KU-LOCAL ZETA-FUNCTIONS OF FINITE CW-COMPLEXES.

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Abstract. Begin with the Hasse-Weil zeta-function of a smooth projective variety over $\mathbb{Q}$. Replace the variety with a finite CW-complex, replace étale cohomology with complex K-theory KU^*, and replace the p-Frobenius operator with the pth Adams operation on K-theory. This simple idea yields a kind of "KU-local zeta-function" of a finite CW-complex. For a wide range of finite CW-complexes X with torsion-free K-theory, we show that this zeta-function admits analytic continuation to a meromorphic function on the complex plane, with a nice functional equation, and whose special values in the left half-plane recover the KU-local stable homotopy groups of X away from 2.

We then consider a more general and sophisticated version of the KU-local zeta-function, one which is suited to finite CW-complexes X with nontrivial torsion in their K-theory. This more sophisticated KU-local zeta-function involves a product of L-functions of complex representations of the torsion subgroup of $KU^0(X)$, similar to how the Dedekind zeta-function of a number field factors as a product of Artin L-functions of complex representations of the Galois group. For a wide range of such finite CW-complexes X, we prove analytic continuation of the zeta-function, and we show that the special values in the left half-plane recover the KU-local stable homotopy groups of X away from 2 if and only if the skeletal filtration on the torsion subgroup of $KU^0(X)$ splits completely.

1. Introduction.

1.1. The main ideas and results.

1.1. The main ideas and results. Recall (e.g. Theorem 8.10 from [26]) the calculation of the $KU[1/2]$-local stable homotopy groups of spheres[1] by Adams-Baird and Ravenel:

Theorem 1.1. *The ring of homotopy groups $\pi_*(L_{KU[1/2]}S^0)$ of the $KU[1/2]$-local sphere is determined by the following:*

- $\pi_0(L_{KU[1/2]}S^0) \cong \mathbb{Z}[1/2]$.
- $\pi_{-1}(L_{KU[1/2]}S^0) \cong 0$.
- $\pi_{-2}(L_{KU[1/2]}S^0) \cong (\mathbb{Q}/\mathbb{Z})[1/2]$.
- *For all $n > 0$, $\pi_{2n-1}(L_{KU[1/2]}S^0)$ is a cyclic group of order equal to the denominator[2] of the rational number $\zeta(1-n)$, up to a power of 2. Here $\zeta(1-n)$ is the value of the Riemann zeta-function at $1-n$.*
- *For all $n > 0$, $\pi_{2n}(L_{KU[1/2]}S^0)$ is trivial.*
- *For each integer n, the multiplication map $\pi_n(L_{KU[1/2]}S^0) \times \pi_{-2-n}(L_{KU[1/2]}S^0) \to \pi_{-2}(L_{KU[1/2]}S^0) \cong (\mathbb{Q}/\mathbb{Z})[1/2]$ is a perfect pairing. In particular, for positive n, $\pi_{-1-2n}(L_{KU[1/2]}S^0)$ is also cyclic of order equal to the denominator of $\zeta(1-n)$, up to a power of 2.*
- *For each n, the multiplication map $\pi_n(L_{KU[1/2]}S^0)\times\pi_{j-n}(L_{KU[1/2]}S^0) \to \pi_j(L_{KU[1/2]}S^0)$ is zero unless $j = -2$ or $j = n$ or $n = 0$.*

Date: June 2023.

[1] The reader who is not already familiar with stable homotopy groups of Bousfield localizations is advised to skip ahead to section 1.2, below, where we give a brief introduction to the idea, and its place within computational stable homotopy theory.

[2] We adopt the convention that the denominator of $0 \in \mathbb{Q}$ is 1.

It is appealing to be able to describe the orders of Bousfield-localized stable homotopy groups of a finite CW-complex in terms of special values of a zeta-function of some kind. Very few results of this kind are already known[3]: the only other example is [29], where for odd primes p, it was shown that the orders of the KU-local stable homotopy groups of the mod p Moore spectrum S^0/p are given by denominators of special values of $\zeta_F(s)/\zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function, and $\zeta_F(s)$ is the Dedekind zeta-function of the smallest subfield F of the cyclotomic field $\mathbb{Q}(\zeta_{p^2})$ in which p is wildly ramified.

In this paper we prove that the orders of the KU-local stable homotopy groups of a much wider class of finite CW-complexes are given by special values of certain zeta-functions. The most important definition is Definition 3.8, in which we define a *KU-local zeta-function*, written $\zeta_{KU}(s,X)$, for every finite CW-complex X such that

- the complex K-theory $KU^*(X)$ is concentrated in even degrees, and
- the order of the torsion subgroup of $KU^0(X)$ is square-free[4].

The zeta-function $\zeta_{KU}(s,X)$ is defined as a product of two factors, a "provisional KU-local zeta-function" $\dot{\zeta}_{KU}(s,X)$, and a "torsion KU-local L-function" $L_{\mathrm{tors}\,KU}(s,X)$:

$$\zeta_{KU}(s,X) = \dot{\zeta}_{KU}(s,X)\cdot L_{\mathrm{tors}\,KU}(s,X), \quad \text{where}$$

$$\dot{\zeta}_{KU}(s,X) = \prod_p \frac{1}{\det\left(\mathrm{id} - p^{-s}\Psi^p \mid_{\hat{KU}^0_{\ell_p}(X)}\right)}, \quad \text{and} \tag{1}$$

$$L_{\mathrm{tors}\,KU}(s,X) = \prod_{w\in\mathbb{Z}}\prod_{\rho_w}\prod_p \frac{1}{\det\left(\mathrm{id} - p^{w-s}\Psi_{\rho_w,p,i_w}(p)\right)}. \tag{2}$$

The Euler products (1) and (2) are understood to be valid only in a suitable right-hand half-plane. The former, (1), is a straightforward Euler product of characteristic polynomials of Adams operations acting on K-theory of X completed at a prime at which the K-theory is torsion-free. By contrast, the Euler product (2) is a product of characteristic polynomials of Adams operations acting on the torsion in the K-theory of X, taken not merely over prime numbers p, but also *prime representations* ρ of the torsion subgroup $\mathrm{tors}\,KU^0(X)$ of the K-theory of X; in this paper, a complex representation is said to be *prime* if it is irreducible and its image has prime order. See section 3 for details.

In Proposition 3.6, $\zeta_{KU}(s,X)$ is shown to analytically continue to a meromorphic function on the complex plane. Proposition 3.6 also yields a factorization of $\zeta_{KU}(s,X)$ as a product of shifts (i.e., Tate twists) of L-functions of even Dirichlet characters. Its functional equation is discussed in section 3.2. In Example 2.6 we also see that, for a smooth

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[3]On the other hand, Bousfield-localized stable homotopy groups of *algebraic K-theory spectra* are well-known to admit deep relationships to special values of zeta-functions: see Example 4.8 of [34], for example. But even after a Bousfield localization, algebraic K-theory spectra are almost never finite CW-complexes (except in the case of the algebraic K-theory of a finite field). We are compelled, by the classical topological applications of stable homotopy groups, to try to understand the stable homotopy groups of finite CW-complexes, and most importantly, spheres: for example, the stable homotopy groups of spheres are the attaching maps for stable 2-cell complexes, so to have any hope of solving the fundamental topological problem of classifying the homotopy types of finite CW-complexes, one must determine the stable homotopy groups of spheres! To classify the stable homotopy types of the 3-cell complexes whose 2-skeleton is a fixed 2-cell complex X, this task amounts to calculating the stable homotopy groups of X; and so on. The point is that calculating stable homotopy groups of finite CW-complexes is of fundamental importance. See section 1.2 for further discussion of topological applications of stable homotopy groups of Bousfield-localized finite CW-complexes.

[4]In fact X need only satisfy a slightly weaker condition than this: it suffices that the filtration quotients of $\mathrm{tors}\,KU^0(X)$ by the skeletal filtration have square-free order. See section 3.1 for details.

projective cellular variety V over $\mathbb{Q}$ whose complex analytic space $\mathbb{C}(V)$ has rational cohomology concentrated in even degrees, the provisional KU-local zeta-function $\dot{\zeta}_{KU}(s, \mathbb{C}(V))$ of the *space* $\mathbb{C}(V)$ recovers the Hasse-Weil zeta-function $\zeta_V(s)$ of the *variety* V.

The main result in section 3, and in this paper, is a formula for KU-local stable homotopy groups of X in terms of special values of $\zeta_{KU}(s, X)$ at negative integers. To state the result, we must explain the factorization of $\zeta_{KU}(s, X)$ into *isoweight factors.* The factor $\dot{\zeta}_{KU}(s, X)$ of $\zeta_{KU}(s, X)$, which is sensitive to the *rational* K-theory of X, splits as a product $\prod_{w\in\mathbb{Z}} \dot{\zeta}^{(w)}_{KU}(s, X)$ of weight w factors, one for each integer w. Similarly, the factor $L_{\mathrm{tors}\, KU}(s, X)$ of $\zeta_{KU}(s, X)$, which is sensitive to the *torsion* in the K-theory of X, splits as a product $\prod_{w\in\mathbb{Z}} \prod_{\ell | n_w} L^{(w,\ell)}_{\mathrm{tors}\, KU}(s, X)$ of weight (w, ℓ) factors, where ℓ is an odd prime divisor of n_w, the order of the $2w$th filtration quotient in the skeletal filtration of the torsion subgroup of $KU^0(X)$. We now state Theorem 3.10 and Corollary 3.12, which identify orders of KU-local stable homotopy groups in terms of denominators of special values of these "isoweight" factors:

Theorem. *Let X be a finite CW-complex whose complex K-theory $KU^*(X)$ is concentrated in even degrees, and such that the torsion subgroup of $KU^0(X)$ has square-free order. Let a, b be the least and greatest integers n, respectively, such that $H^{2n}(X; \mathbb{Z})$ is nontrivial. Then the following conditions are equivalent:*

1. *The filtration of* $\mathrm{tors}\, KU^0(X)$ *by the skeleta of the CW-complex X splits completely, i.e., for each n the restriction* $(\mathrm{tors}\, KU^0(X^{2n}))^\wedge_p \to (\mathrm{tors}\, KU^0(X^{2n-2}))^\wedge_p$ *is a split surjection of $\hat{\mathbb{Z}}_p[\hat{\mathbb{Z}}_p^\times]$-modules for each odd prime p. Here $\hat{\mathbb{Z}}_p^\times$ acts on p-complete K-theory via Adams operations.*
2. *For all odd integers $2k-1$ satisfying $2k-1 > 1-2a$, the KU-local stable homotopy group $\pi_{2k-1}(L_{KU}DX)$ of the Spanier-Whitehead dual DX of X is finite, and up to powers of* 2*, its order is equal to the product*
$$\prod_{w\in\mathbb{Z}} \left(\mathrm{denom}\left(\dot{\zeta}^{(w)}_{KU}(1-k, X)\right) \cdot \prod_{\ell | n_w} \mathrm{denom}\left(L^{(w,\ell)}_{\mathrm{tors}\, KU}(1-k, X)\right)\right),$$
up to powers of 2*.*
3. *For all odd integers $2k-1$ satisfying $2k-1 < -2b-3$, the KU-local stable homotopy group $\pi_{2k-1}(L_{KU}DX)$ is finite of order*
$$\prod_{w\in\mathbb{Z}} \left(\mathrm{denom}\left(\dot{\zeta}^{(w)}_{KU}(k+1, X)\right) \cdot \prod_{\ell | n_w} \mathrm{denom}\left(L^{(w,\ell)}_{\mathrm{tors}\, KU}(k+1, X)\right)\right)$$
up to powers of 2*.*

Corollary. *Suppose that the the skeletal filtration of* $\mathrm{tors}\, KU^0(X)$ *splits completely. Write N for the order of the group* $\mathrm{tors}\, KU^0(X)$*. Then, for all odd integers $2k-1 \geq 1-2a$, the order of $\pi_{2k-1}(L_{KU}DX)$ is equal to the denominator of $\zeta_{KU}(1-k, X)$ up to powers of* 2 *and powers of F-irregular primes, where F ranges across the wildly ramified subfields of the cyclotomic field $\mathbb{Q}(\zeta_{N^2})$.*

"F-irregular primes" are studied in algebraic number theory, e.g. in [17] and [20]. The $\mathbb{Q}$-irregular primes are merely the ordinary irregular primes. A brief definition of the F-irregular primes is given preceding Corollary 3.12.

Theorem 3.10 and Corollary 3.12 demonstrate that the Adams-Baird-Ravenel calculation, Theorem 1.1, is not an isolated phenomenon, but in fact, KU-local stable homotopy groups of finite CW-complexes are quite commonly expressible in terms of special values of zeta-functions which are

- constructed in a natural way (mimicking Hasse-Weil zeta-functions—see below),
- have good properties (e.g. analytic continuation due to Proposition 3.6, functional equations due to Theorem 2.7 and the discussion in section 3.2),
- and are directly connected to number theory (since they are nontrivially equal to products of Tate twists of L-functions of Dirichlet characters, by Proposition 3.6) and arithmetic geometry (since $\zeta_V(s) = \dot{\zeta}_{KU}(s, \mathbb{C}(V))$ for many cellular varieties V).

