{\mathbb{T}}$$

and *topological periodic cyclic homology* as the Tate spectrum

$$TP(R) := THH(R)^{t\mathbb{T}} = \left(\widetilde{E}\mathbb{T} \wedge F(E\mathbb{T}_+, THH(R))\right)^{\mathbb{T}}.$$

There is a homological $\mathbb{T}$-homotopy fixed point spectral sequence

$$E_2^{*,*}(R) = H^*(\mathbb{T}, H_*(THH(R);\mathbb{F}_p)) \Rightarrow H_*^c(TC^-(R);\mathbb{F}_p). \tag{21}$$

The notation H_*^c refers to *continuous homology,* and is defined so that

$$H_*^c(TC^-(R);\mathbb{F}_p) = \lim_k H_*(TC^-(R)[k];\mathbb{F}_p),$$

where

$$TC^-(R)[k] = F\left((E\mathbb{T}^{(2k)})_+, THH(R)\right)^{\mathbb{T}},$$

and where $E\mathbb{T}^{(2k)}$ is the $2k$-skeleton of the standard presentation for $E\mathbb{T}$ as an $\mathbb{T}$-CW complex. Explicitly, we let $E\mathbb{T}^{(2k)} = S(\mathbb{C}^{k+1})$, and it is obtained from $E\mathbb{T}^{(2k-2)}$ by attaching a single $\mathbb{T}$-cell of dimension $2k$. We refer the reader to [15, Sec. 2] for a more detailed account.

There is also a homological $\mathbb{T}$-Tate spectral sequence

$$\widehat{E}_2^{*,*}(R) = \hat{H}^*(\mathbb{T}, H_*(THH(R);\mathbb{F}_p)) \Rightarrow H_*^c(TP(R);\mathbb{F}_p), \tag{22}$$

with abutment $H_*^c(TP(R);\mathbb{F}_p) := \lim_k H_*(TP(R)[k];\mathbb{F}_p)$. The filtration

$$\cdots \to TP(R)[1] \to TP(R)[0] \to TP(R)[-1] \to \ldots$$

is the *Greenlees filtration* [19], and is given by

$$TP(R)[k] := \left(\widetilde{E}\mathbb{T}/\widetilde{E}\mathbb{T}_{-2k} \wedge F(E\mathbb{T}_+, THH(R))\right)^{\mathbb{T}}. \tag{23}$$

The definition of the $\mathbb{T}$-equivariant spectrum $\widetilde{E}\mathbb{T}_{2k}$ depends on whether k is negative or not, as follows.

If $k \geqslant 0$: then $\widetilde{E}\mathbb{T}_{2k}$ is the cofiber of the map $E\mathbb{T}_+^{(2k)} \to S^0$, again induced by the unique unpointed $\mathbb{T}$-equivariant map $E\mathbb{T}^{(2k)} \to *$.

If $k < 0$: then $\widetilde{E}\mathbb{T}_{2k}$ is the Spanier-Whitehead dual of $\widetilde{E}\mathbb{T}_{-2k+2}$.

We will need to make use of two more related spectral sequences. By considering the filtrations on $E\mathbb{T}$ and on $\widetilde{E}\mathbb{T}$ only in a range of dimensions, there is also a spectral sequence

$$E_2^{*,*}(R)[k] = P_{k+1}(t) \otimes H_*(THH(R);\mathbb{F}_p) \Rightarrow H_*(TC^-(R)[k];\mathbb{F}_p) \tag{24}$$

called the *approximate homotopy fixed point spectral sequence*. There is also a spectral sequence

$$\widehat{E}_2^{*,*}(R)[k] = P(t^{-1})\{t^k\} \otimes H_*(THH(R);\mathbb{F}_p) \Rightarrow H_*(TP(R)[k];\mathbb{F}_p), \tag{25}$$

called the *approximate Tate spectral sequence*, where

$$P(t^{-1})\{t^k\} = \{xt^k : x \in P(t^{-1})\}.$$

Each of these four spectral sequences strongly converge when $\pi_*(THH(R))$ is bounded below and $H_m(THH(R);\mathbb{F}_p)$ is finite for all m. All of these spectral sequences are spectral sequences of A_*-comodules; this follows from the same proof as given in [15, Prop. 2.1] in the case of the spectral sequence (21). When $H_m(THH(R);\mathbb{F}_p)$ is finite for all m and bounded below, we also know that these are all spectral sequences of $E(Q_m)$-modules, where Q_m is the Milnor primitive [36] in the Steenrod algebra A, dual to the indecomposable $\bar{\tau}_m \in A_*$. In other words, the differentials are Q_m-linear.

More generally, for a generalized homology theory E_* we write

$$E_*^c(TP(R)) := \lim_k E_*(TP(R)[k]) \text{ and}$$
$$E_*^c(TC^-(R)) := \lim_k E_*(TC^-(R)[k]).$$

The mod p homology of topological Hochschild homology comes equipped with an operator σ, sometimes called the *circle operator*. The circle operator σ is induced by the canonical map

$$\sigma\colon\ R \wedge \mathbb{T} \to R \wedge \mathbb{T}_+ \to THH(R)$$

so that $\sigma x \in H_*(THH(R);\mathbb{F}_p)$ is the image of

$$x \otimes \iota \in H_*(R;\mathbb{F}_p) \otimes H_*(\mathbb{T};\mathbb{F}_p) \cong H_*(R \wedge \mathbb{T};\mathbb{F}_p) \subset H_*(R \wedge \mathbb{T}_+;\mathbb{F}_p)$$

where ι is the canonical generator of $H_1(\mathbb{T};\mathbb{F}_p)$. Note that this operator is compatible with the canonical $\mathbb{T}$-action

$$\alpha\colon\ THH(R) \wedge \mathbb{T} \to THH(R) \wedge \mathbb{T}_+ \to THH(R)$$

in the sense that

$$\sigma(x) = \alpha(\eta(x) \otimes \iota)$$

where $\eta\colon\ H_*(R) \to H_*(THH(R))$ is the unit map of R algebra $THH(R)$ and we write α for the induced map

$$\alpha\colon\ H_*(THH(R)) \otimes H_*(\mathbb{T}) \to H_*(THH(R) \wedge \mathbb{T}_+) \to H_*(THH(R))$$

by abuse of notation. Also, by [6, Prop. 5.10] we know that

$$\sigma(x \cdot y) = x \cdot \sigma(y) + (-1)^{|y|}\sigma(x) \cdot y$$

for $x, y \in H_*(R;\mathbb{F}_p)$, so in particular, $\sigma(x^p) = 0$ whenever $|x|$ is even. In particular, $\sigma(1) = 0$.

3.2. **A higher height Mitchell theorem.** We now recall that by Angeltveit–Rognes [6, Thm. 5.12], for odd primes p, there is an isomorphism

$$H_*(THH(BP\langle n\rangle);\mathbb{F}_p) \cong H_*(BP\langle n\rangle;\mathbb{F}_p) \otimes_{\mathbb{F}_p} E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \ldots, \sigma\bar{\xi}_{n+1}) \otimes_{\mathbb{F}_p} P(\sigma\bar{\tau}_{n+1})$$

of $H_*(BP\langle n\rangle;\mathbb{F}_p)$-algebras. The left A_*-coaction

$$\nu_n\colon\ H_*(THH(BP\langle n\rangle);\mathbb{F}_p) \to A_* \otimes_{\mathbb{F}_p} H_*(THH(BP\langle n\rangle);\mathbb{F}_p)$$

is given as follows:

- On elements in $H_*(BP\langle n\rangle;\mathbb{F}_p) \subseteq H_*(THH(BP\langle n\rangle);\mathbb{F}_p)$, ν_n is simply the restriction of the coproduct of A_* to $H_*(BP\langle n\rangle;\mathbb{F}_p) \subset A_*$.
- On elements in $H_*(THH(BP\langle n\rangle);\mathbb{F}_p)$ of the form $\sigma\bar{\xi}_i$ for $1 \leqslant i \leqslant n$, and on the element $\sigma\bar{\tau}_{n+1}$, ν_n is given by the formula

$$\nu_n(\sigma x) = (1 \otimes \sigma)(\nu_n(x)) \tag{26}$$

  from [6, Eq. 511].
- On the remaining elements of $H_*(THH(BP\langle n\rangle);\mathbb{F}_p)$, the coaction ν_n is given by the formula $\nu_n(xy) = \nu_n(x)\nu_n(y)$.

In particular, because $\sigma(\bar{\xi}_i^{p^j}) = 0$ for $j \geqslant 1$, formulas (2) and (26) imply that the elements $\sigma\bar{\xi}_k$ are comodule primitives. Formulas (3) and (26) imply that

$$\nu(\sigma\bar{\tau}_{n+1}) = 1 \otimes \sigma\bar{\tau}_{n+1} + \bar{\tau}_0 \otimes \sigma\bar{\xi}_{n+1}, \tag{27}$$

which also appears in Theorem 5.12 of [6].

Next, we recall the computation of continuous homology of topological negative cyclic homology and topological periodic cyclic homology of $BP\langle n\rangle$. Bruner and Rognes [15, Proposition 6.1] prove that the E_∞ term of the homological homotopy fixed-point spectral sequence (21) admits an isomorphism

$$E_\infty^{*,*}(BP\langle n\rangle) \cong \big(P(t) \otimes_{\mathbb{F}_p} M_1 \otimes_{\mathbb{F}_p} M_2\big) \oplus T$$

where

(28)

$$M_1 = E(\tau'_{j+1} \mid j \geqslant n+1),$$

(29)

$$M_2 = P(\bar{\xi}_{j+1} | j \geqslant n+1) \otimes P(\bar{\xi}_j^p \mid 1 \leqslant j \leqslant n+1) \otimes E(\bar{\xi}_j^{p-1}\sigma\bar{\xi}_j \mid 1 \leqslant j \leqslant n+1),$$

and where T consists of classes x in filtration $s = 0$ with $tx = 0$ and with

$$\tau'_{k+1} = \bar{\tau}_{k+1} - \bar{\tau}_k(\sigma\bar{\tau}_k)^{p-1}$$

for $k \geqslant m+1$.

One can then easily deduce the computation of the E_∞ term

$$\widehat{E}_\infty^{*,*}(BP\langle n\rangle) \cong P(t, t^{-1}) \otimes_{\mathbb{F}_p} M_1 \otimes_{\mathbb{F}_p} M_2$$

of the homological $\mathbb{T}$-Tate spectral sequence (22) for $BP\langle n\rangle$, where M_1 and M_2 are as defined in (28) and (29). There are no possible additive extensions because the abutment is a graded $\mathbb{F}_p$-vector space. It will also be useful to record the structure of the E_∞-page of the approximate Tate spectral sequence (25). There is an isomorphism of $E(Q_m)$-modules, for all $m \geqslant n+2$,

$$\widehat{E}_\infty^{*,*}(BP\langle n\rangle)[k] = \big(P(t^{-1})\{t^{k-1}\} \otimes_{\mathbb{F}_p} M_1 \otimes_{\mathbb{F}_p} M_2\big) \oplus V_k(n)\{t^k\} \tag{30}$$

where

$$V_k(n) = H_*(THH(BP\langle n\rangle); \mathbb{F}_p)/\operatorname{im}(d_2^{2-2k,*}). \tag{31}$$

Here $d_2^{2-2k,*}$ denotes the differential in the approximate Tate spectral sequence (25), regarded as a map

$$d_2^{2-2k,*}\colon\ H_*(THH(BP\langle n\rangle; \mathbb{F}_p) \to H_{*+1}(THH(BP\langle n\rangle; \mathbb{F}_p).$$

Note that the spectral sequence (25) collapses at the E_3-term as a consequence of [15, Prop. 6.1], so we do not need to consider the image of longer differentials in our description of $V_k(n)$.

Now we prove a lemma that will be useful for the main Margolis homology calculation.

Lemma 3.4. *Suppose k is an integer, and suppose that*

$$\begin{array}{ccccccc} \dots & \hookrightarrow & F_2 & \hookrightarrow & F_1 & \hookrightarrow & F_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \hookrightarrow & F'_2 & \hookrightarrow & F'_1 & \hookrightarrow & F'_0 \end{array}$$

is a map of filtered graded $E(Q_m)$-modules satisfying the following conditions:

1. $F_j = 0$ *for* $j \geqslant k+2$,
2. $F'_j = 0$ *for* $j \geqslant k+1$,
3. $F_j = F'_j$ *for* $j < k$,
4. *and* $H(F_j/F_{j+1}, Q_m)$ *and* $H(F'_j/F'_{j+1}, Q_m)$ *are trivial for all* $j \leqslant k$.

Then the map of graded $\mathbb{F}_p$-vector spaces induced in Margolis homology

$$H(F_0, Q_m) \longrightarrow H(F'_0, Q_m)$$

is the zero map.

Proof. By the long exact sequences in Margolis homology induced by the short exact sequences

$$0 \longrightarrow F_{j+1} \longrightarrow F_j \longrightarrow F_j/F_{j+1} \longrightarrow 0$$

$$0 \longrightarrow F'_{j+1} \longrightarrow F'_j \longrightarrow F'_j/F'_{j+1} \longrightarrow 0$$

and the assumptions that $H(F_j/F_{j+1}, Q_m) = H(F'_j/F'_{j+1}, Q_m) = 0$ for $j \leqslant k$, we know that the induced maps $H(F_{j+1}, Q_m) \xrightarrow{\cong} H(F_j, Q_m)$ and $H(F'_{j+1}, Q_m) \xrightarrow{\cong} H(F'_j, Q_m)$ are isomorphisms for $j \leqslant k$. When $j = k$, the map of long exact sequences in Margolis homology associated to the map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{k+1} & \longrightarrow & F_k & \longrightarrow & F_k/F_{k+1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & F'_k & \longrightarrow & F'_k & \longrightarrow & 0 \end{array}$$

has a subdiagram of the form

$$\begin{array}{ccccc} H(F_{k+1},Q_m) & \xrightarrow{\cong} & H(F_k,Q_m) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H(F'_k,Q_m) & \xrightarrow{\cong} & H(F'_k,Q_m) \end{array}$$

so the map $H(F_k,Q_m) \to H(F'_k,Q_m)$ is the zero map. Now the compatible isomorphisms

$$H(F_k,Q_m) \cong H(F_{k-1},Q_m) \cong H(F_{k-2},Q_m) \cong \dots \quad \text{and}$$
$$H(F'_k,Q_m) \cong H(F'_{k-1},Q_m) \cong H(F'_{k-2},Q_m) \cong \dots$$

yield that $H(F_j,Q_m) \longrightarrow H(F'_j,Q_m)$ is also the zero map for all $j<k$. □

We now apply this lemma in the main example of interest.

Proposition 3.5. *Let p be odd, and let $B\langle n\rangle$ be a p-primary E_2 form of $BP\langle n\rangle$. There is an isomorphism of pro-objects in abelian groups*

$$\{H(H_*(TP(B\langle n\rangle)[k];\mathbb{F}_p),Q_m)\}_{k\in\mathbb{Z}} \cong 0$$

for all $m \geqslant n+2$. Consequently, there is an isomorphism

$$\lim_k H(H_*(TP(B\langle n\rangle)[k];\mathbb{F}_p),Q_m) \cong 0.$$

Proof. We will show that each of the induced maps

$$H(H_*(TP(B\langle n\rangle)[k];\mathbb{F}_p),Q_m) \to H(H_*(TP(B\langle n\rangle)[k-1];\mathbb{F}_p),Q_m)$$

is the zero map for $m \geqslant n+2$, which implies the result.

First, there is a map of $E(Q_m)$-module spectral sequences

$$\begin{array}{ccc} P(t^{\pm 1}) & \longrightarrow & P(t^{-1})\{t^k\} \otimes_{\mathbb{F}_p} H_*(THH(B\langle n\rangle;\mathbb{F}_p) \\ \Downarrow & & \Downarrow \\ H^c_*(TP(S);\mathbb{F}_p) & \longrightarrow & H_*(TP(B\langle n\rangle;\mathbb{F}_p)[k]) \end{array}$$

where the first spectral sequence collapses for bidegree reasons. The Q_m-action is trivial on t^j in $H^c_*(TP(S);\mathbb{F}_p)$, because $H^c_*(TP(S);\mathbb{F}_p)$ is concentrated in even degrees. Consequently $Q_m(t^j)=0$ in the abutment $H_*(TP(B\langle n\rangle;\mathbb{F}_p)[k])$ of the approximate Tate spectral sequence (25) for all m and all $k \geqslant j > -\infty$.

Our next task is to compute the Margolis homology $H(\mathbb{F}_p\{t^j\}\otimes_{\mathbb{F}_p}M_1\otimes_{\mathbb{F}_p}M_2,Q_m)$ where M_1 and M_2 are the $E(Q_m)$-modules defined in (28) and (29), and where $\mathbb{F}_p\{t^j\}$ is the graded $\mathbb{F}_p$-vector subspace of $P(t^{-1})\{t^k\}$ spanned by t^j. The $\mathbb{F}_p$-vector space $\mathbb{F}_p\{t^j\}$ is equipped with the $E(Q_m)$-module structure in which Q_m acts trivially. The tensor product $\mathbb{F}_p\{t^j\}\otimes_{\mathbb{F}_p} M_1 \otimes_{\mathbb{F}_p} M_2$ is then a $E(Q_m)$-module using the usual action of Q_m on a tensor product of $E(Q_m)$-modules.

Recall that $M_1 \cong E(\tau'_{n+1},\tau'_{n+2},\dots)$. Write $\overline{M}_1$ for the exterior algebra on the generators $\tau'_{n+1},\tau'_{n+2},\dots$ *except* for τ'_m. We claim that $M_1 \cong E(\tau'_m)\otimes\overline{M}_1$ as $E(Q_m)$-modules. This tensor splitting follows from the formula

$$Q_m(\tau'_k) = \bar{\xi}_{k-m}^{p^{k-m}}, \tag{32}$$

which in turn follows from the A_*-coaction on τ'_k being given by

$$\nu(\tau'_k) = 1 \otimes \tau'_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}, \tag{33}$$

as a consequence of (27).

As another consequence of (32), $E(Q_m)$ acts freely on $E(\tau'_m)$. Hence $\mathbb{F}_p\{t^j\} \otimes_{\mathbb{F}_p} M_1 \otimes_{\mathbb{F}_p} M_2$ has a free $E(Q_m)$-module as a tensor factor. This is enough to conclude that $\mathbb{F}_p\{t^j\} \otimes_{\mathbb{F}_p} M_1 \otimes_{\mathbb{F}_p} M_2$ is free over Q_m, hence its Margolis Q_m-homology is trivial.