We remark that the definitions and results of section 3 are more general than what is stated here in the introduction. In section 3, we allow for a set of primes P at which $KU^*(X)$ is *not* necessarily concentrated in even degrees, and has torsion subgroup which does not necessarily have square-free order. We define an "away-from-P" KU-local zeta-function $\zeta_{KU[P^{-1}]}(s, X)$ such that the denominators of the special values of $\zeta_{KU[P^{-1}]}(s, X)$ at negative integers are related to the orders of the KU-local stable homotopy groups of the Spanier-Whitehead dual of X, *up to powers of* 2 *and primes in* P. Consequently, the methods and results of section 3 can be applied to *all* finite CW-complexes whose rational cohomology is concentrated in even degrees.

In this paper, the exposition is oriented toward building up the theory of KU-local zeta functions of finite CW-complexes in an incremental way: in section 2, we begin with the Hasse-Weil zeta-function of a smooth projective variety over $\mathbb{Q}$, and we make the simple change of replacing the Weil cohomology with complex K-theory, and replacing the p-Frobenius operator on the Weil cohomology with the pth Adams operator on K-theory. This yields the "provisional KU-local zeta-function" $\dot{\zeta}_{KU}(s, X)$ which, as mentioned above, is sensitive only to information visible to the rational K-theory of X—and consequently sensitive only to the *rational cohomology* of X. From there, we make amendments and improvements to the provisional KU-local zeta-function, eventually arriving at the (non-"provisional") KU-local zeta-function $\zeta_{KU}(s, X)$ in section 3. The author hopes that this step-by-step method of exposition makes the definitions seem more natural, makes the motivations more obvious, and makes it more satisfying when we find (in Theorems 2.8 and 3.10) that the special values of these zeta-functions count orders of KU-local stable homotopy groups.

The subject matter of this paper has significant overlap with that of [38], but the constructions and results in this paper are quite different from those of [38], and the aim of [38] is the opposite of what this paper sets out to do. In [38], Zhang begins with a (p-adic) Dirichlet character χ, and from that character, constructs a KU-local spectrum whose homotopy groups are describable in terms of the denominators of the L-values of χ at negative integers. However, most KU-localizations of finite CW-complexes—e.g. the KU-localization of any spectrum whose rational homotopy groups have rank > 1—do not arise from a character in this way. By contrast, in the present paper, we begin with a finite CW-complex, and from it, we build a zeta-function (which, in the end, will always be equal to a product of shifts of Dirichlet L-functions, but this is nontrivial!) such that the denominators of its L-values recover the orders of the KU-local homotopy groups of X. So the "dictionary" we construct goes in the reverse direction from that of [38]. Having "dictionaries" in both directions is worthwhile, and we hope the reader agrees that this paper and [38] complement each other.

1.2. **The broader program.** Given a generalized homology theory E_* and a spectrum X, by [7] there exists a *Bousfield localization of X at E_**, written $L_E X$. The spectrum $L_E X$ is defined by a certain universal property; see section 1 of [26] for a very approachable introduction to L_E and its basic properties. In the case that E_* is $\pi_*(-)[P^{-1}]$, i.e., stable homotopy groups with some set P of prime numbers inverted, the effect of the Bousfield localization $L_E X$ on stable homotopy groups is merely to invert the primes in P. That is, $\pi_*\left(L_{\pi_*(-)[P^{-1}]}X\right) \cong \pi_*(X)[P^{-1}]$. In that sense, Bousfield localization generalizes the classical theory of localization in algebra, by which one inverts a set of primes.

However, there are many more generalized homology theories E_* than those of the form $\pi_*(-)[P^{-1}]$ for a set of primes P. For many choices of E_*, the relationship between $\pi_* X$ and $\pi_* L_E X$ is more mysterious and subtle than any classical algebraic localization, but yields data of great topological importance.

Here is the most pressing class of examples. For each prime p and each nonnegative integer n, there exists a generalized homology theory $E(n)_*$, the *p-primary height n Johnson-Wilson theory*. In the base case $n = 0$, $E(0)_*$ is merely classical rational homology, $H_*(-;\mathbb{Q})$, regardless of the prime p. On the other hand, for all positive integers n, the generalized homology theory $E(n)_*$ depends on the choice of prime number p, but the choice of p is traditionally suppressed from the notation $E(n)_*$.

In the case $n = 1$, the generalized homology theory $E(1)_*$ is the Adams summand of p-local complex K-theory, and consequently we have isomorphisms[5]

$$\pi_*(L_{E(1)}X) \simeq \pi_*(L_{KU_{(p)}}X) \simeq \pi_*(L_{KU}X)_{(p)}. \tag{3}$$

The KU-local and $E(1)$-local stable homotopy groups have been studied systematically in [8] and [9]. They have been useful: for example, Thomason [34] showed that KU-local stable homotopy groups of certain algebraic K-theory spectra agree with the étale K-theory groups of [14] and [15], and used this fact to great effect in calculations.

The $E(n)$-localizations $L_{E(n)}X$ for $n > 1$ are less familiar, but also important. Fix a prime p and a finite CW-complex X. The *chromatic tower* is a tower of spectra

$$\cdots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X$$

whose homotopy limit is weakly equivalent [27, Theorem 7.5.7] to the p-localization of X. The calculation of the stable homotopy groups of each individual stage $L_{E(n)}X$ is, at least in principle, approachable by a sequence of spectral sequences and homotopy fiber squares ("fracture squares") which begins with the continuous cohomology of the profinite automorphism group of a formal group law of height h over a finite field [12], for each $h \leq n$.

In the base case, $X = S^0$, the chromatic tower plays a central role in many algebraic topologists' understanding of the stable homotopy groups of spheres, i.e., the "stable stems": the p-local stable stems are recoverable from the infinite sequence $\cdots \to \pi_* L_{E(2)}S^0 \to \pi_* L_{E(1)}S^0 \to \pi_* L_{E(0)}S^0$, and each stage in the sequence is, at least in principle, calculable starting from certain cohomology calculations arising from formal group laws.

For $n \geq 2$, the stable homotopy groups of $L_{E(n)}X$ have generally been so complicated that the outcome of making a long and difficult calculation of $\pi_* L_{E(n)}X$ is a theorem whose *statement*—let alone the proof!—is prohibitively long and complicated; see for example the discussion in [5] of the pioneering calculations of [31]. It puts the subject in a difficult position when the outcomes of deep and important fundamental calculations are theorems which are extremely cumbersome to state, even to an audience of experts.

[5] To be clear: the expression $\pi_*(L_{KU}X)_{(p)}$ in (3) means the localization, in the classical sense, of the graded abelian group $\pi_*(L_{KU}X)$ at the prime p.

The description of $\pi_*(L_{E(1)}S^0)$ in terms of special values of $\zeta(s)$ (i.e., the p-local version of Theorem 1.1) suggests a way forward: *rather than attempt to describe $\pi_*(L_{E(n)}X)$ in elementary terms for each m, we might write down a description of the order of $\pi_m(L_{E(n)}X)$, or of various natural summands of $\pi_m(L_{E(n)}X)$, in terms of special values of various L-functions or zeta-functions*. While $\pi_m(L_{E(n)}X)$ would remain a somewhat mysterious object, at least the mystery would be fruitfully identified with some other compelling and well-studied mystery. Theorem 1.1 demonstrates that even in the case $n = 1$ and $X = S^0$, this approach already yields rewards, in the form of shorter and more natural *statements* of theorems.

The main theorem of this paper, Theorem 3.10, demonstrates that in the case $n = 1$, there is a simple and natural special-values description of $\pi_*(L_{KU[1/2]}X)$ for a wide range of finite CW-complexes X, not merely the case $X = S^0$ demonstrated by Adams-Baird and Ravenel, and not merely the case $X = S^0/p$ demonstrated in [29]. The author regards this as incremental progress toward more fully realizing the perspective described in the previous paragraph.

1.3. **The intended audience and scope of this paper.** The author has made an effort to write this paper so that it can be read by algebraic topologists who do not already know a lot about zeta-functions, and also number theorists who do not already know a lot about stable homotopy theory. As a consequence, sometimes a notion is explained which is very elementary and well-known to one audience, but not to the other. The author apologizes to any reader who finds this annoying.

Beginning in section 3, the author assumes that the reader is comfortable with basic notions about Dirichlet characters, e.g. primitivity and conductors. Such material is covered in many textbooks, like [3], but section 3 of the recent paper [29] is intended to be a "crash course" in those ideas specifically suited for an audience of algebraic topologists, so perhaps that reference can be particularly useful to some readers.

Remark 1.2. The results in this paper admit generalizations in several directions. One direction of generalization is to higher heights, i.e., describing E-local zeta-functions of finite CW-complexes, where E is the spectrum representing a complex-orientable cohomology theory whose p-height is > 1 for some primes p. However, that direction of generalization is entirely outside the scope of this paper: it is important to get the KU-local story right first, and that is what this paper tries to do.

Even within the KU-local story, in some places it is possible to generalize the results beyond what is described in this paper. For instance, beginning in section 3.1, we restrict our attention to CW-complexes whose K-theory is concentrated in even degrees. This is not strictly necessary, but yields cleaner statements and proofs, and makes the ideas and their motivations more obvious. In this paper the author has chosen to prioritize simplicity and well-motivatedness of the constructions, rather than reaching for the very greatest generality.

Remark 1.3. An alternative approach to producing KU-local L-functions and zeta-functions of finite CW-complexes is to construct *p-adic* L-functions using Iwasawa's machinery. Here is a sketch of the construction for $p > 2$. One begins with a finite CW-complex X, splits the group $\hat{\mathbb{Z}}_p^\times$ of p-adic Adams operations as $\mathbb{F}_p^\times \times \hat{\mathbb{Z}}_p$, then considers the ω^i-eigenspace $e_i\hat{KU}_p^0(X)$ of the action of a generator for $\mathbb{F}_p^\times$ on $\hat{KU}_p^0(X)$, where ω is a fixed primitive $(p-1)$th root of unity in $\hat{\mathbb{Z}}_p$. Using the structure theory for Iwasawa modules (see Theorem 13.12 of [35] for a textbook account) and the finiteness of X, each $e_i\hat{KU}_p^0(X)$

is pseudo-isomorphic to a direct sum $\bigoplus_{j=1}^{d_i} \Lambda/f_{i,j}(T)$ of quotients of the Iwasawa algebra $\Lambda \cong \hat{\mathbb{Z}}_p[[T]]$ by characteristic polynomials $f_{i,1}(T), \dots, f_{i,d_i}(T)$. If $d_i = 1$ for all i, then we define $L_p(s, \omega^i) := f_i(q^{1-s} - 1)$, with q a principal unit in $\hat{\mathbb{Z}}_p^{\times}$ (see discussion preceding Theorem 11.6.7 of [23] for a precise account), obtaining a set of $(p-1)$ p-adic L-functions of X; compare with Iwasawa's theorem $e_i(U/C) \sim \Lambda/f_i(T)$ as in Theorem 11.6.18 of [23]. The resulting "KU-local p-adic L-functions of finite CW-complexes" have some good properties but are not guaranteed to be the p-adic interpolations of complex-analytic functions. More fundamentally, this Iwasawa-theoretic way of building a p-adic L-function yields a p-adic power series which is only defined up to multiplication by an invertible series in $\hat{\mathbb{Z}}_p[[T]]^{\times}$, so without further work to pin down this indeterminacy, only the *p-adic valuations* of the special values are well-defined. These functions do not quite have well-defined special values at negative integers (or anywhere else)!

The author and his student A. Maison are investigating the resulting theory of p-adic L-functions for finite CW-complexes, but we regard that theory as beyond the scope of this paper, which is concerned with the complex-analytic case instead.

Conventions 1.4. Throughout this paper, all our CW-complexes will be implicitly understood to be *pointed*, and all our generalized homology and cohomology theories, including ordinary homology and K-theory and stable homotopy, are implicitly understood to be *reduced*. There is only one place in this paper, Lemma 2.3, where we think these conventions might lead to any confusion. We include a reminder of these conventions immediately before the statement of that lemma.

Furthermore, all our topological constructions on CW-complexes are *stable* constructions. Consequently, for the sake of the definitions and theorems in this paper, the reader is welcome to mentally replace every instance of a CW-complex X in this paper with the suspension spectrum of X, and more generally, to allow a finite *spectrum* (i.e., an arbitrary desuspension of a finite CW-complex) in place of X. For example, spheres of negative dimension are perfectly acceptable finite CW-complexes for the purposes of all the definitions and theorems in this paper!

We also consistently write π_* for *stable* homotopy groups. There are no unstable homotopy groups appearing in this paper.

I am grateful to an anonymous referee for helpful comments on this paper, especially for noticing and pointing out that the version of Definition 3.2 that appeared in the first draft of this paper did not make sense.

2. The provisional KU-local zeta-function of a finite CW-complex.

2.1. **Defining the provisional KU-local zeta-function.** We begin with a cursory account of what global Hasse-Weil zeta-functions (henceforth simply called "Hasse-Weil zeta-functions") are. For a fuller story, Weil's original article [36] is a very good starting place.

When constructed via cohomology, the Hasse-Weil zeta-function of a smooth projective variety X over $\mathbb{Q}$ begins as an Euler product over primes p of good reduction for X:

$$\prod_{p \notin P} \prod_{n \geq 0} \det(\mathrm{id} - p^{-s} \operatorname{Fr}_p |_{H^n_{\text{ét}}(X/\overline{\mathbb{F}}_p; \mathbb{Q}_\ell)})^{(-1)^{n+1}}, \tag{4}$$

where P is the set of primes of bad reduction, and where $H^n_{\text{ét}}(X/\overline{\mathbb{F}}_p; \mathbb{Q}_\ell)$ is ℓ-adic étale cohomology, for a prime $\ell \neq p$, applied to $\overline{\mathbb{F}}_p$-points of an integral model for X. The notation Fr_p denotes the pth-power relative Frobenius operator acting on the étale cohomology. The Euler product (4) converges for complex numbers s with $\Re\mathfrak{e}(s) >> 0$, and is then (when all goes well, depending on X!) analytically continued to a meromorphic function on the complex plane. With a bit more trouble, one can also include appropriate p-local Euler factors at the primes $p \in P$ of bad reduction. Since the set P is finite, such changes only affect finitely many Euler factors, hence do not affect many of the important analytic properties of the zeta-function. Special values of the resulting function $\zeta_X(s)$ are of deep interest in arithmetic geometry; for example, the strong form of the Birch-Swinnerton-Dyer conjecture predicts the behavior of $\zeta_E(s)$ at $s = 1$, for E an elliptic curve.

The element Fr_p is a topological generator for the profinite Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. The definition (4) makes good sense because the ℓ-adic étale cohomology of the variety X has, at each prime $p \neq \ell$, a natural action of the topologically cyclic profinite group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

Described in that way, étale cohomology resembles complex topological K-theory, KU^*. Recall that, for each prime number ℓ, the ℓ-adically complete complex topological K-theory $\hat{KU}_\ell$ admits an natural action of the profinite group $\hat{\mathbb{Z}}_\ell^\times$ of units in the ℓ-adic integers. This is the action by *Adams operations,* introduced in [1].

A comparison between Frobenius actions on ℓ-adic étale cohomology and Adams operations on ℓ-adic K-theory is, of course, not a new idea at all: see Sullivan's paper [32], for example, or work of Quillen in [25] establishing that, for $p \neq \ell$, the action of Ψ^p on the ℓ-completed complex K-theory spectrum $\hat{KU}_\ell$ agrees with the action of the p-Frobenius operator on the ℓ-completed algebraic K-theory spectrum $\mathcal{K}(\overline{\mathbb{F}}_p)\hat{}_\ell$ under the Suslin rigidity equivalence $\mathcal{K}(\overline{\mathbb{F}}_p)\hat{}_\ell \simeq \hat{KU}_\ell$. In that sense, Ψ^p really *is* the p-Frobenius operator.

Our aim in this section is to take that idea seriously, by mimicking the construction of Hasse-Weil zeta-functions of varieties but using Adams operations in place of Frobenius operations, to yield some kind of Hasse-Weil-like zeta-function of a *topological space X* (not a variety!), and to derive useful topological invariants of X from the special values of that zeta-function. The idea is simple: write down the Euler product (4) for the Hasse-Weil zeta-function, but

- rather than X being a smooth projective variety, let X be a finite CW-complex,
- for each prime ℓ, replace ℓ-adic étale cohomology with ℓ-adic complex topological K-theory,
- for each prime p, replace the Frobenius operator Fr_p on ℓ-adic étale cohomology with the pth Adams operation Ψ^p on ℓ-adic complex topological K-theory,
- and, since complex topological K-theory is 2-periodic, rather than taking the Euler product over all degrees, we will merely take the Euler product over degrees 0 and 1.

Here is the resulting definition:

Definition 2.1. *Let X be a finite CW-complex. Let P be the set of primes consisting of* 2 *and all of the primes at which the cohomology of X has p-torsion, i.e.,*

$$P = \{2\} \cup \{p \textit{ prime} : H^*(X;\mathbb{Z}) \textit{ has nontrivial p-torsion}\}.$$

For each prime number p, suppose we have chosen a prime number $\ell_p \notin P$ such that p is a topological generator for the group of ℓ_p-adic units $\hat{\mathbb{Z}}_{\ell_p}^\times$. Then the KU-theoretic Euler product for *X is defined as the product over all primes p:*

$$\prod_p \prod_{n\in\{0,1\}} \det(\mathrm{id} - p^{-s}\Psi^p \mid_{\hat{KU}^n_{\ell_p}(X)})^{(-1)^{n+1}} = \prod_p \frac{\det\left(\mathrm{id} - p^{-s}\Psi^p \mid_{\hat{KU}^1_{\ell_p}(X)}\right)}{\det\left(\mathrm{id} - p^{-s}\Psi^p \mid_{\hat{KU}^0_{\ell_p}(X)}\right)}. \tag{5}$$

The prime 2 will have to be avoided or excluded in various small ways later in this paper, simply due to the slightly different properties of 2-local K-theory as compared to K-theory localized at an odd prime (compare Theorem 8.10 to Theorem 8.15 in [26], for example). This happened in a small way in Definition 2.1 by including 2 in the set of primes P. Note that this did not cause any missing Euler factors in the Euler product (5): the 2-local Euler factor is still present in (5). The author suspects that the prime 2 can be incorporated elegantly into the theory presented in this paper, by using methods along the lines of Bousfield's "united K-theory" [9]. That extension of the theory goes beyond the scope of this paper, though.

In (5), we chose to use ℓ_p-adically completed K-theory to maintain the similarity with the Euler product (4) of a Hasse-Weil zeta-function. It would have worked just as well to use $KU[P^{-1}, p^{-1}]^*$, that is, complex K-theory with:

- p inverted, so that the stable Adams operation Ψ^p is defined, and
- all the primes in P inverted, so that $KU^*(X)[P^{-1}, p^{-1}]$ is a torsion-free (hence free) $\mathbb{Z}[P^{-1}, p^{-1}]$-module[6].

This change would not affect the determinants in (4), hence would not affect the resulting zeta-function or the theorems we prove about it, below.

Remark 2.2. The author would like to emphasize how naïve the Euler product (5) is: it comes from blindly mimicking the Euler product of the Hasse-Weil zeta-function, and in the process, losing hold of any clear interpretation in terms of Lefschetz fixed-point theory.

Recall from Conventions 1.4 that our CW-complexes in this paper are understood to be pointed, and our generalized (co)homology theories are understood to be reduced. Consequently $KU^*(X \vee Y) \cong KU^*(X) \oplus KU^*(Y)$. The reader who dislikes working with pointed spaces and reduced theories is welcome to drop the basepoints and work with non-reduced theories throughout this paper; the one place to be careful is that the wedge product in the statement of Lemma 2.3 would become a disjoint union.

Lemma 2.3. *The KU-theoretic Euler product of a wedge sum $X \vee Y$ is equal to the product of the KU-theoretic Euler products of X and of Y.*

Proof. The splitting $KU^*(X \vee Y) \cong KU^*(X) \oplus KU^*(Y)$ respects the Adams operations. Hence each of the characteristic polynomials $\det\left(\mathrm{id} - p^{-s}\Psi^p \mid_{\hat{KU}^n_{\ell_p}(X \vee Y)}\right)$ in (5) splits as the product of the characteristic polynomials $\det\left(\mathrm{id} - p^{-s}\Psi^p \mid_{\hat{KU}^n_{\ell_p}(X)}\right)$ and $\det\left(\mathrm{id} - p^{-s}\Psi^p \mid_{\hat{KU}^n_{\ell_p}(Y)}\right)$. □

Theorem 2.4 establishes the basic properties of the KU-theoretic Euler product, which are all straightforward consequences of the basic properties of the Chern character:

Theorem 2.4. *Let X be a finite CW-complex. Let P be as in Definition 2.1. Write b for the greatest integer n such that at least one of the two vector spaces $H^{2n}(X; \mathbb{Q}), H^{2n+1}(X; \mathbb{Q})$ is nontrivial. Then the following claims are each true:*

- *The KU-theoretic Euler product* (5) *converges absolutely for all complex numbers s with $\mathfrak{Re}(s) > 1 + b$.*
- *The KU-theoretic Euler product* (5) *analytically continues to a meromorphic function $\dot{\zeta}_{KU}(s, X)$ on the complex plane.*

[6] If $KU^*(X)[P^{-1}, p^{-1}]$ were not free, we would have to exercise a bit of care about what the determinant of Ψ^p acting on $KU^n(X)[P^{-1}, p^{-1}]$ ought to mean. We generalize and extend these ideas to handle torsion in K-theory starting in section 3.

- *The meromorphic function $\dot{\zeta}_{KU}(s,X)$ is equal to the L-function of a cellular motive. That is, $\dot{\zeta}_{KU}(s,X)$ is equal to a product of "shifts" of the Riemann zeta-function $\zeta(s)$. The product is as follows:*

$$\dot{\zeta}_{KU}(s,X) = \prod_{w\in\mathbb{Z}} \zeta(s-w)^{\beta_{2w}(X)-\beta_{2w+1}(X)}, \tag{6}$$

where $\beta_n(X) = \dim_{\mathbb{Q}} H^n(X;\mathbb{Q})$ is the nth Betti number of X.

- *The order of vanishing of $\dot{\zeta}_{KU}(s,X)$ at an integer $s = m$ is equal to*

$$-\beta_{2m-2}(X) + \beta_{2m-1}(X) + \sum_{k\geq 1} \left(\beta_{2m+4k}(X) - \beta_{2m+4k+1}(X)\right). \tag{7}$$

If the rational cohomology of X is concentrated in even degrees, then all *poles of $\dot{\zeta}_{KU}(s,X)$ occur at integers, and their orders are calculated by formula* (7).

Proof. Of course the determinant of the linear operator $\mathrm{id} - p^{-s}\Psi^p \mid_{(\hat{KU}_{\ell_p})^n(X)}$ acting on the free $\hat{\mathbb{Z}}_{\ell_p}$-module $(\hat{KU}_{\ell_p})^n(X)$ is unchanged by first passing to the fraction field $\mathbb{Q}_{\ell_p}$ of $\hat{\mathbb{Z}}_{\ell_p}$. However, since all the attaching maps in a minimal $S[P^{-1}]$-cell decomposition for $X[P^{-1}]$ are rationally trivial, the action of the Adams operation Ψ^p on the rationalized ℓ_p-adic K-groups

$$\mathbb{Q}\otimes_{\mathbb{Z}} (\hat{KU}_{\ell_p})^n(X) \cong (\ell_p^{-1}\hat{KU}_{\ell_p})^n(X)$$

agrees with the Adams operations on the rationalized ℓ_p-adic K-groups of a wedge of spheres. We have Adams-operation-preserving isomorphisms

$$(\ell_p^{-1}\hat{KU}_{\ell_p})^0(X) \cong \coprod_{n\in\mathbb{Z}} \ell_p^{-1}(\hat{KU}_{\ell_p})^0\left(S^{2n}(H^{2n}(X;\mathbb{Q}))\right) \quad\text{and}$$
$$(\ell_p^{-1}\hat{KU}_{\ell_p})^1(X) \cong \coprod_{n\in\mathbb{Z}} \ell_p^{-1}(\hat{KU}_{\ell_p})^1\left(S^{2n+1}(H^{2n+1}(X;\mathbb{Q}))\right),$$

where $S^m(V)$ denotes the wedge of spheres whose rational homology in degree m is V, and whose homology in all other degrees is trivial[7].

The point is that the KU-theoretic Euler product for X agrees with the KU-theoretic Euler product for a wedge of spheres. The number of n-spheres in this wedge is equal to the $\mathbb{Q}$-linear dimension of $H^n(X;\mathbb{Q})$, i.e., β_n. By Lemma 2.3, the KU-theoretic Euler product of X is equal to the product of the KU-theoretic Euler products of each of the spheres in that wedge. The Adams operation Ψ^p acts on $(\hat{KU}_{\ell_p})^0(S^{2n}) \cong \hat{\mathbb{Z}}_{\ell_p}$ as multiplication by p^n. Hence the Euler factor for S^{2n} at the prime p is $\frac{1}{1-p^{-s}p^n} = \frac{1}{1-p^{n-s}}$, i.e., it is the Euler factor of S^0 with $s-n$ "plugged in" for s. Similarly, the Euler factor for S^{2n+1} is $1-p^{n-s}$. This yields that the Euler product for X factors as $\prod_{w\in\mathbb{Z}}\prod_p \left(\frac{1}{1-p^{w-s}}\right)^{\beta_{2w}(X)-\beta_{2w+1}(X)}$, which is precisely the Euler product for (6). The third claim is now proven.

The product (6) converges absolutely for all complex numbers s with $\mathfrak{Re}(s) > 1+b$, since the Riemann zeta-function converges absolutely and is zero-free to the right of the line $\mathfrak{Re}(s) = 1$. Each of the factors $\zeta(s-w)$ in (6) analytically continues to a meromorphic function on the complex plane, so the finite product (6) of those factors also analytically continues to the plane. By uniqueness of analytic continuation, the meromorphic function $\dot{\zeta}_{KU}(s,X)$ is well-defined. This proves the first and second claims.

---

[7]This kind of observation is, of course, quite old: it is essentially just the statement of what the Chern character does to Adams operations, and it is why Bousfield imposes "rational diagonalizability conditions" on the objects of his categories $\mathcal{A}(p)$ and $\mathcal{B}(p)$ constructed in [8], for example.