We then write $F_\bullet^{(k)}$ for the abutment of the approximate Tate spectral sequence (25), regarded as a filtered graded $E(Q_m)$-module. Consider the map of filtered graded $E(Q_m)$-modules $F_\bullet^{(k+1)} \to F_\bullet^{(k)}$. By the calculation of Bruner and Rognes (see (34), above), for any integers j, k satisfying $j < k$, there are isomorphisms

$$F_j^{(k)}/F_{j+1}^{(k)} \cong \mathbb{F}_p\{t^j\} \otimes_{\mathbb{F}_p} M_1 \otimes_{\mathbb{F}_p} M_2. \tag{34}$$

Hence there are identifications $F_j^{(k+1)} \cong F_j^{(k)}$ for $j < k$, and $F_j^{(k)} = 0$ for $j > k$, and in the preceding paragraphs of this proof, we showed that the Margolis Q_m-homology of the quotients $F_j^{(k)}/F_{j+1}^{(k)}$ is trivial for $j \leqslant k$. That confirms all the hypotheses of Lemma 3.4. The lemma then implies the claimed result. □

Lemma 3.6. *Let p be odd. Fix a positive integer k. Suppose L is a graded A_*-comodule such that $L \cong P(t^{-1})\{t^k\} \otimes_{\mathbb{F}_p} B \otimes_{\mathbb{F}_p} E$, where*

- *B is isomorphic to a graded comodule subalgebra of the dual Steenrod algebra A_*,*
- *E is a graded A_*-comodule, finite-dimensional as a $\mathbb{F}_p$-vector space, and consisting only of A_*-comodule primitives,*
- *and we have an isomorphism $P(t^{-1})\{t^k\} \cong H_*(TP(S)[k]; \mathbb{F}_p)$ of A_*-comodules.*

Then the A_*-comodule primitives in L are bounded above by $\sum_{i \in I} |e_i|$, where $\{e_i : i \in I\}$ is a homogeneous $\mathbb{F}_p$-linear basis for E, and where $|e_i|$ is the degree of e_i.

Proof. Recall that the j-th stage in the Greenlees filtration of $TP(S)$ has homology

$$H_*(TP(S)[j]; \mathbb{F}_p) \cong P(t^{-1})\{t^j\} = \{xt^j : x \in P(t^{-1})\}$$

as a quotient of $H_*(TP(S); \mathbb{F}_p) \cong P(t^{\pm 1})$. Hence we may consider L as a filtered graded $\mathbb{F}_p$-vector space

$$L = L_k \supset L_{k-1} \supset \cdots \supset L_{-\infty} = 0,$$

by defining L_j as

$$L_j = H_*(TP(S)[j]; \mathbb{F}_p) \otimes_{\mathbb{F}_p} B \otimes_{\mathbb{F}_p} E.$$

We refer to this filtration as the t-weight filtration on L. Specifically, t^j has t-weight j. (Note that higher t-weight corresponds to lower degree, since t^j is in degree $-2j$ in homology.)

We may regard the $\mathbb{F}_p$-vector space quotient L_j/L_{j-1} of L_j also as a graded $\mathbb{F}_p$-submodule of L, by identifying the quotient L_j/L_{j-1} with

$$B \otimes_{\mathbb{F}_p} E\{t^j\} = \{zt^j : z \in B \otimes_{\mathbb{F}_p} E\} \subset L.$$

Since B is isomorphic to a comodule subalgebra of the dual Steenrod algebra, the only A_*-comodule primitive of B is the element 1. Writing

$$\psi_B \colon B \to A_* \otimes_{\mathbb{F}_p} B$$

for the A_*-coaction, we have

$$\psi_B(x_i) = 1 \otimes x_i + \sum_{j \geqslant 0} y^i_j \otimes z^i_j.$$

If $x_i \in B$ is not a $\mathbb{F}_p$-linear multiple of 1, then the sum $\sum_{j \geqslant 0} y^i_j \otimes z^i_j$ must be nonzero, or otherwise x_i would be an A_*-comodule primitive. We also know that $t \in P(t^{-1})\{t^k\}$ has coaction[6]

$$\psi_k(t) = \sum_{j \geqslant 0} \bar{\xi}_j \otimes t^{p^j} \tag{35}$$

in which terms involving a power t^{p^j} with $p^j > k$ are omitted. More generally, we can compute the coaction on $P(t^{-1})\{t^k\}$ by the formula

$$\psi_k(t^i) = (\sum_{j \geqslant 0} \xi_j \otimes t^{p^j})^i$$

in which the sum is taken in the graded $\mathbb{F}_p$-algebra $\mathbb{F}_p[t^{\pm 1}]$, and then terms with t-weight $> k$ are omitted.

Since E consists only of A_*-comodule primitives, the natural map

$$\mathrm{Hom}_{A_*}(\mathbb{F}_p, M) \otimes_{\mathbb{F}_p} E \to \mathrm{Hom}_{A_*}(\mathbb{F}_p, M \otimes_{\mathbb{F}_p} E)$$

is an isomorphism for any A_*-comodule M. Hence there is an isomorphism

$$\mathrm{Hom}_{A_*}(\mathbb{F}_p, P(t^{-1})\{t^k\} \otimes_{\mathbb{F}_p} B \otimes_{\mathbb{F}_p} E) \cong \mathrm{Hom}_{A_*}(\mathbb{F}_p, P(t^{-1})\{t^k\} \otimes_{\mathbb{F}_p} B) \otimes_{\mathbb{F}_p} E. \tag{36}$$

The highest-degree element in the graded comodule E is the product $e_1 \cdot \ldots \cdot e_n$ of all the generators of the exterior algebra E. This product has degree $\sum_{i=1}^n |e_i|$. If we can show that $P(t^{-1})\{t^k\} \otimes_{\mathbb{F}_p} B$ has no nonzero A_*-comodule primitives in positive degrees, then isomorphism (36) will tell us that L has no A_*-comodule primitives in degree greater than $\sum_{i \in I} |e_i|$, finishing the proof of the theorem.

Even better: when $k \geqslant 2p - 2$, neither $P(t^{-1})\{t^k\}$ nor B have any comodule primitives in positive degrees. To see that this is the case for $P(t^{-1})\{t^k\}$, it is a matter of simple calculation, using the formula (35). To see that B has no comodule primitives in positive degrees, one need only remember that B was assumed to be a comodule subalgebra of the Steenrod algebra.

Continue to assume $k \geqslant 2p - 2$. While neither $P(t^{-1})\{t^k\}$ nor B have any comodule primitives in positive degrees, we need to verify that their tensor product A_*-comodule $P(t^{-1})\{t^k\} \otimes_{\mathbb{F}_p} B$ also has no comodule primitives in positive degrees. Suppose we have a homogeneous, positive-degree sum of the form $\sum_i t^i \otimes x_i \in P(t^{-1})\{t^k\} \otimes_{\mathbb{F}_p} B$, with $x_i \in B$, all but finitely many x_i are zero, and the sum satisfies $\sum_i t^i \otimes x_i \notin P(t^{-1})\{t^k\}$ and $\sum_i t^i \otimes x_i \notin B$. We need to verify that $\sum_i t^i \otimes x_i$ is not an A_*-comodule primitive.

[6] The coaction (35) is extremely classical, but curiously, we know very few places where it appears explicitly in the literature. One place is page 14 of [3]. It is, however, derivable from Milnor's formula $\lambda^*(t) = \sum_{j \geqslant 0} t^{p^j} \otimes \xi_j$ for the adjoint A_*-coaction on the *cohomology* of $\mathbb{C}P^\infty$, from Lemma 6 in section 5 of [36]. The idea is to identify positive-degree Tate cohomology with negative-degree homology, in such a way that the A_*-coaction on negative-degree homology of $\mathbb{C}P^\infty_{-\infty}$ agrees with the adjoint A_*-coaction on cohomology of $\mathbb{C}P^\infty$.

Write ψ_B for the A_*-coaction map on B. Write $1\otimes x_i+\sum_{j\geqslant 0} y_j^i\otimes z_j^i$ for $\psi_B(x_i)$, and write

$$\psi_{k,B}\colon\ P(t^{-1})\{t^k\}\otimes_{\mathbb{F}_p} B\to A_*\otimes_{\mathbb{F}_p} P(t^{-1})\{t^k\}\otimes_{\mathbb{F}_p} B$$

for the A_*-coaction on $P(t^{-1})\{t^k\}\otimes_{\mathbb{F}_p} B$. The coaction map $\psi_{k,B}$ satisfies

$$\begin{aligned}\psi_{\ell,B}(x_it^i)&=\psi_B(x_i)\cdot\psi_\ell(t^i)\\&=(1\otimes x_i+\sum_{j\geqslant 0}y_j^i\otimes z_j^i)(\sum_{j\geqslant 0}\overline{\xi}_j\otimes t^{p^j})^i\end{aligned}$$

in which the sum is taken in the graded $\mathbb{F}_p$-algebra $\mathbb{F}_p[t^{\pm 1}]\otimes_{\mathbb{F}_p} B$, and then terms with t-weight $>k$ are omitted. The t-weight filtration on L is a filtration by A_*-subcomodules, i.e.,

$$\psi_{\ell,B}(L_\ell)\subset A_*\otimes_{\mathbb{F}_p} L_\ell$$

for each ℓ. Consequently, nonzero elements in distinct t-weight filtrations in L cannot cancel each other.