The fourth claim, about the poles of $\dot{\zeta}_{KU}(s,X)$, follows from the fact that the only pole of the Riemann zeta-function in the complex plane occurs at $s = 1$, and the fact that the only poles of $\frac{1}{\zeta(s)}$ at integers are the trivial zeroes of $\zeta(s)$, which are simple, and occur at all negative even integers. □

Theorem 2.4 suggests that we define the KU-local zeta-function of a finite CW-complex as the analytic continuation of the KU-theoretic Euler product (5). This seems natural from the point of view of Hasse-Weil zeta-functions: after all, we were led to the KU-theoretic Euler product by a very simple analogy with the Hasse-Weil zeta-function. It is reasonable to study the analytic continuation described in Theorem 2.4, but when we reach section 3, we will have good reason to define the KU-local zeta-function of a finite CW-complex as a modification and refinement of that analytic continuation, one which pays attention to torsion in K-theory. For now, we will take the analytic continuation of (5) as a *provisional* version of the KU-local zeta-function:

Definition 2.5. *Let P be a finite set of primes, with $2 \in P$. Suppose that X is a finite CW-complex. We refer to the meromorphic function $\dot{\zeta}_{KU}(s,X)$ of Theorem 2.4 as the* provisional KU-local zeta function of X. *That is,*

$$\dot{\zeta}_{KU}(s,X) = \prod_{w\in\mathbb{Z}} \zeta(s-w)^{\beta_{2w}(X)-\beta_{2w+1}(X)}. \tag{8}$$

Finally, for an integer w, we write

$$\dot{\zeta}^{(w)}_{KU}(s,X) = \zeta(s-w)^{\beta_{2w}(X)-\beta_{2w+1}(X)},$$

and we refer to $\dot{\zeta}^{(w)}_{KU}(s,X)$ as the weight w factor in $\dot{\zeta}_{KU}(s,X)$. *Any single such factor will be called an* isoweight *factor.*

The provisional KU-local zeta-function is clearly a very crude invariant of a finite CW-complex. If X and Y are finite CW-complexes whose rational cohomology is concentrated in even degrees (which will be the case of greatest interest for much of the rest of this paper), then $\dot{\zeta}_{KU}(s,X) = \dot{\zeta}_{KU}(s,Y)$ if and only if X and Y are *rationally* stably homotopy-equivalent. **In other words, $\dot{\zeta}_{KU}(s,-)$ is really a $H\mathbb{Q}$-local invariant, not only a KU-local invariant.** It is a peculiar fact that, on the finite CW-complexes whose cohomology is torsion-free and concentrated in even degrees, the orders of the KU-local stable homotopy groups are *also* a $H\mathbb{Q}$-local invariant, and in fact are recoverable from the special values of $\dot{\zeta}_{KU}(s,-)$: see Theorem 2.4, below.

Example 2.6. Here is a very simple example of a provisional KU-local zeta-function of a finite CW-complex. It is easy to use the methods in this section to see that, for any integer n, the complex projective space $\mathbb{C}P^n$ satisfies[8]

$$\dot{\zeta}_{KU}(s,\mathbb{C}P^n) = \prod_{w=0}^{n} \zeta(s-w). \tag{9}$$

This is precisely the Hasse-Weil zeta-function of the projective space P^n regarded as a variety over $\mathbb{Q}$.

[8]To be careful about basepoints: the left-hand side of equation (9) is the provisional KU-local zeta-function of $\mathbb{C}P^n$ *regarded as a spectrum*, i.e., the suspension spectrum of $\mathbb{C}P^n$ with a disjoint basepoint adjoined. Similarly, in the left-hand side of (10), we are taking the provisional KU-local zeta-function of the suspension spectrum of $\mathbb{C}(V)$ with a disjoint basepoint adjoined.

A trivial zero in one zeta-factor in (9) may cancel with the pole in another zeta-factor, occasionally yielding amusing calculations like

$$\begin{aligned}\dot{\zeta}_{KU}(1, \mathbb{C}P^3) &= \zeta(1)\zeta(0)\zeta(-1)\zeta(-2)\\ &= \frac{-\gamma\cdot\zeta(3)}{96\pi^2},\end{aligned}$$

where γ is the Euler constant. This example demonstrates that, while $\zeta(s)$ has a pole at $s = 1$, it is not always true that $\dot{\zeta}_{KU}(s, X)$ has a pole at $s = 1$.

More generally, suppose V is a smooth projective *cellular* variety over $\mathbb{Q}$ with associated complex analytic space $\mathbb{C}(V)$. Suppose that the cohomology $H^*(\mathbb{C}(V); \mathbb{Q})$ is concentrated in even degrees. Then the provisional KU-local zeta-function of the *space* $\mathbb{C}(V)$ recovers the Hasse-Weil zeta-function of the *variety* V:

$$\dot{\zeta}_{KU}\,(s, \mathbb{C}(V)) = \zeta_V(s). \tag{10}$$

One cannot expect (10) to generalize to non-cellular varieties V, like elliptic curves, since the zeta-functions of such varieties are not products of shifts of $\zeta(s)$. Put another way, the Hasse-Weil zeta-function of a non-cellular variety captures genuinely *arithmetic* information about the variety, not merely topological information. Hence for non-cellular V one cannot expect to recover $\zeta_V(s)$ from a topological invariant of $\mathbb{C}(V)$ like $\dot{\zeta}_{KU}\,(s, \mathbb{C}(V))$.

2.2. **The functional equation of the provisional KU-local zeta-function.** For each given weight w, the weight w factor $\dot{\zeta}^{(w)}_{KU}(s, X)$ satisfies a functional equation relating $\dot{\zeta}^{(w)}_{KU}(s, X)$ to $\dot{\zeta}^{(w)}_{KU}(1 + w - s, X)$. This functional equation is easily extracted from the functional equation from the Riemann zeta-function:

(11)

$$\dot{\zeta}^{(w)}_{KU}(s, X) = \left(2^{s-w}\pi^{s-w-1}\sin\left(\frac{\pi(s-w-1)}{2}\right)\Gamma(1+w-s)\right)^{\beta_{2w}(X)-\beta_{2w+1}(X)}\dot{\zeta}^{(w)}_{KU}(1+w-s, X).$$

The functional equation (11) is of the usual form of the functional equation of the L-function of a weight w motive M, which relates $L(s, M)$ to $L(1 + w - s, M)$.

One ought to compare this situation to the situation of the classical Hasse-Weil zeta-function of a smooth projective variety over $\mathbb{Q}$. That Hasse-Weil zeta-function *does* admit a functional equation—it is not only that each of its isoweight factors has a functional equation, but the whole zeta-function itself also does. But the existence of that functional equation relies crucially on the (Weil) cohomology of a smooth projective variety satisfying Poincaré duality.

As an example, consider the case of the projective line. An elementary and classical calculation of the Hasse-Weil zeta-function yields $\zeta_{\mathbb{P}^1}(s) = \zeta(s)\zeta(s-1)$. If we agree to write $\hat{\zeta}_{\mathbb{P}^1}(s)$ for the completed zeta-function $\hat{\zeta}(s)\hat{\zeta}(s-1)$, where $\hat{\zeta}(s)$ is the completed Riemann zeta-function satisfying $\hat{\zeta}(s) = \hat{\zeta}(1-s)$, then we have

$$\begin{aligned}\hat{\zeta}_{\mathbb{P}^1}(s) &= \hat{\zeta}(s)\hat{\zeta}(s-1)\\ &= \hat{\zeta}(1-s)\hat{\zeta}(2-s)\\ &= \hat{\zeta}_{\mathbb{P}^1}(2-s),\end{aligned} \tag{12}$$

yielding a functional equation for $\zeta_{\mathbb{P}^1}(s)$. However, the equation (12) exchanged the weight zero $\hat{\zeta}$-factor coming from the Weil $H^0(\mathbb{P}^1)$ with its Poincaré dual weight 1 $\hat{\zeta}$-factor coming from the Weil $H^2(\mathbb{P}^1)$!

In the general story of KU-local zeta-functions of finite CW-complexes, we have not restricted our attention to finite CW-complexes satisfying any form of Poincaré duality, so we cannot expect to get a simple functional equation relating $\dot{\zeta}_{KU}(s, X)$ to $\dot{\zeta}_{KU}(n - s, X)$ for any single value of n. Without assuming some kind of self-duality properties on X, it seems we must compromise:

- either settle for having only a functional equation for each isoweight factor of $\dot{\zeta}_{KU}(s, X)$,
- or settle for having a functional equation relating $\dot{\zeta}_{KU}(s, X)$ to $\dot{\zeta}_{KU}(s, X^*)$ for some kind of dual X^* of X.

We already gave the outcome of the first compromise above, in (11).

The second compromise yields a better result. One formulation of the functional equation for motivic L-functions is as follows: if M is a motive, with dual motive $M^\vee$, then a functional equation relates $L(s, M)$ to $L(1 - s, M^\vee)$. The functional equation for $\dot{\zeta}_{KU}(s, X)$ has an especially nice expression in similar terms. Let us write $\hat{\dot{\zeta}}_{KU}(s, X)$ for the *completed provisional KU-local zeta-function of X,* which we define naïvely by replacing each zeta-factor $\zeta(s - w)$ in (8) with its completion $\hat{\zeta}(s - w)$ in the classical sense:

$$\begin{aligned}\hat{\dot{\zeta}}_{KU}(s, X) &:= \prod_{w\in\mathbb{Z}} \hat{\zeta}(s - w)^{\beta_{2w}(X)-\beta_{2w+1}(X)}\\ &= \prod_{w\in\mathbb{Z}} \left(\pi^{-(s-w)/2}\Gamma(\frac{s-w}{2})\zeta(s - w)\right)^{\beta_{2w}(X)-\beta_{2w+1}(X)}.\end{aligned}$$

Theorem 2.7. *If we write D for the Spanier-Whitehead dual of a finite spectrum X with torsion-free cohomology concentrated in even degrees, then we have the functional equation*

$$\hat{\dot{\zeta}}_{KU}(s, X) = \hat{\dot{\zeta}}_{KU}(1 - s, DX). \tag{13}$$

Proof. Elementary from (11), Theorem 2.4, and the fact that $H^n(DX; \mathbb{Q}) \cong H^{-n}(X; \mathbb{Q})$. □

2.3. **Special values of the provisional KU-local zeta-function.** We now study the special values of the provisional KU-local zeta-function of a finite CW-complex X at negative integers. Theorem 2.8 generalizes Theorem 1.1, the Adams-Baird-Ravenel calculation of the $KU[1/2]$-local stable homotopy groups of spheres in terms of special values of $\zeta(s)$.

Theorem 2.8. *Let X be a finite CW-complex, and let P be the set of primes defined in Definition 2.1. Suppose that $H^*(X; \mathbb{Z}[P^{-1}])$ is concentrated in even dimensions. Write DX for the Spanier-Whitehead dual of X. Write a, b for the least and greatest integers n, respectively, such that $H^{2n}(X; \mathbb{Q})$ is nontrivial. Then we have the following consequences:*

- *The KU-local stable homotopy group $\pi_{2k}(L_{KU}DX)[P^{-1}]$ is trivial if either $2k \geq 1 - 2a$ or $2k < -2b - 2$.*
- *If $2k - 1 \geq 1 - 2a$, then the group $\pi_{2k-1}(L_{KU}DX)$ is finite of order equal to*[9]

$$\prod_{w\in\mathbb{Z}} \mathrm{denom}\left(\dot{\zeta}^{(w)}_{KU}(1 - k, X)\right), \tag{14}$$

up to factors of primes in P. We furthermore have an equality

$$|\pi_{2k-1}(L_{KU}DX)| = \mathrm{denom}\left(\dot{\zeta}_{KU}(1 - k, X)\right) \tag{15}$$

[9]The integer (14) is indeed well-defined, i.e., the special value $\dot{\zeta}^{(w)}_{KU}(1 - k, X)$ of the weight w factor of $\dot{\zeta}_{KU}(1 - k, X)$ is rational for all $2k - 1 \geq 1 - 2a$. Similar observations yield well-definedness of (15), (16), and (17).

up to powers of irregular primes[10] *and primes in P.*

- *If $2k-1 < -2b-3$, the group $\pi_{2k-1}(L_{KU}DX)$ is finite of order equal to*

$$\prod_{w\in\mathbb{Z}} \mathrm{denom}\left(\dot{\zeta}_{KU}^{(w)}(1+k,X)\right). \tag{16}$$

up to factors of primes in P. We furthermore have an equality

$$|\pi_{2k-1}(L_{KU}DX)| = \mathrm{denom}\left(\dot{\zeta}_{KU}(1+k,X)\right) \tag{17}$$