We now consider the coaction on $\sum_i x_i\cdot t^i$ where we assume that all x_i are nonzero and the sum is therefore a finite sum. Then

$$\psi_{\ell,B}(\sum_i t^i\cdot x_i)=\sum_i(1\otimes x_i+\sum_{j\geqslant 0}y_j^i\otimes z_j^i)(\sum_{j\geqslant 0}\xi_j\otimes t^{p^j})^i$$

in which the sum is taken in the graded $\mathbb{F}_p$-algebra $\mathbb{F}_p[t^{\pm 1}]\otimes_{\mathbb{F}_p} B$, and then terms with t-weight $>k$ are omitted. As nonzero terms of distinct t-weight cannot cancel each other, it suffices to show that the reduction

$$\overline{\psi}_{k,B}(\sum_i x_i\cdot t^i)\subset A_*\otimes_{\mathbb{F}_p} L_i/L_{i+1}$$

of $\psi_{k,B}(\sum_i x_i\cdot t^i)$ is nontrivial after subtracting $1\otimes\sum_i x_i\cdot t^i$. We therefore need to show that

$$\overline{\psi}_{k,B}(\sum_i x_i\cdot t^i)-1\otimes\sum_i x_i\cdot t^i\neq 0\in A_*\otimes_{\mathbb{F}_p} L_i/L_{i-1}.$$

By inspection,

$$\overline{\psi}_{k,B}(\sum_i x_i\cdot t^i)-1\otimes\sum_i x_i\cdot t^i=(\sum_{j\geqslant 0}y_j^i\otimes t^j\cdot z_j^i)=(1\otimes t^j)\cdot(\sum_{j\geqslant 0}y_j^i\otimes z_j),$$

but we know that $(\sum_{j\geqslant 0}y_j^i\otimes z_j)\neq 0$, since B has no nonzero A_*-comodule primitives in positive degrees. We also know that $z\cdot t^j\neq 0$ when $z\neq 0\in B$ and $j\in\mathbb{Z}$, so this sum must be nontrivial. The conclusion is that $\sum_{i\geqslant 0}x_i\cdot t^i$ cannot be a comodule primitive, as desired. □

Lemma 3.7. *Let $B\langle n\rangle$ be a p-primary E_2 form of $BP\langle n\rangle$ satisfying Running Assumption 1.3. Suppose p is odd. For each n, there exists a positive integer $M(n)$, independent of k, such that the spectrum $TP(B\langle n\rangle)[k]$ satisfies condition $H(M(n))$ for every $k\geqslant 1$.*

Proof. To prove the lemma, it suffices to check that the graded $\mathbb{F}_p$-vector space of A_*-comodule primitives in $H_*(TP(BP\langle n\rangle)[k];\mathbb{F}_p)$ is bounded above. Recall from (30) that the E_∞-term of the approximate Tate spectral sequence (25) is

$$\widehat{E}_\infty^{*,*}(BP\langle n\rangle)[k]\cong\big(P(t^{-1})\{t^{k-1}\}\otimes_{\mathbb{F}_p} M_1\otimes_{\mathbb{F}_p} M_2\big)\oplus V_k(n)\{t^k\},$$

and it strongly converges to $H_*(TP(BP\langle n\rangle)[k];\mathbb{F}_p)$. The graded A_*-comodules M_1, M_2 and $V_k(n)$ were defined in formulas (28), (29) and (31) respectively. We first claim that that the A_*-comodule primitives in this E_∞-term are bounded above by the integer $M(n)=\sum_{j=1}^n|\bar{\xi}_j^{p-1}\sigma\bar{\xi}_j|$. To prove this it suffices to prove that the graded A_*-subcomodule of A_*-comodule primitives in $P(t^{-1})[t^{k-1}]\otimes_{\mathbb{F}_p}M_1\otimes_{\mathbb{F}_p}M_2$ and $V_k(n)\{t^k\}$ are each trivial in grading degrees greater than $M(n)$.

We have an isomorphism of graded A_*-comodules $M_1\otimes_{\mathbb{F}_p}M_2\cong B\otimes_{\mathbb{F}_p}E$, where B and E are as follows:

B is

$$E(\tau'_{j+1}|j\geqslant n+1)\otimes P(\bar{\xi}_{j+1}|k\geqslant n+1)\otimes P(\bar{\xi}_j^p|1\leqslant j\leqslant n+1),$$

regarded as a graded A_*-subcomodule of the dual Steenrod algebra by the map $B\to A_*$ sending τ'_{j+1} to τ_{j+1} (this is a consequence of (33)), sending $\bar{\xi}_{j+1}$ to $\bar{\xi}_{j+1}$, and sending $\bar{\xi}_j^p$ to $\bar{\xi}_j^p$.

E is $E(\bar{\xi}_j^{p-1}\sigma\bar{\xi}_j|1\leqslant j\leqslant n)$ with trivial A_*-coaction, i.e., every element of E is a A_*-comodule primitive. This is a consequence of the calculation of the homology of $THH(BP\langle n\rangle)$ in [6] together with the calculation of the homological approximate Tate spectral sequence for $THH(BP\langle n\rangle)$ in [15]: one just needs to keep track of the A_*-coaction while running the spectral sequence. We give a bit of detail of how this works, by explaining how to verify that $\bar{\xi}_1^{p-1}\sigma\bar{\xi}_1$ is an A_*-comodule primitive; the same verification for the other classes in E is left to the interested reader as an exercise.

As shown in [15], the homological approximate Tate spectral sequence for $BP\langle n\rangle$ collapses at the E_3-page. Consequently $\widehat{E}_\infty^{*,*}(BP\langle n\rangle)[k]$ is a subquotient of E_2. The A_*-coaction on $H_*(TP(BP\langle n\rangle)[k];\mathbb{F}_p)$ is computed by first restricting the coaction on the E_2-page to the A_*-subcomodule consisting of the d_2-cycles, and then reducing modulo d_2-boundaries. On the E_2-term we have

$$\psi(\bar{\xi}_1^{p-1}\sigma\bar{\xi}_1)=(\bar{\xi}_1\otimes 1+1\otimes\bar{\xi}_1)^{p-1}(1\otimes\sigma\bar{\xi}_1)=\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i\otimes\bar{\xi}_1^{p-1-i}\sigma\bar{\xi}_1.$$

and this remains the coaction after restricting to d_2-cycles. Every summand except $1\otimes\bar{\xi}_1^{p-1}\sigma\bar{\xi}_1$ is a d_2-boundary. Consequently $\bar{\xi}_1^{p-1}\sigma\bar{\xi}_1$ is an A_*-comodule primitive.

It therefore follows that the A_*-subcomodule of primitives in

$$P(t^{-1})[t^{k-1}]\otimes_{\mathbb{F}_p}M_1\otimes_{\mathbb{F}_p}M_2$$

is trivial in grading degrees greater than $M(n)=\sum_{j=1}^n|\bar{\xi}_j^{p-1}\sigma\bar{\xi}_j|$ by Lemma 3.6.

From Theorem 5.12 of [6], we know that

$$H_*(THH(BP\langle n\rangle);\mathbb{F}_p)\cong E(\sigma\bar{\xi}_1,\dots,\sigma\bar{\xi}_{n+1})\otimes_{\mathbb{F}_p}P(\sigma\overline{\tau}_{n+1})\otimes_{\mathbb{F}_p}H_*(BP\langle n\rangle;\mathbb{F}_p)$$

and by formulas (3) and (26) the A_*-comodule primitives in $H_*(THH(BP\langle n\rangle);\mathbb{F}_p)\{t^k\}$ are contained in $\ker d_2^{-2k,*}$, where

$$d_2^{-2k,*}\colon\ H_*(THH(BP\langle n\rangle;\mathbb{F}_p)\{t^k\}\to H_*(THH(BP\langle n\rangle;\mathbb{F}_p)\{t^{k+1}\}$$

is the d_2-differential in the approximate Tate spectral sequence. Consequently, the map

$$\mathrm{Hom}_{A_*}(\mathbb{F}_p,\ker(d_2^{-2k,*}))\to\mathrm{Hom}_{A_*}(\mathbb{F}_p,H_*(THH(BP\langle n\rangle);\mathbb{F}_p))$$

induced by the canonical inclusion is an epimorphism. We then consider the map of long exact sequences

$$\begin{array}{ccc}
\mathrm{Hom}_{A_*}(\mathbb{F}_p, \ker(d_2^{-2k,*})) & \twoheadrightarrow & \mathrm{Hom}_{A_*}(\mathbb{F}_p, H_*(THH(BP\langle n\rangle);\mathbb{F}_p)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{A_*}(\mathbb{F}_p, \ker(d_2^{-2k,*})/\operatorname{im}(d_2^{2-2k,*})) & \longrightarrow & \mathrm{Hom}_{A_*}(\mathbb{F}_p, H_*(THH(BP\langle n\rangle);\mathbb{F}_p)/\operatorname{im}(d_2^{2-2k,*})) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{A_*}(\mathbb{F}_p, \operatorname{im}(d_2^{2-2k,*})) & \xrightarrow{\mathrm{id}} & \mathrm{Ext}^1_{A_*}(\mathbb{F}_p, \operatorname{im}(d_2^{2-2k,*})) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{A_*}(\mathbb{F}_p, \ker(d_2^{-2k,*})) & \hookrightarrow & \mathrm{Ext}^1_{A_*}(\mathbb{F}_p, H_*(THH(BP\langle n\rangle);\mathbb{F}_p))
\end{array}$$

and observe that

$$V_k(n)\{t^k\} \cong \left(H_*(THH(BP\langle n\rangle);\mathbb{F}_p)\{t^k\}\right)/\operatorname{im}(d_2^{2-2k,*}).$$

We already checked that the A_*-comodules in $\ker d^{-2k,*}/\operatorname{im}\, d^{2-2k,*} = M_1 \otimes_{\mathbb{F}_p} M_2$ are bounded above by $M(n)$, so we conclude that the set of A_*-comodule primitives in $V_k(n)$ is also bounded above by $M(n)$ by the 4-lemma.

In sum, this proves that the associated graded of a filtration[7] on

$$H_*(TP(BP\langle n\rangle)[k];\mathbb{F}_p)$$

has comodule primitives bounded above by $M(n)$. Because we are working with the *approximate* Tate spectral sequence, this filtration is finite in each grading degree: it is of the form

$$H_*(TP(BP\langle n\rangle)[k];\mathbb{F}_p) \supset H_*(TP(BP\langle n\rangle)[k-1];\mathbb{F}_p) \supset \cdots \supset 0$$

where $H_s(TP(BP\langle n\rangle)[k];\mathbb{F}_p) = 0$ for $k > -s/2$ so in a fixed grading degree s it is the filtration

$$H_s(TP(BP\langle n\rangle)[k];\mathbb{F}_p) \supset \cdots \supset H_s(TP(BP\langle n\rangle)[-\lceil s/2\rceil];\mathbb{F}_p) \supset 0 = 0 = \dots$$

which is a finite filtration, where $\lceil s/2\rceil$ is the integer ceiling of $s/2$.

Hence, if there were a nontrivial comodule primitive $z \in F_i$ in internal degree $> M(n)$, then either it would map to a comodule primitive in F_i/F_{i-1}, or it would pull back to an element in F_{i-1}. By a finite downward induction on filtration degree, z must be a comodule primitive in F_j/F_{j-1} for some $\ell < j < i$. Therefore, it suffices to check that there are no comodule primitives in internal degrees $> M(n)$ in the associated graded of this filtration on $H_*(TP(R)[k];\mathbb{F}_p)$, which we have already done. □

Lemma 3.8 must surely be known in some form or another: it is quite close to well-developed ideas dating back to Bousfield and Kan's book [14]. We provide a proof since we have not been able to locate the result in the existing literature. We will use the following notation: given a prime number p and a spectrum X, we write $\gamma_p X$ for the homotopy fiber of the p-completion map $X \to X_p^\wedge$.

[7] Namely, the filtration on the abutment of the approximate Tate spectral sequence (25), whose associated graded is $\hat{E}_\infty^{*,*}(R)[k]$.

Lemma 3.8. *Let p be a prime number, and let $\{Y_i\}$ be a sequence of p-local spectra. Suppose that, for each integer n, the abelian group $\pi_*(Y_n)$ is p-reduced. Suppose furthermore that the derived limit $R^1 \lim_n \pi_*(Y_n)$ is trivial. Then the homotopy limit $\operatorname{holim}_i \gamma_p(Y_i)$ is S/p-acyclic, i.e., $(\operatorname{holim}_i \gamma_p(Y_i))^{\wedge}_p$ is contractible.*

Proof. Write $\Lambda : \operatorname{Mod}(\mathbb{Z}_{(p)}) \to \operatorname{Mod}(\mathbb{Z}_{(p)})$ for the p-adic completion functor, i.e., $\Lambda M = \lim_n M/p^n M$. It is well-known that Λ is neither left nor right exact, and consequently its 0th left-derived functor $L_0\Lambda$ may fail to coincide with Λ itself; see [47] or the appendix of [27] for surveys of these ideas. By a special case of Harrison duality[8] [23], the group $L_n\Lambda M$ is naturally isomorphic to $\operatorname{Ext}^{1-n}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$, hence vanishes if $n > 1$.

For each integer j, we have a commutative diagram with exact columns

$$\begin{array}{ccc}
\hom_{\mathbb{Z}_{(p)}}(\mathbb{Q}, \pi_j(X)) & & \\
\downarrow & & \\
\hom_{\mathbb{Z}_{(p)}}\left(\mathbb{Z}_{(p)}, \pi_j(X)\right) & \xrightarrow{\cong} & \pi_j(X) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^1_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^\infty, \pi_j(X)) & \xrightarrow{\cong} & L_0\Lambda\pi_j(X) \\
\downarrow & & \\
\operatorname{Ext}^1_{\mathbb{Z}_{(p)}}(\mathbb{Q}, \pi_j(X)) & & \\
\downarrow & & \\
0 & &
\end{array}$$

induced by the short exact sequence of abelian groups

$$0 \to \mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{Z}/p^\infty \to 0.$$

The group $\operatorname{Ext}^1_{\mathbb{Z}_{(p)}}(\mathbb{Q}, M)$ is uniquely p-divisible[9] for any $\mathbb{Z}_{(p)}$-module M. Consequently the cokernel of the derived completion map

$$\pi_j(X) \to L_0\Lambda\pi_j(X) \tag{37}$$

is uniquely p-divisible. Since $\pi_j(X)$ was assumed to be p-reduced, the kernel of (37) is trivial.

It is classical (see Proposition 2.5 in [13]) that, for any spectrum X, we have a short exact sequence

$$0 \to L_0\Lambda\pi_j(X) \to \pi_j\left(X^{\wedge}_p\right) \to L_1\Lambda\pi_{j-1}(X) \to 0$$

for each integer j. For each i, the group $L_1\Lambda\pi_* Y_i \cong \hom_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_*(Y_i))$ is trivial since $\pi_*(Y_i)$ was assumed to be p-reduced.

[8] Also a special case of dualities that generalize Harrison duality, e.g. Matlis duality [35] and Greenlees-May duality [20].

[9] This is of course classical. An easy way to see that it is true: first, note that $\hom_{\mathbb{Z}_{(p)}}(U, D)$ is uniquely p-divisible whenever D is p-divisible and U is uniquely p-divisible. Now embed M into an injective $\mathbb{Z}_{(p)}$-module I, so that I and I/M are both p-divisible. Then $\operatorname{Ext}^1_{\mathbb{Z}_{(p)}}(\mathbb{Q}, M)$ is the cokernel of a homomorphism $\hom_{\mathbb{Z}_{(p)}}(\mathbb{Q}, I) \to \hom_{\mathbb{Z}_{(p)}}(\mathbb{Q}, I/M)$ whose domain and codomain are both uniquely p-divisible.

Hence $\pi_*(\gamma_p Y_i)$ is uniquely p-divisible for all i. A limit and derived limit of uniquely p-divisible $\mathbb{Z}_{(p)}$-modules remains uniquely p-divisible, since such a limit and derived limit can simply be calculated in $\mathbb{Q}$-vector spaces. Hence $\operatorname{holim}_i \gamma_p Y_i$ is an $H\mathbb{Q}$-module spectrum, hence is S/p-acyclic. □

Lemma 3.9. *Suppose R is an $\mathbb{E}_1$-ring spectrum satisfying*

$$\operatorname{Ext}^{*,*}_{A_*}(\mathbb{F}_p, H_*(R)) \cong \mathbb{F}_p[x_i : 1 \leqslant i \leqslant n]$$

with x_i in even, nonnegative internal degrees for $i \geqslant 1$. Then, there exists an integer L such that $\pi_s(\mathrm{TP}(R)/p)$ is finite for all $s \geqslant L$.

Proof. Consider the functor $\mathbb{Z}^{\mathrm{op}} \to \mathrm{Sp}$ defined by $\mathrm{fil}^q_{\mathbb{F}_p}(R) = \mathrm{Tot}\left(\mathrm{Wh}^q(R \wedge H\mathbb{F}_p^{\wedge \bullet+1})\right)$ with associated graded $\mathrm{gr}^s_{\mathbb{F}_p}(R) = \mathrm{Tot} H\pi_s(R \wedge H\mathbb{F}_p^{\wedge\bullet+1})$. Then as in [2], we consider the filtered object $\mathrm{TP}(\mathrm{fil}^\bullet_{\mathbb{F}_p}(R))/p$, where we take the quotient in filtered spectra with p in filtration 0. By [2, Theorem 3.3.10] (cf. [29, Corollary 4.14]), we know that there is an $\mathbb{T}$-equivariant equivalence

$$\mathrm{gr}^*\mathrm{THH}(\mathrm{fil}^\bullet_{\mathbb{F}_p}(R)) \simeq \mathrm{THH}(\mathrm{gr}^*_{\mathbb{F}_p}(R)).$$

Also, the functor gr^* is a left adjoint by [21, Lemma 3.30] so it commutes with homotopy colimits and we have

$$\mathrm{gr}^*\left(\mathrm{THH}(\mathrm{fil}^\bullet_{\mathbb{F}_p}(R))_{h\mathbb{T}}\right) \simeq \left(\mathrm{gr}^*\mathrm{THH}(\mathrm{fil}^*_{\mathbb{F}_p}(R))\right)_{h\mathbb{T}} \simeq \mathrm{THH}(\mathrm{gr}^*_{\mathbb{F}_p}(R))_{h\mathbb{T}}.$$