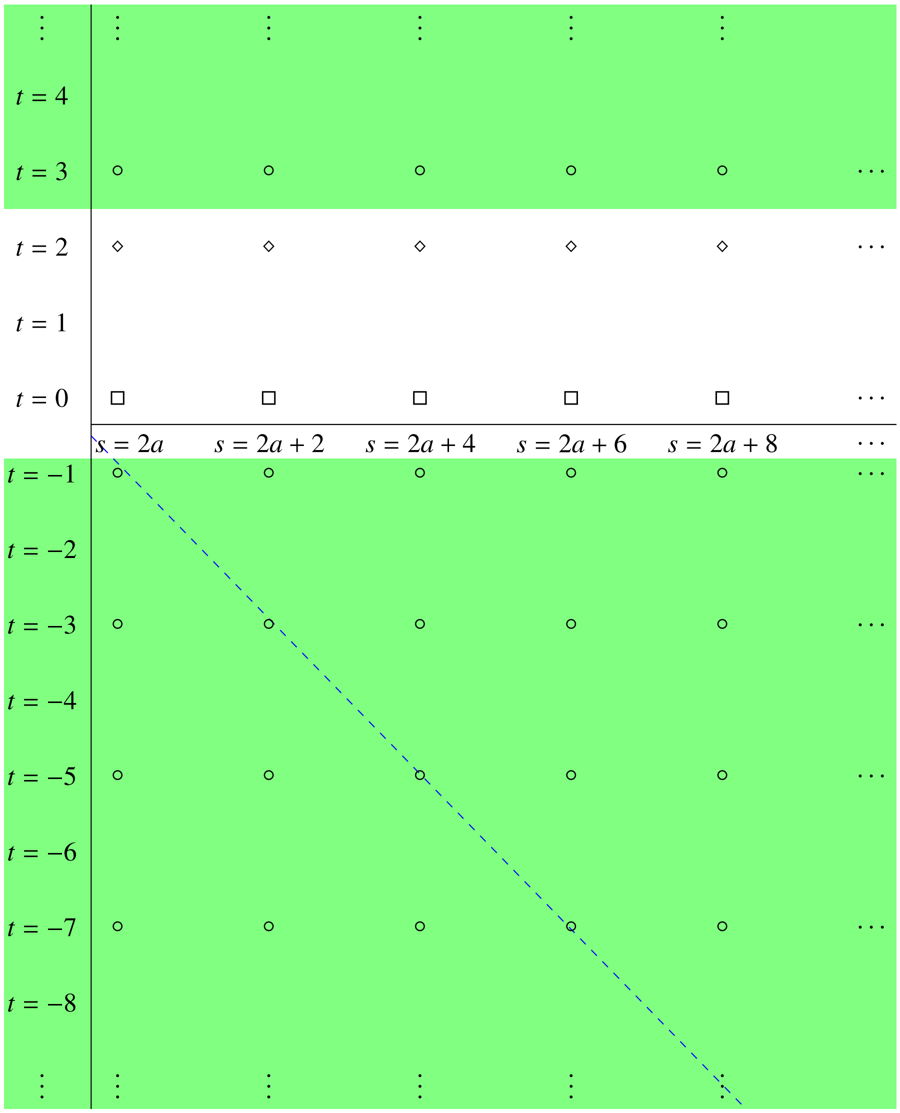
up to powers of irregular primes and primes in P.

Proof. A simple spectral sequence argument suffices. Because P includes all primes at which the cohomology of X has nontrivial torsion, the Atiyah-Hirzebruch spectral sequence for the generalized cohomology theory represented by $L_{KU[1/2]}S^0$ takes the form

$$\begin{aligned} E_2^{s,t} \cong H^s(X;\mathbb{Z}[P^{-1}]) \otimes_{\mathbb{Z}[1/2]} \pi_{-t}(L_{KU[1/2]}S^0) &\Rightarrow \pi_{-s-t}F\left(X[P^{-1}], L_{KU[1/2]}S^0\right) \\ &\cong \pi_{-s-t}\left(L_{KU[1/2]}DX\right)[P^{-1}] \\ d^r : E_r^{s,t} &\to E_r^{s+r,t-r+1}. \end{aligned}$$

The spectral sequence converges strongly, since X is finite. Plotted with the Serre conventions, the spectral sequence E_2-term is of the following form:

[10]In number theory, regular primes (and their complement, the irregular primes) are classical and well-studied, but since some of the imagined audience for this paper includes topologists who may not be familiar with regular primes, here is a quick recap. A prime p is *irregular* if and only if p divides the class number of the cyclotomic field $\mathbb{Q}(\zeta_p)$. Kummer showed that p is irregular if and only if p divides the numerator of the Bernoulli number $B_{2n} = -2n\cdot\zeta(1-2n)$ for some $n \in \{1,\dots,\frac{p-3}{2}\}$. Equivalently, by the Kummer congruences on Bernoulli numbers: p is irregular if and only if p divides the numerator of B_{2n} for some $n \geq 1$.



- In the bidegrees marked with squares (i.e., the $t = 0$ row, in the even-numbered columns), we have a direct sum of copies of $\mathbb{Z}[P^{-1}]$.
- In the bidegrees marked with diamonds (i.e., the $t = 2$ row, in the even-numbered columns), we have a direct sum of copies of $\mathbb{Q}/\mathbb{Z}[P^{-1}]$.
- The white-colored region is trivial in all other bidegrees.
- In the green-colored regions, in each bidegree in an even-numbered column and an odd-numbered row, we have a finite abelian group (perhaps trivial, depending on P and the bidegree). Those bidegrees are marked with a circle.
- The green-colored regions are trivial in all other bidegrees.

In particular, on and below the blue dashed line (i.e., the line $s + t = 2a - 1$), the E_2-term is concentrated in even-numbered columns and odd-numbered rows. All differentials originating below the blue-dashed line are zero for degree reasons, and no differentials originating above the blue-dashed line can hit elements below the blue-dashed line.

Hence the elements below the blue-dashed line in the E_2-term survive unchanged to the E_∞-term. The bidegrees below the blue-dashed line are precisely those which contribute, in the abutment, to $\pi_n(L_{KU[P^{-1}]}DX)$ with $n \geq 1-2a$.

An analogous argument shows that there can be no nonzero differentials involving bidegrees strictly above the line $s+t=2b+2$. Consequently the E_∞-term coincides with the E_2-term in all those bidegrees which contribute in the abutment to $\pi_n(L_{KU[P^{-1}]}DX)$ with $n < -2b-2$.

Depending on the attaching maps in the CW-complex X, there may be additive filtration jumps, so that the abutment is not simply the direct sum of the bidegrees in the E_∞-page. However, such filtration jumps do not affect the *number* of elements in a given degree. Hence, if $n < -2b-2$ or $n \geq 1-2a$, then the order of the homotopy group $\pi_n(L_{KU}DX)[P^{-1}]$ is equal to the product of the orders of the groups $H^t(X;\mathbb{Z}[P^{-1}]) \otimes_{\mathbb{Z}[1/2]} \pi_{n+t}(L_{KU[1/2]}S^0))$ for all integers t. The calculation of the orders of the groups $\pi_n(L_{KU[1/2]}S^0))$ for $n < -2$, from Theorem 1.1, then yields the first claim in the statement of the theorem: if $n \geq 1-2a$, then the total order of the bidegrees (s,t) below the dashed blue line contributing to $\pi_n(L_{KU}DX)[P^{-1}]$ is equal to

$$\prod_j \left|\pi_{2j+n}(L_{KU[1/2]}S^0)\right|^{\beta_{2j}(X)} = \begin{cases} 1 & \text{if } 2 \mid n \\ \prod_j \operatorname{denom}(\zeta(-j-\frac{n-1}{2}))^{\beta_{2j}(X)} & \text{if } 2 \nmid n \end{cases}$$

up to powers of primes in P. By a similar argument, if $n < -2b-3$, then the total order of the bidegrees (s,t) below the dashed blue line contributing to $\pi_n(L_{KU[1/2]}DX)$ is equal to

$$\begin{aligned}\prod_j \left|\pi_{2j+n}(L_{KU[1/2]}S^0)\right|^{\beta_{2j}(X)} &= \prod_j \left|\pi_{-2-2j-n}(L_{KU[1/2]}S^0)\right|^{\beta_{2j}(X)} \\ &= \begin{cases} 1 & \text{if } 2 \mid n \\ \prod_j \operatorname{denom}(\zeta(j+\frac{n+3}{2}))^{\beta_{2j}(X)} & \text{if } 2 \nmid n \end{cases}\end{aligned}$$

up to powers of primes in P.

Consequently:

- $\pi_n(L_{KU[P^{-1}]}DX)$ vanishes for even integers n satisfying $n \geq 1-2a$ or $n < -2b-3$. This proves the first claim.
- If $2k-1 \geq 1-2a$, then $\pi_{2k-1}(L_{KU[P^{-1}]}DX)$ is finite of order $\prod_j \operatorname{denom}(\zeta(1-j-k))^{\beta_{2j}(X)}$, up to factors of primes in P. This yields formula (14) in the second claim.
- If $2k-1 < -2b-3$, then $\pi_{2k-1}(L_{KU[P^{-1}]}DX)$ is finite of order $\prod_j \operatorname{denom}(\zeta(j+k+1))^{\beta_{2j}(X)}$, up to factors of primes in P. This yields formula (16) in the third claim.

The product (14) is equal to the denominator of $\dot{\zeta}_{KU}(1-k,X)$ up to primes in P and primes which occur in *numerators* of values of $\dot{\zeta}^{(w)}_{KU}(1-k,X)$, since such factors in numerators may plausibly cancel with factors in denominators of $\dot{\zeta}^{(w')}_{KU}(1-k,X)$ for some weights $w' \neq w$. By classical work of Kummer and the relationship between Riemann zeta special-values and Bernoulli numbers, it is precisely the irregular primes which occur as numerators of special values of $\zeta(s)$ at negative integers[11]. Formulas (15) and (17) follow. □

[11]It is possible to improve on this theorem by showing that under some conditions on X, there is no cancellation between factors in numerators of special values and factors in denominators of special values. The relevant tool here is the Herbrand-Ribet theorem [28], which, for a given irregular prime p, gives an algebraic characterization of *which* special values of $\zeta(s)$ will have numerator divisible by p. These methods are highly compatible with those in this paper, but the resulting conditions on X are more meticulous and technical to state. We leave off that direction of investigation for a later time.

Corollary 2.9. *Let V be a smooth projective cellular variety over $\mathbb{Q}$ of positive dimension, with Hasse-Weil zeta-function $\zeta_V(s)$. Let $\mathbb{C}(V)$ denote the complex analytic space associated to V. Suppose that the cohomology $H^*(\mathbb{C}(V);\mathbb{Q})$ is concentrated in even degrees. Let P be the set of primes*

$$P = \{2\} \cup \{p \text{ prime} : H^*(\mathbb{C}(V);\mathbb{Z}) \text{ has nontrivial } p\text{-torsion}\}.$$

Then, for each positive odd integer $2k-1$, the denominator of the Hasse-Weil zeta-value $\zeta_V(1-k)$ is equal to the order of the KU-local stable homotopy group $\pi_{2k-1}(L_{KU}D(\mathbb{C}(V)))$ of the Spanier-Whitehead dual $D(\mathbb{C}(V))$, up to powers of irregular primes and primes in P.

Proof. Consequence of Example 2.6 and Theorem 2.8. □

Remark 2.10. The statement of Theorem 2.8 suggests that, at irregular primes, there may be a discrepancy between the denominator of $\dot{\zeta}_{KU}(1-k, X)$ and the product (14), i.e., the order of $\pi_{2k-1}(L_{KU}DX)$. This indeed can happen. As an amusing example, let X be the cofiber of the map $S^{1355} \to S^0$ given by the 227-primary $\alpha_3 \in \pi_{1355}(S^0)$, i.e., a generator for the 227-torsion in the third stable stem in which nonzero 227-torsion appears[12]. Then $\pi_{-25}L_{KU}DX$ is cyclic of order equal to

$$\begin{aligned}\mathrm{denom}(\zeta(-11)) \cdot \mathrm{denom}(\zeta(-689)) &= 32760 \cdot 387923085396 \\ &= 2^5 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 31 \cdot 47 \cdot 139 \cdot 691\end{aligned}$$

up to a power of 2, while the denominator of $\dot{\zeta}_{KU}(-11, X)$ is equal to

$$\mathrm{denom}(\zeta(-11) \cdot \zeta(-689)) = 2^5 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 31 \cdot 47 \cdot 139.$$

The discrepancy is because the prime 691 is irregular: $\zeta(-11) = \frac{691}{32760}$, and the factor of 691 in the numerator of $\zeta(-11)$ cancels with the factor of 691 in the denominator of $\zeta(-689)$.

Here is one more comment on the "crudeness" of the provisional KU-local zeta-function.

Definition 2.11. *Suppose E is a spectrum such that $\pi_n(L_E S^0)$ is finite for all $n << 0$. Let say that finite spectra X and Y are E-*locally numerically equivalent *if there exists some integer N such that*

- *for all $n < N$, the abelian groups $\pi_n(L_E X)$ and $\pi_n(L_E Y)$ are each finite,*
- *and for all $n < N$, $|\pi_n(L_E X)| = |\pi_n(L_E Y)|$.*

If X and Y are E-locally equivalent (in the sense of Bousfield [7]), then X and Y are clearly also E-locally numerically equivalent. However, the converse is not true: we see from Theorem 2.8 that, for finite spectra X and Y with torsion-free cohomology concentrated in even degrees, all that is necessary for X and Y to be numerically KU-locally equivalent is for X and Y to be *rationally* equivalent. Of course there are many such examples: for example, the cofiber of $\alpha_1 \in \pi_{2p-3}(S^0)$ is a finite spectrum X whose cohomology is torsion-free and concentrated in even degrees. This spectrum X is *not* KU-locally equivalent to the wedge sum $S^0 \vee S^{2p-2}$, and yet X and $S^0 \vee S^{2p-2}$ are rationally equivalent, and KU-locally numerically equivalent. One can replace the role of α_1 here with any one of the divided alpha elements in the stable stems, and arrive at the same conclusion.