Since $B\mathbb{T}$ has finite skeleta $\mathrm{sk}_n B\mathbb{T}$ using the standard simplicial CW filtration on $B\mathbb{T} = \mathbb{C}P^\infty$, we know that

$$\left(\mathrm{gr}^* \lim_{\mathrm{sk}_n B\mathbb{T}}\left(\mathrm{THH}(\mathrm{fil}^\bullet_{\mathbb{F}_p}(R))\right)\right) \simeq \lim_{\mathrm{sk}_n B\mathbb{T}} \mathrm{gr}^*\left(\left(\mathrm{THH}(\mathrm{fil}^*_{\mathbb{F}_p}(R))\right)\right) \simeq \lim_{\mathrm{sk}_n B\mathbb{T}} \mathrm{THH}(\mathrm{gr}^*_{\mathbb{F}_p}(R)).$$

Finally, the canonical map

$$\mathrm{gr}^s\left(\left(\mathrm{THH}(\mathrm{fil}^\bullet_{\mathbb{F}_p}(R))\right)^{h\mathbb{T}}\right) \longrightarrow \lim_n\left(\mathrm{gr}^s \lim_{\mathrm{sk}_n B\mathbb{T}}\left(\mathrm{THH}(\mathrm{fil}^\bullet_{\mathbb{F}_p}(R))\right)\right)$$

is an equivalence because

$$\lim_n \mathrm{gr}^s F_n(R) \simeq 0$$

where

$$F_n := \mathrm{fib}\left((\mathrm{THH}(\mathrm{fil}^\bullet_{\mathbb{F}_p}(R)))^{h\mathbb{T}} \to \lim_{\mathrm{sk}_n B\mathbb{T}}(\mathrm{THH}(\mathrm{fil}^\bullet_{\mathbb{F}_p}(R)))\right).$$

We then use the fiber sequence

$$\mathrm{THH}(\mathrm{fil}^q_{\mathbb{F}_p}(R))^{h\mathbb{T}} \to \mathrm{THH}(\mathrm{fil}^q_{\mathbb{F}_p}(R))^{t\mathbb{T}} \to \Sigma^2\mathrm{THH}(\mathrm{fil}^q_{\mathbb{F}_p}(R))_{h\mathbb{T}}$$

to conclude that

$$\mathrm{gr}^*\left(\mathrm{THH}(\mathrm{fil}^\bullet_{\mathbb{F}_p}(R))^{t\mathbb{T}}\right) \simeq \mathrm{THH}(\mathrm{gr}^*_{\mathbb{F}_p}(R))^{t\mathbb{T}}.$$

Finally, there is an equivalence

$$\mathrm{TP}(\mathrm{gr}^*_{\mathbb{F}_p}(R))/p \simeq \mathrm{gr}^*\left(\mathrm{TP}(\mathrm{fil}^*_{\mathbb{F}_p}(R))/p\right).$$

For conditional convergence, we observe that

$$\lim_s \mathrm{TP}(\mathrm{fil}^s_{\mathbb{F}_p}(R))/p = 0.$$

This follows because $\mathrm{THH}(\mathrm{fil}^s_{\mathbb{F}_p}(R))$ is increasingly connective as $s \to \infty$, implying that

$$\lim_s (\mathrm{THH}(\mathrm{fil}^s_{\mathbb{F}_p}(R))^{h\mathbb{T}}) = (\lim_s \mathrm{THH}(\mathrm{fil}^s_{\mathbb{F}_p}(R))^{h\mathbb{T}}) = 0$$

and

$$\lim_s (\mathrm{THH}(\mathrm{fil}^s_{\mathbb{F}_p}(R))_{h\mathbb{T}}) = (\lim_s \mathrm{THH}(\mathrm{fil}^s_{\mathbb{F}_p}(R))_{h\mathbb{T}}) = 0$$

where we also use the fact that $B\mathbb{T}$ has finite skeleta for the last identification. It is also clear that

$$\operatorname{colim}_s \mathrm{TP}(\mathrm{fil}^s_{\mathbb{F}_p}(R))/p = \mathrm{TP}(R)/p$$

for similar reasons, so the associated spectral sequence

$$E_1^{s,2t-s} = \pi_s([\mathrm{TP}(\mathrm{gr}^*_{\mathbb{F}_p}(R))/p]^t) \implies \pi_s \mathrm{TP}(R)/p$$

conditionally converges in the sense of [12].

By [39, Lemma IV.4.12], there is an equivalence

$$\mathrm{TP}(\mathrm{gr}^*_{\mathbb{F}_p}(R))/p \simeq \mathrm{THH}(\mathrm{gr}^*_{\mathbb{F}_p}(R))^{tC_p}$$

and by [22, Proposition 4.2.2] we know that the Frobenius map

$$\varphi_p \colon \mathrm{THH}(\mathrm{gr}^*_{\mathbb{F}_p}(R)) \to \mathrm{THH}(\mathrm{gr}^*_{\mathbb{F}_p}(R))^{tC_p}$$

is an isomorphism on π_s for all $s \geqslant L$ for some integer L. It is clear that $\pi_s \mathrm{THH}(\mathrm{gr}^*_{\mathbb{F}_p}(R))$ is a finitely type graded $\mathbb{F}_p$-vector space for each s, by two standard Künneth spectral sequence arguments. Consequently, we conclude that the $\mathbb{F}_p$-vector space $\pi_s(\mathrm{TP}(\mathrm{gr}^*_{\mathbb{F}_p}(R))/p)$ is finite for each $s \geqslant L$. Finally, we observe that we can apply the Beilinson t-structure [24, II.2.1] to produce a filtered object

$$\tau^{\mathrm{Bei}}_{\geqslant L} \mathrm{TP}(\mathrm{fil}^s_{\mathbb{F}_p}(R))$$

whose associated spectral sequence has E_1-term

$$E_1^{s,*} = \begin{cases} \mathrm{THH}(\mathrm{gr}^*_{\mathbb{F}_p}(R))^{tC_p} & \text{if } s \geqslant L \\ 0 & \text{otherwise} \end{cases}$$

and converges to $\pi_* \mathrm{Wh}^L \mathrm{TP}(R)/p$. Consequently, this spectral sequence strongly converges and we conclude that the groups $\pi_s(\mathrm{TP}(R)/p)$ are finite for all $s \geqslant L$ as desired. □

Theorem 3.10. *Let m, n be nonnegative integers such that $m \geqslant n+2$. Let R be a p-primary E_2 form of $BP\langle n\rangle$ satisfying Running Assumption 1.3. Suppose p is odd. Then there are isomorphisms*

$$K(m)_*(TP(R)) \cong 0 \text{ and}$$
$$K(m)_*(TC^-(R)) \cong 0.$$

Proof. We claim that the sequence of spectra

$$\cdots \to TP(R)[1]^\wedge_p \to TP(R)[0]^\wedge_p \to TP(R)[-1]^\wedge_p \to \dots$$

is $K(m)$-amenable. We check the conditions for $K(m)$-amenability, given in Definition 2.20, as follows:

(1) It is straightforward from the definitions that $TP(R)[k]^\wedge_p$ is p-complete and bounded below.
(2) The homology $H_*(TP(R)[k]^\wedge_p; \mathbb{F}_p)$ is finite-type, for all k.

(3) The pro-triviality of the sequence $\{H(H_*(TP(R)[\bullet]^\wedge_p;\mathbb{F}_p),Q_n)\}$ is Proposition 3.5.
(4) There exists an integer L such that the groups $(\pi_s TP(R))/p$ are finite for $s \geqslant L$ by Lemma 3.9.
(5) There exists an integer M such that the spectrum $TP(R)[k]^\wedge_p$ satisfies condition $H(M)$ for all k, by Lemma 3.7.

Hence $\operatorname{holim}_k \left(TP(R)[k]^\wedge_p\right)$ is $K(m)$-acyclic, by Theorem 2.23.

Consequently have a commutative square of spectra whose rows and columns are homotopy fiber sequences:

(38)

$$\begin{array}{ccccc} \gamma_p \operatorname{holim}_k \gamma_p\left(TP(R)[k]\right) & \xrightarrow{\simeq} & \gamma_p TP(R) & & \\ \downarrow \simeq & & \downarrow & & \\ \operatorname{holim}_k \gamma_p\left(TP(R)[k]\right) & \longrightarrow & TP(R) & \longrightarrow & \operatorname{holim}_k \left(TP(R)[k]^\wedge_p\right) \\ & & \downarrow & & \downarrow \simeq \\ & & \left(TP(R)\right)^\wedge_p & \xrightarrow[\simeq]{} & \left(\operatorname{holim}_k \left(\left(TP(R)[k]\right)^\wedge_p\right)\right)^\wedge_p . \end{array}$$

Every S/p-local equivalence is also a $K(m)$-local equivalence, so the spectra in the top row of (38) are all $K(m)$-acyclic. We have already shown that $\operatorname{holim}_k \left(TP(R)[k]^\wedge_p\right)$ is $K(m)$-acyclic. Consequently $TP(R)$ and $(TP(R))^\wedge_p$ are each $K(m)$-acyclic.

Since $\operatorname{holim}_k TP(R)[k] \simeq TP(R) \simeq THH(R)^{t\mathbb{T}}$, the vanishing of

$$K(m)_* \operatorname{holim}_k TP(R)[k]$$

for $m \geqslant n+2$ together with the fiber sequence

$$\Sigma THH(R)_{h\mathbb{T}} \to THH(R)^{h\mathbb{T}} \to THH(R)^{t\mathbb{T}}$$

implies that the map $\Sigma THH(R)_{h\mathbb{T}} \to THH(R)^{h\mathbb{T}}$ is a $K(m)$-local equivalence for $m \geqslant n+2$.