All those examples are merely 2-cell complexes, and of course attaching more cells to kill KU-locally nontrivial torsion elements in π_* in odd degrees yields many more examples. Our point is simply that there is an ample supply of KU-local stable homotopy types to which Theorem 2.8, and more generally the results of this section, apply.

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[12] To be clear: the numbers 227 and 691 appearing in this example are indeed prime.

Remark 2.12. Of course the group $\hat{\mathbb{Z}}_p^\times$ of Adams operations on p-complete K-theory is abstractly isomorphic to the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$. Sullivan's approach to the Adams conjecture [32] involved producing a *particular* such isomorphism. Consequently, if we take the K-theory of a finite CW-complex and tensor it with the complex numbers, the resulting complex vector space $KU^0(X) \otimes_{\mathbb{Z}} \mathbb{C}$ is a representation of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$, hence by Kronecker-Weber, a direct sum of degree 1 representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The provisional zeta function $\dot{\zeta}_{KU}(s, X)$ agrees with the L-function of that Galois representation.

It is not hard to formulate the ideas in this section as a kind of "class field theory of spectra." Recall that the local Langlands correspondence for GL_n establishes an L-function-preserving bijection between certain irreducible representations of $GL_n(K)$ and certain irreducible degree n representations of $\mathrm{Gal}(\overline{K}/K)$. The $n = 1$ case amounts to class field theory: the Dirichlet characters (or more generally, Hecke characters) on the "automorphic side" of the correspondence are matched up with degree 1 Galois representations on the "spectral side" of the correspondence. The Dirichlet (or Hecke) L-functions on the automorphic side are equal to the Artin L-functions on the spectral side.

The complex K-theory of a finite CW-complex, tensored up to $\mathbb{C}$, yields representations of the abelianized Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. One can match up these Galois representations with Dirichlet characters in a way that preserves the L-functions. This is fine, but in the state described in this section, it is not a very good theory. There are simply too few Galois representations which arise as $KU^0(X) \otimes_{\mathbb{Z}} \mathbb{C}$ for X a finite CW-complex.

Any genuinely *useful* class field theory of spectra ought to involve many more Galois representations than the ones arising in this section. In the next section, we give a more general construction of Galois representations associated to finite CW-complexes. That construction yields a much richer supply of Galois representations and corresponding L-functions. The essential idea is not merely to tensor the complex K-theory $KU^*(X)$ with $\mathbb{C}$, which destroys all information about torsion elements in $KU^*(X)$. Instead, we must find a way to use the *torsion* in K-theory in the process of defining KU-local L-functions and zeta-functions.

3. The KU-local zeta-function of a finite CW-complex with squarefree torsion in K-theory.

3.1. **Defining the KU-local torsion L-function and KU-local zeta-function.**

In this section, we will modify and improve the "provisional KU-local zeta-function" $\dot{\zeta}_{KU}(s, X)$ of a finite CW-complex X. It is now time to restrict the level of generality, in order to tidy up the theory and make it far more usable. The theorems proven in section 2.3 applied only to finite CW-complexes whose cohomology (hence also K-theory), after inverting P, is concentrated in even degrees. Earlier, in section 2, we defined the provisional KU-local zeta-function in such a way so as not to disallow CW-complexes with K-theory in odd degrees, in order to maintain similarity with the classical story of Hasse-Weil zeta-functions. But, as the reader can see from section 2.3, when the cohomology $H^*(X; \mathbb{Z}[P^{-1}])$ is not concentrated in even degrees, we were able to prove much less about $\dot{\zeta}_{KU}(s, X)$.

It will simplify our theory dramatically if, from now on, we only work with CW-complexes whose cohomology is concentrated in even degrees, after inverting a set of primes P. Beginning in this section, we restrict to that level of generality[13].

[13] For the reader who may find this restricted generality disappointing, we remark that it is probably sensible to think about *two* KU-local zeta-functions for a given finite CW-complex, one for the even K-groups and one for the odd K-groups, and to simply treat them as a pair, rather than as a single function. Something roughly

In Definition 3.8, below, we will give our improved version of the "provisional KU-local zeta-function." We will also need an auxiliary definition of something like a "KU-local Dirichlet L-function." The idea is to modify our definition of the provisional KU-local zeta-function with some extra factors that keep track of Adams operations on torsion in the K-theory of the CW-complex X. We cannot simply take the determinant of the action of the Adams operations on torsion in K-theory, since we would not get well-defined complex numbers that way. Instead, we must think of the various ways of embedding the torsion in $KU^0(X)$ into the complex numbers. That is, we must consider the various *complex representations* of the torsion subgroup of $KU^0(X)$. To each such representation (satisfying some hypotheses) we will associate an L-function.

For a finite CW-complex X with torsion-free cohomology (hence torsion-free K-theory) concentrated in even degrees, the filtration of K-theory by the skeleta of X coincides with the filtration of K-theory defined by the eigenvalues of the Adams operations on $KU^*(X)$. The agreement of these two filtrations is lost when one begins to study torsion in K-theory: the filtration by Adams eigenvalues is distinct from the skeletal filtration. It is the skeletal filtration which has good properties, when considering torsion in K-theory. Here is the relevant definition:

Definition 3.1. *Let P be a set of primes*[14]*, including 2. Let X be a finite CW-complex with cohomology $H^*(X;\mathbb{Z}[P^{-1}])$ concentrated in even degrees. Let ρ be a complex representation of the torsion subgroup of $KU^0(X)[P^{-1}]$.*

- *We will say that ρ has* skeletal weight *w if w is the least integer n such that the composite map*

$$\operatorname{tors} KU^0(X/X^{2n}) \to \operatorname{tors} KU^0(X)[P^{-1}] \xrightarrow{\rho} GL(V)$$

is trivial. Here X^{2n} denotes the $2n$-skeleton of X.

similar is done already in the study of p-adic L-functions, where one begins with a classical L-function $L(s)$ and p-adically interpolates its values $L(-a), L(1-p-a), L(2-2p-a), L(3-3p-a), \dots$ *separately* for each individual residue class a modulo $p-1$.

[14]Here is an explanation of the role of P here and throughout this section. Later, in Theorem 3.10, we will be able to prove good properties of a certain zeta-function associated to a finite CW-complex X with cohomology concentrated in even degrees, under a hypothesis which is a bit weaker than asking that the torsion subgroup of $KU^0(X)$ has square-free order. We might reasonably want to apply these methods and results to finite CW-complexes which fail to satisfy that hypothesis. Since X is a *finite* CW-complex, there is some finite set of primes P such that, after inverting the primes in P, X *does* satisfy the hypotheses of Theorem 3.10. So the role of P throughout this section is that it is a set of "bad primes" which we will invert, so that the results of this section can be applied to *any* finite CW-complex X.

- *We will say that* the skeletal filtration of $\operatorname{tors} KU^0(X)[P^{-1}]$ splits additively *if each of the subgroup inclusions in the skeletal filtration*

$$\begin{tikzcd} \vdots \arrow[d, hook] \\ \ker\left(\operatorname{tors} KU^0(X)[P^{-1}] \to \operatorname{tors} KU^0(X^6)[P^{-1}]\right) \arrow[d, hook] \\ \ker\left(\operatorname{tors} KU^0(X)[P^{-1}] \to \operatorname{tors} KU^0(X^4)[P^{-1}]\right) \arrow[d, hook] \\ \ker\left(\operatorname{tors} KU^0(X)[P^{-1}] \to \operatorname{tors} KU^0(X^2)[P^{-1}]\right) \arrow[d, hook] \\ \vdots \arrow[d, hook] \\ \operatorname{tors} KU^0(X)[P^{-1}] \end{tikzcd} \tag{18}$$

of $\operatorname{tors} KU^0(X)[P^{-1}]$ *is a split monomorphism of abelian groups. We will say that* the skeletal filtration of $\operatorname{tors} KU^0(X)[P^{-1}]$ splits completely *if, after completing at each prime ℓ, each of the subgroup inclusions in* (18) *is a split monomorphism of* $\hat{\mathbb{Z}}_\ell[\hat{\mathbb{Z}}_\ell^\times]$*-modules.*
- *Regardless of whether the skeletal filtration of* $\operatorname{tors} KU^0(X)[P^{-1}]$ *splits, we will write* $\operatorname{tors}^{(w)} KU^0(X)[P^{-1}]$ *for the skeletal filtration w subquotient of* $\operatorname{tors} KU^0(X)[P^{-1}]$*, i.e.,*

$$\operatorname{tors}^{(w)} KU^0(X)[P^{-1}] = \frac{\ker\left(\operatorname{tors} KU^0(X)[P^{-1}] \to \operatorname{tors} KU^0(X^{2w})[P^{-1}]\right)}{\ker\left(\operatorname{tors} KU^0(X)[P^{-1}] \to \operatorname{tors} KU^0(X^{2w+2})[P^{-1}]\right)}.$$

Now we can define an Euler product which, again, cleaves as closely as possible to the Hasse-Weil Euler product (4), but this time, our Euler product will "pay attention" to torsion in K-theory. Let P again be a set of primes, including 2. Suppose that X is a finite CW-complex whose cohomology $H^*(X;\mathbb{Z}[P^{-1}])$ is concentrated in even degrees. Furthermore, suppose that, for each integer w, the order of $\operatorname{tors}^{(w)} KU^0(X)[P^{-1}]$ is square-free. Let n_w denote the order of $\operatorname{tors}^{(w)} KU^0(X)[P^{-1}]$. The group of units $\mathbb{Z}/n_w^2\mathbb{Z}^\times$ decomposes canonically as the product $\prod_{\ell \mid n_w} \mathbb{Z}/\ell^2\mathbb{Z}^\times$ taken over all the primes ℓ dividing n_w. The group $\operatorname{tors}^{(w)} KU^0(X)[P^{-1}]$ is non-canonically isomorphic to the product, over all such primes ℓ, of the ℓ-Sylow subgroup of $\mathbb{Z}/\ell^2\mathbb{Z}^\times$. Write $\mathrm{Syl}(n_w)$ for that product of ℓ-Sylow subgroups, and choose an isomorphism $i_w : \mathrm{Syl}(n_w) \xrightarrow{\cong} \operatorname{tors}^{(w)} KU^0(X)[P^{-1}]$.

Now, given a complex representation ρ of $\mathrm{tors}^{(w)}\, KU^0(X)[P^{-1}]$, we have group homomorphisms

$$\begin{aligned} \mathrm{Syl}(n_w) &\xrightarrow{i_w} \mathrm{tors}^{(w)}\, KU^0(X)[P^{-1}] \\ &\xrightarrow{\Psi^p} \mathrm{tors}^{(w)}\, KU^0(X)[P^{-1}] \\ &\xrightarrow{\rho} GL(V) \end{aligned}$$

where p is any prime not dividing n_w. The resulting homomorphism $\rho \circ \Psi^p \circ i_w$ extends canonically to a group homomorphism

$$\Psi_{\rho,p,i_w} : \mathbb{Z}/n_w\mathbb{Z}^\times \to GL(V)$$

which

- agrees with $\rho \circ \Psi^p \circ i_w$ on the summand $\mathrm{Syl}(n_w)$ of $\mathbb{Z}/n_w\mathbb{Z}^\times$,
- and is trivial on the complementary summand of $\mathrm{Syl}(n_w)$ in $\mathbb{Z}/n_w\mathbb{Z}^\times$.

For any such prime p, the group $\mathbb{Z}/n_w\mathbb{Z}^\times$ furthermore has a particular element named p—simply the integer p in $\mathbb{Z}$, regarded as a residue class in $\mathbb{Z}/n_w\mathbb{Z}^\times$. We may evaluate Ψ_{ρ,p,i_w} at p to get an element of $GL(V)$. Consider the Euler product

$$\prod_p \det\left(\mathrm{id} - p^{w-s}\Psi_{\rho,p,i_w}(p)\right)^{-1} \tag{19}$$

taken over all primes p which do not divide n_w. The Euler product (19) mimics the Euler product of the Dirichlet L-function of a Dirichlet character χ,

$$\prod_p \left(1 - p^{-s}\chi(p)\right)^{-1}$$

as well as the Euler product of the Artin L-function of a Galois representation $\rho : \mathrm{Gal}(F/\mathbb{Q}) \to GL(V)$,

$$\prod_p \det\left(\mathrm{id} - p^{-s}\rho\left(\mathrm{Fr}\mid_{V^{I_p}}\right)\right)^{-1},$$

where Fr is a lift of the Frobenius element in $\mathrm{Gal}((O_F/\mathfrak{p})/\mathbb{F}_p)$ to an element of $\mathrm{Gal}(F/\mathbb{Q})$, and I_p is the p-inertia subgroup of $\mathrm{Gal}(F/\mathbb{Q})$. The power $w-s$, rather than $-s$, in (19) simply arranges for the skeletal weight w torsion in K-theory to contribute to weight w factors in the L-function. In the torsion-free case (as in Definition 2.1), this was unnecessary, since these weights all agree automatically: the action of Adams operations on rational K-groups naturally recovers the skeletal weight, by the relationship between the skeletal filtration and the filtration by Adams eigenvalues, discussed before Definition 3.1. Since that relationship is lost when one considers the torsion in K-theory, the L-factors coming from torsion in K-theory must be put into the correct weight "by hand," i.e., by having a factor of p^{w-s} rather than p^{-s} in (19).