Consequently, if we prove that $THH(R)_{h\mathbb{T}}$ is also $K(m)$-acyclic, then $TC^-(R) \simeq THH(R)^{h\mathbb{T}}$ must also be $K(m)$-acyclic. This, however, is quite straightforward: when $m \geqslant n+1$, the vanishing of $K(m)_*(BP\langle n\rangle)$ was proven in [42, Thm. 2.1(d),(f),(i)]. Hence $THH(R)$ is an algebra over a $K(m)$-acyclic ring spectrum (namely, R), hence $THH(R)$ is $K(m)$-acyclic. Smashing with $K(m)$ commutes with homotopy colimits, so

$$K(m) \wedge \left(THH(R)_{h\mathbb{T}}\right) \simeq \left(K(m) \wedge THH(R)\right)_{h\mathbb{T}} \simeq 0$$

for $m \geqslant n+1$. □

Corollary 3.11. *Suppose p is odd. If $B\langle n\rangle$ is a p-primary E_2 form of $BP\langle n\rangle$ satisfying Running Assumption 1.3, then there are weak equivalences*

$$L_{K(m)} K(B\langle n\rangle) \simeq 0$$

for $m \geqslant n+2 \geqslant 2$.

Proof. We first show that there is an equivalence

$$L_{K(m)} TC(B\langle n\rangle) \simeq 0$$

for $m \geqslant n+2 \geqslant 2$. This follows by Theorem 3.10 together with the long exact sequence in Morava K-theory associated to the homotopy fiber sequence

$$TC(B\langle n\rangle)^{\wedge}_p \to TC^{-}(B\langle n\rangle)^{\wedge}_p \xrightarrow{\mathrm{can}-\varphi_p^{h\mathbb{T}}} TP(B\langle n\rangle)^{\wedge}_p$$

of [39]. The fact that

$$L_{K(m)}K(\mathbb{Z}_{(p)}) = 0$$

for $m \geqslant 2$ then follows by [25, Theorem D]. For $n > 0$, the result follows from Theorem 3.10 by the Dundas–Goodwillie–McCarthy theorem [17, Theorem 7.2.2.1] and the $n = 0$ case, which together imply that

$$L_{K(m)}K(BP\langle n\rangle) \simeq L_{K(m)}TC(BP\langle n\rangle)$$

for $m \geqslant n+2 \geqslant 3$. □

Appendix A. Brief review of Margolis homology.

This appendix, which does not logically rely on anything earlier in the paper, consists of material that is well-known to users of Margolis homology. This material is not difficult and certainly not new. Nevertheless we include it in this paper because, for some of this material, we do not know of a clear and straightforward account in the existing literature.

Given a graded ring R, we write $\mathrm{gr}\,\mathrm{Mod}(R)$ for the category of graded R-modules and grading-preserving R-module homomorphisms.

Definition A.1. *Let k be a field and let $E(Q)$ be the exterior k-algebra on a single homogeneous generator Q in an odd grading degree $|Q|$. By* Margolis Q-homology *we mean the functor $H(-;Q)\colon \mathrm{gr}\,\mathrm{Mod}(E(Q)) \to \mathrm{gr}\,\mathrm{Mod}(E(Q))$ given on a graded $E(Q)$-module M by the quotient*

$$H(M;Q) = \left(\ker(M \xrightarrow{Q} \Sigma^{-|Q|}M)\right) / \left(\mathrm{im}\,(\Sigma^{|Q|}M \xrightarrow{Q} M)\right).$$

It is routine to verify that $H(-;Q_n)$ sends short exact sequences of graded $E(Q_n)$-modules to long exact sequences.

Here is a quick note on gradings; it is extremely elementary, but not taking a moment to "fix notations" on this point tends to lead to sign errors in the gradings.

Conventions A.2. Given a graded ring R and graded R-modules M and N, we write $\mathrm{Hom}_R(M,N)$ for the degree-preserving R-linear morphisms $M \to N$, and we write $\underline{\mathrm{Hom}}_R(M,N)$ for the graded abelian group whose degree n summand is $\mathrm{Hom}_R(\Sigma^n M, N)$. We write $\mathrm{Ext}_R^{s,*}(M,N)$ for the graded abelian group whose degree t summand is $\mathrm{Ext}_R^{s,t}(M,N)$, and we refer to this grading as the *internal* or *topological* grading, to distinguish it from the *cohomological* degree given by s.

In particular, the k-linear dual of a graded k-vector space has the signs of the gradings reversed, i.e.,

$$(\Sigma^n V)^* = \underline{\mathrm{Hom}}_k(\Sigma^n V, k) \cong \Sigma^{-n}(V^*).$$

Now given a spectrum X, the action of Q_n on $H^*(X;\mathbb{F}_p)$ is the one induced in homotopy by the map of function spectra $F(X, H\mathbb{F}_p) \to F(X, \Sigma^{2p^n-1}H\mathbb{F}_p)$ induced by the composite (4). Somewhat less famous than the action of Steenrod operations on mod p cohomology, we have also the dual action of Steenrod operations on mod p homology: the action of Q_n on $H_*(X;\mathbb{F}_p)$ is the one induced in homotopy by the map of spectra $X \wedge H\mathbb{F}_p \to X \wedge \Sigma^{2p^n-1}H\mathbb{F}_p$ induced by the composite (4). These operations are $\mathbb{F}_p$-linearly dual under the isomorphism $H^i(X;\mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(H_i(X;\mathbb{F}_p),\mathbb{F}_p)$; see Proposition III.13.5 of [1] or Theorem IV.4.5 of [18].

Lemma A.3. *Let $k, E(Q), |Q|$ be as in Definition A.1. Let M be a graded $E(Q)$-module. Then we have an isomorphism of graded $E(Q)$-modules*

$$\underline{\mathrm{Hom}}_{E(Q)}(M, E(Q)) \cong \underline{\mathrm{Hom}}_k(M, \Sigma^{|Q|}k)$$

natural in the choice of M.

Proof. Since each of the functors $\underline{\mathrm{Hom}}_{E(Q)}(-, E(Q))$ and $\underline{\mathrm{Hom}}_k(-, \Sigma^{|Q|}k)$ send colimits to limits, it suffices to prove the isomorphism for finitely generated $E(Q)$-modules. Since $E(Q)$ is a PID, all finitely generated $E(Q)$-modules are direct sums of suspensions of $E(Q)$ and k. Therefore, it suffices to prove the statement for suspensions of $E(Q)$ and of k, but in this case it is clear that the statement holds. □

Proposition A.4. *Let $k, E(Q), |Q|$ be as in Definition A.1. Then we have natural isomorphisms of graded $E(Q)$-modules:*

$$H(M^*;Q) \cong H(M;Q)^*, \tag{39}$$

$$\mathrm{Ext}^{s,*}_{E(Q)}(k, M) \cong \Sigma^{-s|Q|}H(M;Q) \quad \textit{if } s>0, \tag{40}$$

$$\mathrm{Ext}^{s,*}_{E(Q)}(M, k) \cong \Sigma^{-(s+1)|Q|}H\left(\underline{\mathrm{Hom}}_{E(Q)}(M, E(Q)); Q\right) \quad \textit{if } s>0, \tag{41}$$

$$\cong \Sigma^{-s|Q|}\left(H\left(M;Q\right)^*\right) \quad \textit{if } s>0. \tag{42}$$

natural in the choice of graded $E(Q)$-module M. (The notation M^* is for the graded k-linear dual of M, i.e., $M^* = \underline{\mathrm{Hom}}_k(M,k)$.)

Proof. We handle each of the isomorphisms (39) through (42) in turn:

The Q-Margolis homology of M is the homology of the chain complex

$$\dots \xrightarrow{Q} \Sigma^{|Q|}M \xrightarrow{Q} M \xrightarrow{Q} \Sigma^{-|Q|}M \xrightarrow{Q} \dots \tag{43}$$

and so, since the k-linear dual of the multiplication-by-Q map on a $E(Q)$-module is the multiplication-by-Q map on the k-linear dual of that module, the cohomology of the k-linear dual of the chain complex (43) is $H(M^*;Q)$. So the classical universal coefficient sequence for chain complexes (e.g. as in 3.6.5 of [46]) yields the isomorphism (39).

Applying $\underline{\mathrm{Hom}}_{E(Q)}(-, M)$ to the projective graded $E(Q)$-module resolution of k

$$0 \leftarrow E(Q) \xleftarrow{Q} \Sigma^{|Q|}E(Q) \xleftarrow{Q} \Sigma^{2|Q|}E(Q) \xleftarrow{Q} \dots \tag{44}$$

yields the cochain complex

$$0 \to M \xrightarrow{Q} \Sigma^{-|Q|}M \xrightarrow{Q} \Sigma^{-2|Q|}M \xrightarrow{Q} \dots$$

whose homology is $\Sigma^{-s|Q|}H(M;Q)$ in each cohomological degree $s>0$. This gives us isomorphism (40). (See Convention A.2 for the sign change in grading degrees.)

We take advantage of the fact that $E(Q)$ is self-injective, so that

$$0 \to \Sigma^{-|Q|}E(Q) \xrightarrow{Q} \Sigma^{-2|Q|}E(Q) \xrightarrow{Q} \Sigma^{-3|Q|}E(Q) \xrightarrow{Q} \dots \tag{45}$$

is an injective graded $E(Q)$-module resolution of k. Applying $\underline{\mathrm{Hom}}_{E(Q)}(M,-)$ to (45) yields the cochain complex

$$0 \to \underline{\mathrm{Hom}}_{E(Q)}(M, \Sigma^{-|Q|}E(Q)) \xrightarrow{Q} \underline{\mathrm{Hom}}_{E(Q)}(M, \Sigma^{-2|Q|}E(Q)) \xrightarrow{Q} \dots, \tag{46}$$

hence isomorphism (41).

Isomorphism (42) then follows from the chain of isomorphisms

$$\begin{aligned}\Sigma^{-(s+1)|Q|}H\left(\underline{\mathrm{Hom}}_{E(Q)}(M, E(Q)); Q\right) &\cong \Sigma^{-(s+1)|Q|}H\left(\underline{\mathrm{Hom}}_{k}(M, \Sigma^{|Q|}k); Q\right)\\ &\cong \Sigma^{-(s+1)|Q|}H\left(\Sigma^{|Q|}M^*; Q\right)\\ &\cong \Sigma^{-s|Q|}\left(H\left(M;Q\right)^*\right),\end{aligned}$$

due to Lemma A.3. □

Proposition A.5 is a simple cohomological duality. For clarity, we drop the gradings:

Proposition A.5. *Let $k, E(Q), |Q|$ be as in Definition A.1. For each $E(Q)$-module M, we have an isomorphism of graded $E(Q)$-modules*

$$\mathrm{Ext}^s_{E(Q)}(M,k) \cong \mathrm{Ext}^s_{E(Q)}(k, M^*) \tag{47}$$

for each integer s. If $s > 0$, then each side of (47) is furthermore isomorphic to $\mathrm{Ext}^{s,*}_{E(Q)}(k,M)^*$.

Proof. The $s = 0$ case of isomorphism (47) is immediate. Consequently, for the rest of this proof we assume $s > 0$. The hypotheses of Proposition A.4 are then fulfilled. Stringing together isomorphisms from Proposition A.4:

$$\begin{aligned}\mathrm{Ext}^s_{E(Q)}(M,k) &\cong \Sigma^{-s|Q|}\left(H(M;Q)^*\right)\\ &\cong \Sigma^{-s|Q|}H(M^*;Q)\\ &\cong \mathrm{Ext}^s_{E(Q)}(k,M^*).\end{aligned} \tag{48}$$

The right-hand side of (48) is also isomorphic to $\mathrm{Ext}^s_{E(Q)}(k,M)^*$, by isomorphism (40). □

Corollary A.6. *Fix $s \geqslant 1$ and suppose $H_*(X;\mathbb{F}_p)$ is finite type. Then the graded $\mathbb{F}_p$-vector space $\mathrm{Ext}^{s,*}_{A_*}(\mathbb{F}_p, H_*(k(n) \wedge X; \mathbb{F}_p))$ is isomorphic to a suspension of $H(H_*(X;\mathbb{F}_p);Q_n)$.*

Proof. Fix $s \geqslant 1$. We apply the Künneth isomorphism and the change-of-rings isomorphism to produce the isomorphism

$$\mathrm{Ext}^{s,-*}_{A_*}(\mathbb{F}_p, H_*(k(n)\wedge X;\mathbb{F}_p)) = \mathrm{Ext}^{s,-*}_{E(Q_n)_*}(\mathbb{F}_p, H_*(X;\mathbb{F}_p))$$

Since $E(Q_n)_*$ is a self-dual finite-dimensional Hopf algebra and $H_*(X;\mathbb{F}_p)^* = H^*(X;\mathbb{F}_p)$, we can apply Proposition A.5 to identify $\mathrm{Ext}^{s,-*}_{E(Q_n)_*}(\mathbb{F}_p, H_*(X;\mathbb{F}_p))$ with $\mathrm{Ext}^{s,*}_{E(Q_n)}(H^*(X;\mathbb{F}_p),\mathbb{F}_p)$. Now Proposition A.5 finishes the job. □

References

[1] J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original.

[2] Gabe Angelini-Knoll and Andrew Salch. A May-type spectral sequence for higher topological Hochschild homology. *Algebr. Geom. Topol.*, 18(5):2593–2660, 2018.

[3] Gabriel Angelini-Knoll. Detecting β elements in iterated algebraic K-theory. *Trans. Amer. Math. Soc.*, 376(4):2657–2692, 2023.

[4] Gabriel Angelini-Knoll and J. D. Quigley. Chromatic complexity of the algebraic k-theory of $y(n)$. Preprint, 2019.

[5] Vigleik Angeltveit and John A. Lind. Uniqueness of $BP\langle$nrangle. *J. Homotopy Relat. Struct.*, 12(1):17–30, 2017.

[6] Vigleik Angeltveit and John Rognes. Hopf algebra structure on topological Hochschild homology. *Algebr. Geom. Topol.*, 5:1223–1290, 2005.

[7] M. Artin and B. Mazur. *Etale homotopy*. Lecture Notes in Mathematics, No. 100. Springer-Verlag, Berlin-New York, 1969.

[8] Christian Ausoni and John Rognes. Algebraic K-theory of topological K-theory. *Acta Math.*, 188(1):1–39, 2002.

[9] Christian Ausoni and John Rognes. The chromatic red-shift in algebraic K-theory. *Enseignement Mathématique*, 54(2):13–15, 2008.

[10] Andrew Baker and Birgit Richter. On the Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory. *Comment. Math. Helv.*, 80(4):691–723, 2005.

[11] Maria Basterra and Michael A. Mandell. The multiplication on BP. *J. Topol.*, 6(2):285–310, 2013.

[12] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.

[13] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[14] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.

[15] Robert R. Bruner and John Rognes. Differentials in the homological homotopy fixed point spectral sequence. *Algebr. Geom. Topol.*, 5:653–690, 2005.

[16] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. Descent and vanishing in chromatic algebraic K-theory via group actions. *arXiv e-prints*, page arXiv:2011.08233, November 2020.

[17] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013.

[18] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[19] J. P. C. Greenlees. Representing Tate cohomology of G-spaces. *Proc. Edinburgh Math. Soc. (2)*, 30(3):435–443, 1987.

[20] J. P. C. Greenlees and J. P. May. Derived functors of I-adic completion and local homology. *J. Algebra*, 149(2):438–453, 1992.

[21] O. Gwilliam and D. Pavlov. Enhancing the filtered derived category. *ArXiv e-prints*, February 2016.

[22] Jeremy Hahn and Dylan Wilson. Redshift and multiplication for truncated Brown-Peterson spectra. *Ann. Math. (2)*, 196(3):1277–1351, 2022.

[23] D. K. Harrison. Infinite abelian groups and homological methods. *Ann. of Math. (2)*, 69:366–391, 1959.

[24] Alice Hedenlund. *Multiplicative Tate Spectral Sequences*. PhD thesis, University of Oslo, 2020.

[25] Lars Hesselholt and Ib Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[26] Michael Hill and Tyler Lawson. Automorphic forms and cohomology theories on Shimura curves of small discriminant. *Adv. Math.*, 225(2):1013–1045, 2010.

[27] Mark Hovey, John H. Palmieri, and Neil P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.
[28] I. M. James. Reduced product spaces. *Ann. of Math. (2)*, 62:170–197, 1955.
[29] Liam Keenan. The May filtration on THH and faithfully flat descent. *arXiv e-prints*, page arXiv:2004.12600, April 2020.
[30] Markus Land, Akhil Mathew, Lennart Meier, and Georg Tamme. Purity in chromatically localized algebraic K-theory. *arXiv e-prints*, page arXiv:2001.10425, January 2020.
[31] Tyler Lawson. Secondary power operations and the Brown-Peterson spectrum at the prime 2. *Ann. of Math. (2)*, 188(2):513–576, 2018.
[32] Tyler Lawson and Niko Naumann. Strictly commutative realizations of diagrams over the Steenrod algebra and topological modular forms at the prime 2. *Int. Math. Res. Not.*, 2014(10):2773–2813, 2014.
[33] Sverre Lunø e Nielsen and John Rognes. The topological Singer construction. *Doc. Math.*, 17:861–909, 2012.
[34] Mark Mahowald and Charles Rezk. Brown-Comenetz duality and the Adams spectral sequence. *Amer. J. Math.*, 121(6):1153–1177, 1999.
[35] Eben Matlis. Cotorsion modules. *Mem. Amer. Math. Soc. No.*, 49:66, 1964.
[36] John Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958.
[37] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.
[38] S. A. Mitchell. The Morava K-theory of algebraic K-theory spectra. *K-Theory*, 3(6):607–626, 1990.
[39] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.
[40] Eric Peterson. Coalgebraic formal curve spectra and spectral jet spaces. *Geom. Topol.*, 24(1):1–47, 2020.
[41] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.
[42] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984.
[43] Hal Sadofsky. The homology of inverse limits of spectra. *unpublished*.
[44] Hal Sadofsky. Morava k-theory of homotopy inverse limits of spectra. *unpublished*.
[45] Andrew Senger. The Brown-Peterson spectrum is not $E_{2(p^2+2)}$ at odd primes. *arXiv e-prints*, page arXiv:1710.09822, October 2017.
[46] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
[47] Amnon Yekutieli. On flatness and completion for infinitely generated modules over Noetherian rings. *Comm. Algebra*, 39(11):4221–4245, 2011.