Since $\mathbb{Z}/n_w\mathbb{Z}^\times$ is abelian, its representations split as direct sums of one-dimensional representations. For such a one-dimensional representation π, Ψ_{π,p,i_w} is a homomorphism

$$\mathbb{Z}/n_w\mathbb{Z}^\times \to \mathbb{C}^\times,$$

i.e., a Dirichlet character of modulus n_w. It is not necessarily the case that the Euler product (19) is the Dirichlet L-function of any single Dirichlet character, though. The trouble is that, due to the effect of the Adams operations, for distinct primes p_1 and p_2 it may happen that the p_1-local Euler factor in (19) is the p_1-local Euler factor of one Dirichlet character, while the p_2-local Euler factor in (19) is the p_2-local Euler factor of some *other* Dirichlet character.

The solution is simply to take the product over *sufficiently many* representations: the Euler product ought not to merely be a product over prime *numbers,* but also a product over prime *representations.*

Definition 3.2. *Let G be a finite cyclic group. A representation $\rho : G \to GL(V)$ of G is* prime *if it is irreducible and the image of ρ has prime order.*

Definition 3.3. *Given a positive integer n, we let $\mathrm{Dir}_{prime}(n^2)$ be the set of Dirichlet characters of modulus n^2 which*

- *have conductor equal to ℓ^2 for some prime divisor ℓ of n, and*
- *are trivial on the complementary summand of $\mathrm{Syl}_\ell(\mathbb{Z}/n^2\mathbb{Z}^\times) \subseteq \mathrm{Syl}(n)$ in $\mathbb{Z}/n^2\mathbb{Z}^\times$.*

We call the Dirichlet characters in $\mathrm{Dir}_{prime}(n^2)$ the prime Dirichlet characters of modulus n^2.

Examples 3.4. Given a positive integer n, the number of prime representations of a cyclic group of order n is $\sum_{p|n}(p-1)$, where the sum is taken over all prime divisors of n. This is also the number of prime Dirichlet characters of modulus n^2. It is straightforward to construct a bijection between the set of prime representations of a cyclic group of order n and the set of prime Dirichlet characters of modulus n^2, but we hope the reader will forgive some very elementary examples to demonstrate how it works. We think these examples help to make it clear how Definitions 3.2 and 3.3 play out in practice, and they make the proof of Proposition 3.6 more transparent.

- Suppose G is a cyclic group of order 3. It is straightforward to see that G has 2 prime representations. Similarly, while there are $\phi(3^2) = 6$ Dirichlet characters of modulus 3^2, only four have conductor equal to 9, and of those, precisely two are *even*, i.e., satisfy $\chi(-1) = 1$. The complementary summand of the 3-Sylow subgroup of $\mathbb{Z}/9\mathbb{Z}^\times$ is generated by -1, so the prime Dirichlet characters of modulus 9 are precisely the even Dirichlet characters of modulus 9. Since each such Dirichlet character is determined by its value on an element of order 3 in $\mathbb{Z}/9\mathbb{Z}^\times$, the prime Dirichlet characters of modulus 9 are in bijection with the prime representations of a cyclic group of order 3.
- Now suppose that G is a cyclic group of order $15 = 3 \cdot 5$. Then G has 2 irreducible representations whose image has order 3, and 4 irreducible representations whose image has order 5, for a total of 6 prime representations. Similarly, there are precisely $\phi(15^2) = 120$ Dirichlet characters of modulus 15^2, but only two[15] of conductor 9 which vanish on the complementary summand in $\mathbb{Z}/225\mathbb{Z}^\times$ of the 3-Sylow subgroup of $\mathbb{Z}/9\mathbb{Z}^\times \subseteq \mathbb{Z}/225\mathbb{Z}^\times$, and only four[16] of conductor 25 which vanish on the complementary summand in $\mathbb{Z}/225\mathbb{Z}^\times$ of the 5-Sylow subgroup of $\mathbb{Z}/15\mathbb{Z}^\times \subseteq \mathbb{Z}/225\mathbb{Z}^\times$. Dirichlet characters of the former kind are determined by their value on an element of order 3 in $\mathbb{Z}/225\mathbb{Z}^\times$, while Dirichlet characters of the latter kind are determined by their value on an element of order 5 in $\mathbb{Z}/225\mathbb{Z}^\times$. It is easy to see the one-to-one correspondence between the prime representations of G and the prime Dirichlet characters of modulus 225.
- For the rest of this paper, we will only use Definitions 3.2 and 3.3 in the case where the order n of the cyclic group is square-free, but here we will give a non-square-free example just to demonstrate how the correspondence works. Consider the

[15]Specifically, these are the two Dirichlet characters χ of modulus 225 such that $\chi(101)$ is a primitive third root of unity and $\chi(127) = 1$.

[16]Specifically, these are the four Dirichlet characters χ of modulus 225 such that $\chi(101) = 1$ and $\chi(127)$ is a primitive fifth root of unity.

case where G is cyclic of order 9. Then G still has only 2 prime representations. Among the $\phi(9^2) = 54$ Dirichlet characters of modulus 9^2, there are 27 which vanish on the complementary summand of the 3-Sylow subgroup of $\mathbb{Z}/81\mathbb{Z}^\times$, and of those 27, there are precisely two of conductor 9, namely, those Dirichlet characters χ in which $\chi(2)$ is a primitive cube root of unity. Each such Dirichlet character is determined by its value on a 3-torsion element of $\mathrm{Syl}_3(\mathbb{Z}/81\mathbb{Z}^\times)$, i.e., it corresponds to a prime representation of a cyclic group of order 3.

Now consider the Euler product

$$\prod_{w\in\mathbb{Z}}\prod_{\rho_w}\prod_{p}\det\left(\mathrm{id} - p^{w-s}\Psi_{\rho_w,p,i_w}(p)\right)^{-1} \tag{20}$$

where:

- the product $\prod_{\rho_w}$ ranges over all the *prime* representations of $\mathrm{tors}^{(w)}\, KU^0(X)[P^{-1}]$,
- and the product $\prod_p$ is taken over all primes p not dividing the order of $\mathrm{tors}\, KU^0(X)[P^{-1}]$.

Of course the product $\prod_{w\in\mathbb{Z}}$ in (20) is finite, since X is a finite CW-complex, so $\mathrm{tors}^{(w)}\, KU^0(X)[P^{-1}]$ is trivial for all but finitely many skeletal weights w.

We reiterate our assumptions so far in this section:

Assumptions 3.5.

- P is a set of primes, including 2,
- X is a finite CW-complex with cohomology $H^*(X;\mathbb{Z}[P^{-1}])$ concentrated in even dimensions,
- and, for each integer w, the skeletal weight w subquotient of $\mathrm{tors}\, KU^0(X)[P^{-1}]$ has square-free order n_w.

We also adopt the following notation: given a Dirichlet character χ, we write $\tilde{\chi}$ for its associated primitive character, i.e., $\tilde{\chi}$ is a Dirichlet character of modulus equal to the conductor of χ, and χ is induced up from $\tilde{\chi}$.

With those assumptions and that notation, we have the following result:

Proposition 3.6. *The Euler product* (20) *is equal to*

$$\prod_{w\in\mathbb{Z}}\ \prod_{\chi\in\mathrm{Dir}_{prime}(n_w^2)} L(s-w,\tilde{\chi}). \tag{21}$$

Consequently the Euler product (20) *does not depend on the choice of i_w, and it converges absolutely for all $s\in\mathbb{C}$ such that $\Re\mathfrak{e}(s) > 1 + c$, where c is*[17] *the greatest integer N such that* $\mathrm{tors}\, KU^0(X)[P^{-1}]$ *has a nontrivial summand of skeletal weight N. Furthermore,* (20) *analytically continues to a meromorphic function on the complex plane.*

Proof. Fix an integer w. Let p be a prime not dividing n_w. For each prime ℓ not dividing n_w, we know from the Atiyah-Hirzebruch spectral sequence

$$H^*(X;\hat{KU}^*_\ell(S^0)[P^{-1}]) \Rightarrow \hat{KU}^*_\ell(X)[P^{-1}]$$

that the $\hat{\mathbb{Z}}_\ell[\hat{\mathbb{Z}}_\ell^\times]$-module $\mathrm{tors}^{(w)}\, \hat{KU}^0_\ell(X)[P^{-1}]$ is a subquotient of $\hat{KU}^0_\ell(X^{2w}/X^{2w-1})$, i.e., the ℓ-complete K-theory of a wedge of $2w$-dimensional spheres, on which the Adams operation Ψ^p acts by multiplication by p^w. The Euler product (20) is taken only over those primes p which do not divide n_w, so the action of Ψ^p on $\mathrm{tors}^{(w)}\, KU^0(X)[P^{-1}]$ is by an automorphism. This automorphism depends on p and on w, but of course it does not depend on ρ_w. Hence, as explained in Examples 3.4, as ρ ranges over the set of prime representations

[17] A simple upper bound for c is the greatest integer N such that $H^{2N}(X;\mathbb{Z}[P^{-1}])$ is nontrivial. Hence, if we call the latter integer b, then (20) converges absolutely for all $s\in\mathbb{C}$ such that $\Re\mathfrak{e}(s) > 1 + b$.

of $\mathrm{tors}^{(w)} KU^0(X)[P^{-1}]$, the representations Ψ_{ρ,p,i_w} range over the elements of $\mathrm{Dir}_{prime}(n_w^2)$. That is, given an element $\chi \in \mathrm{Dir}_{prime}(n_w^2)$, there exists precisely one prime representation ρ_w such that $\chi = \Psi_{\rho_w,p,i_w}$.

Now consider the p-local Euler factor $(1 - p^{w-s}\tilde{\chi}(p))^{-1}$ in the Dirichlet L-series of $\tilde{\chi}$ evaluated at $w - s$. This L-factor occurs precisely once as an L-factor in (20), as $\det\left(\mathrm{id} - p^{w-s}\Psi_{\rho_w,p,i_w}(p)\right)^{-1}$ for precisely that prime representation ρ_w such that $\chi = \Psi_{\rho_w,p,i_w}$. The product formula (21) follows immediately. $\square$

Remark 3.7. Choose a Dirichlet character χ of conductor ℓ^2 appearing in (21). Then χ is trivial on the complementary summand $\mathbb{F}_\ell^\times$ of the ℓ-Sylow subgroup of $\mathbb{Z}/\ell^2\mathbb{Z}^\times$. This is precisely the *opposite* of the behavior of the mod ℓ cyclotomic character and its nontrivial powers.

The Dirichlet L-functions considered in Mitchell's paper [22] are L-functions only of powers of cyclotomic characters. For that reason, the relationships between $K(1)$-local homotopy theory and special values of L-functions considered in [22] are orthogonal to the relationships considered in this paper. (The paper [22] also is about $K(1)$-local algebraic K-theory, rather than localizations of finite CW-complexes. This difference is important and fundamental, as described in a footnote in section 1.1.)

Definition 3.8. *Let P be a set of primes and let X be a finite CW-complex satisfying Assumptions 3.5.*

- *The* torsion $KU[P^{-1}]$-local L-function of X*, written $L_{\mathrm{tors}\, KU[P^{-1}]}(s, X)$, is the meromorphic function on $\mathbb{C}$ given by the analytic continuation of the Euler product $\prod_{w\in\mathbb{Z}} \prod_{\rho_w} \prod_p \det\left(\mathrm{id} - p^{w-s}\Psi_{\rho_w,p,i_w}(p)\right)^{-1}$ from* (20)*.*
- *The* $KU[P^{-1}]$-local zeta-function of X*, written $\zeta_{KU[P^{-1}]}(s, X)$, is the product of the provisional KU-local zeta-function of X (defined in Definition 2.5) with the torsion $KU[P^{-1}]$-local L-function of X:*

$$\zeta_{KU[P^{-1}]}(s, X) = \dot{\zeta}_{KU}(s, X) \cdot L_{\mathrm{tors}\, KU[P^{-1}]}(s, X).$$

- *Given an integer w, we will write $L^{(w)}_{\mathrm{tors}\, KU[P^{-1}]}(s, X)$ for the weight w factor*

$$\prod_{\rho_w} \prod_p \det\left(\mathrm{id} - p^{w-s}\Psi_{\rho_w,p,i_w}(p)\right)^{-1}$$

  of $L_{\mathrm{tors}\, KU[P^{-1}]}(s, X)$. We will write $\zeta^{(w)}_{KU[P^{-1}]}(s, X)$ for the weight w factor

$$\dot{\zeta}^{(w)}_{KU[P^{-1}]}(s, X) \cdot L^{(w)}_{\mathrm{tors}\, KU[P^{-1}]}(s, X)$$

  of $\zeta_{KU[P^{-1}]}(s, X)$.
- *Finally, given an integer w and a prime divisor ℓ of the order n_w of* $\mathrm{tors}^{(w)} KU^0[P^{-1}]$*, we write $L^{(w,\ell)}_{\mathrm{tors}\, KU[P^{-1}]}(s, X)$ for the factor*

$$L^{(w,\ell)}_{\mathrm{tors}\, KU[P^{-1}]}(s, X) = \prod_\chi L(s - w, \tilde{\chi}), \tag{22}$$

  of $L^{(w)}_{\mathrm{tors}\, KU[P^{-1}]}(s, X)$, where the product (22) *is taken over all the characters $\chi \in \mathrm{Dir}_{prime}(n_w^2)$ of conductor equal to ℓ^2.*

As a consequence of Definition 2.5 and Proposition 3.6, we have

$$\zeta_{KU[P^{-1}]}(s,X) = \prod_{w\in\mathbb{Z}} \left(\zeta(s-w)^{\dim_{\mathbb{Q}} H_{2w}(X;\mathbb{Q})} \cdot \prod_{\chi\in \mathrm{Dir}_{prime}(n_w^2)} L(s-w,\tilde{\chi}) \right), \tag{23}$$

where again n_w is the order of the skeletal weight w summand of $\mathrm{tors}\, KU^0(X)[P^{-1}]$.

Proposition 3.9. *The L-function $L_{\mathrm{tors}\, KU[P^{-1}]}(s,X)$ and the zeta-function $\zeta_{KU[P^{-1}]}(s,X)$ depend only on the set of primes P and the stable homotopy type of X. That is, $L_{\mathrm{tors}\, KU[P^{-1}]}(s,X)$ and $\zeta_{KU[P^{-1}]}(s,X)$ do* not *depend on the choice of CW-decomposition of X.*

Proof. The skeletal filtration of $\mathrm{tors}\, KU^0(X)[P^{-1}]$, defined above in (18), is simply the Atiyah-Hirzebruch filtration on the torsion in the abutment $KU^*(X)[P^{-1}]$ of the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} \cong H^*(X; KU^*[P^{-1}]) \Rightarrow KU^*(X)[P^{-1}]$$

constructed by choosing a CW-decomposition of X, regarding that CW-decomposition as a tower of cofiber sequences, and applying the generalized cohomology theory $KU^*(-)[P^{-1}]$ to that tower of cofiber sequences to get an exact couple $\mathcal{E}$.

However, the same spectral sequence—and consequently the same filtration—is constructible by other means. Let $\mathcal{E}'$ be the exact couple obtained by applying the functor $[\Sigma^* X, -]$ to the Postnikov tower of KU. By a classical theorem of Maunder [21], the derived exact couple of $\mathcal{E}'$ agrees with the derived exact couple of $\mathcal{E}$, i.e., starting from its E_2-page, the Atiyah-Hirzebruch spectral sequence is constructible using the Postnikov filtration on KU rather than the cellular filtration of X. Hence the spectral sequence does not depend on the choice of cellular filtration on X, and hence the skeletal filtration on $\mathrm{tors}\, KU^0(X)[P^{-1}]$ does not depend on the choice of cellular filtration of X. The claim follows. □

3.2. **Functional equations for $L_{\mathrm{tors}\, KU}(s,X)$ and $\zeta_{KU}(s,X)$.** One can use the functional equation for Dirichlet L-functions to prove a functional equation for $L_{\mathrm{tors}\, KU[P^{-1}]}(s,X)$, as follows. The classical completed Dirichlet L-function of an even[18] primitive Dirichlet character of modulus ℓ, $\Lambda(s,\chi) := (\pi/\ell)^{-s/2}\Gamma(s/2)L(s,\chi)$, satisfies the functional equation $\Lambda(s,\chi) = W(\chi)\Lambda(1-s,\bar{\chi})$ (see for example 8.5 of [24]). Here $W(\chi)$ is the *root number* of the Dirichlet character χ, defined as $W(\chi) := \frac{\tau(\chi)}{\sqrt{\ell}}$, where $\tau(\chi)$ is the Gauss sum $\tau(\chi) = \sum_{m=1}^{\ell} \chi(m) e^{m/\ell} \in \mathbb{C}$.

Let $\hat{L}_{\mathrm{tors}\, KU[P^{-1}]}(s,X)$ denote the product $\prod_{w\in\mathbb{Z}} \prod_{\chi\in\mathrm{Dir}_{prime}(n_w^2)} \Lambda(s-w,\tilde{\chi})$ of the completed Dirichlet L-functions of the characters χ appearing in the factorization (21) of $L_{\mathrm{tors}\, KU[P^{-1}]}(s,X)$. Since $W(\chi)W(\overline{\chi}) = 1$, the product of the root numbers $\prod_{\chi\in\mathrm{Dir}_{\mathbb{Q}(\zeta_{\ell^2})[\ell]^*}} W(\chi)$ must be ± 1, yielding a functional equation

$$\hat{L}_{\mathrm{tors}\, KU[P^{-1}]}(s,X) = \pm \hat{L}_{\mathrm{tors}\, KU[P^{-1}]}(1-s, \Sigma^{-1}DX) \tag{24}$$

for $\hat{L}_{\mathrm{tors}\, KU[P^{-1}]}(s,X)$.

Aside from the $\pm$ in (24) arising from the root number, the other obvious difference between the functional equation (24) for $\hat{L}_{\mathrm{tors}\, KU[P^{-1}]}(s,X)$ and the functional equation (13) for $\hat{\hat{\zeta}}_{KU}(s,X)$ is the presence of the desuspension Σ^{-1}. The functional equation (24) relates the completed KU-local torsion L-functions of X and of the desuspended Spanier-Whitehead dual DX, while the functional equation (13) relates the completed KU-local provisional zeta-functions of X and of the Spanier-Whitehead dual DX, *without* any desuspension.

[18] A Dirichlet character χ is *even* if $\chi(-1) = 1$. All Dirichlet characters whose L-functions occur as factors in $L_{\mathrm{tors}\, KU[P^{-1}]}(s,X)$ are even.

The reason for this difference is the shape of the universal coefficient sequence in the complex K-theory of a finite CW-complex X (originally [2], but see [37] for a published source, or Theorem IV.4.5 of [16] for a useful wide generalization):

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(KU_{n-1}(X), \mathbb{Z}) \to KU^n(X) \to \hom_{\mathbb{Z}}(KU_n(X), \mathbb{Z}) \to 0. \tag{25}$$

The left-hand term in (25) is torsion and depends only on the torsion in $KU_{n-1}(X) \cong KU^{1-n}(DX)$, while the right-hand term in (25) is torsion-free and depends only on the torsion-free part of $KU_n(X) \cong KU^{-n}(DX)$. Consequently the effect of Spanier-Whitehead dualization in K-theory is that there is a natural degree shift of 1 in the K-theoretic torsion, while the torsion-free part of K-theory does *not* get shifted in this way[19]. The shift in the K-theoretic torsion is the reason the functional equation for the KU-local torsion L-function must relate X and $\Sigma^{-1}DX$, rather than X and DX.

3.3. **Special values of the KU-local zeta-function.** Finally, we had better see how these constructions relate to orders of homotopy groups, or all these definitions are worth very little. Recall that Theorem 2.8 expressed orders of KU-local stable homotopy groups of some finite CW-complexes in terms of special values of provisional KU-local zeta-functions, *away from* 2 and the primes dividing the order of the torsion subgroup of $KU^0(X)$. Theorem 3.10 extends Theorem 2.8 to those primes p such that the order of the torsion subgroup of $KU^0(X)$ is divisible by p, but not p^2:

Theorem 3.10. *Let P be a set of primes with $2 \in P$, and let X be a finite CW-complex satisfying Assumptions 3.5. Let a, b be the least and greatest integers n, respectively, such that $H^{2n}(X; \mathbb{Z}[P^{-1}])$ is nontrivial. Then the following conditions are equivalent:*

- *The skeletal filtration of* $\mathrm{tors}\, KU^0(X)[P^{-1}]$ *splits completely.*
- *For all odd integers $2k-1$ satisfying $2k-1 > 1-2a$, the KU-local stable homotopy group $\pi_{2k-1}(L_{KU}DX)$ is finite, and up to powers of primes in P, its order is equal to the product*[20]

$$\prod_{w\in\mathbb{Z}} \left(\mathrm{denom}\left(\dot{\zeta}^{(w)}_{KU}(1-k, X)\right) \cdot \prod_{\ell \mid n_w} \mathrm{denom}\left(L^{(w,\ell)}_{\mathrm{tors}\, KU[P^{-1}]}(1-k, X)\right)\right) \tag{26}$$

  of denominators of the isoweight factors in $\zeta_{KU}(s, X)$.
- *For all odd integers $2k-1$ satisfying $2k-1 < -2b-3$, the KU-local stable homotopy group $\pi_{2k-1}(L_{KU}DX)$ is finite of order*

$$\prod_{w\in\mathbb{Z}} \left(\mathrm{denom}\left(\dot{\zeta}^{(w)}_{KU}(k+1, X)\right) \cdot \prod_{\ell \mid n_w} \mathrm{denom}\left(L^{(w,\ell)}_{\mathrm{tors}\, KU[P^{-1}]}(k+1, X)\right)\right)$$

  up to powers of primes in P.

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[19] Because of this difference, the author knows of no single functional equation for the product $\hat{\zeta}_{KU[P^{-1}]}(s, X)$ of $\hat{\dot{\zeta}}_{KU}(s, X)$ and $\hat{L}_{\mathrm{tors}\, KU[P^{-1}]}(s, X)$. One would like to find some kind of dualization functor D' on finite CW-complexes, which has the same effect as the Spanier-Whitehead dualization D on the torsion-free part of K-theory, but which, unlike D, does not introduce a degree shift in the torsion in K-theory. Then one would have a nice functional equation $\hat{\zeta}_{KU[P^{-1}]}(s, X) = \pm\hat{\zeta}_{KU[P^{-1}]}(1-s, D'X)$. The author would be pleased to learn of a way to do this. Perhaps it is possible by modifying D in roughly the way that Anderson [2] modified the Brown-Comenetz dualization functor I.

[20] To be clear, the product $\prod_{\ell \mid n_w}$ in (26) is taken over all *prime* divisors ℓ of the order n_w of $\mathrm{tors}^{(w)}\, KU^0(X)[P^{-1}]$.

Proof. Throughout, we continue to use the notation $\tilde{\chi}$ from section 3.1: if χ is a Dirichlet character, then $\tilde{\chi}$ is its associated primitive character.

Fix an integer w. Since $n_w = \left|\mathrm{tors}^{(w)} KU^0(X)[P^{-1}]\right|$ is assumed square-free, it is a product of distinct primes. Let ℓ be a prime factor of n_w, and let D_ℓ denote the set of Dirichlet characters $\chi \in \mathrm{Dir}_{prime}(n_w^2)$ of conductor ℓ^2. Then $\{\tilde{\chi} : \chi \in D_\ell\}$ is precisely the set $\mathrm{Dir}_{\mathbb{Q}(\zeta_\ell)}(\ell^2)[\ell]^*$ of elements of order exactly ℓ in the group $\mathrm{Dir}_{\mathbb{Q}(\zeta_\ell)}(\ell^2)$ of $\mathbb{Q}(\zeta_\ell)$-valued Dirichlet characters of modulus ℓ^2. Using Carlitz's estimates [10],[11] on p-adic valuations of generalized Bernoulli numbers, it was shown in [29] that, for any positive integer k, the product $\prod_{\tilde{\chi}\in \mathrm{Dir}_{\mathbb{Q}(\zeta_\ell)}(\ell^2)[\ell]^*} L(1-k,\tilde{\chi})$ is a rational number whose denominator is given by:

$$\mathrm{denom}\left(\prod_{\tilde{\chi}\in \mathrm{Dir}_{\mathbb{Q}(\zeta_\ell)}(\ell^2)[\ell]^*} L(1-k,\tilde{\chi})\right) = \begin{cases} 1 & \text{if } \ell-1 \nmid k \\ \ell & \text{if } \ell - 1 \mid k. \end{cases} \tag{27}$$

Consequently, for positive integers k, the denominator of $L^{(w)}_{\mathrm{tors}\, KU^0(X)[P^{-1}]}(1-k, X)$ is equal to the product, over all prime factors ℓ of n_w, of the numbers

$$\begin{cases} 1 & \text{if } \ell-1 \nmid k+w \\ \ell & \text{if } \ell - 1 \mid k+w. \end{cases}$$

We must compare that denominator to the order of $\pi_{2k-1}(L_{KU}DX)$. To do this, we will work one prime at a time. Fix a prime factor ℓ of the order of tors $KU^0(X)[P^{-1}]$. Consider the homotopy fixed-point/descent spectral sequence (this is the $n = 1$ case of the constructions in [12]; see that paper and its discussion of the relationship to the earlier constructions of [25]):

$$\begin{aligned} E_2^{s,t} \cong H^s_c\left(\hat{\mathbb{Z}}_\ell^\times; \hat{KU}_\ell^{-t}(X)\right) &\Rightarrow \pi_{t-s}\left(F(X, \hat{KU}_\ell)^{h\hat{\mathbb{Z}}_\ell^\times}\right) \\ &\cong \pi_{t-s} F\left(X, L_{K(1)}S\right) \\ d_r : E_r^{s,t} &\to E_r^{s+r,t+r-1}, \end{aligned} \tag{28}$$

where $K(1)$ is the ℓ-primary height 1 Morava K-theory spectrum. To be clear, the notation $F(X,Y)$ denotes the function spectrum of maps from X to Y, so that $\pi_n F(X,Y) \cong [\Sigma^n X, Y] \cong Y^{-n}(X)$.

We need to make an analysis of what the E_2-term of spectral sequence (28) looks like, under the stated hypotheses on X. For each integer k, we have the short exact sequence of graded $\hat{\mathbb{Z}}_\ell[\hat{\mathbb{Z}}_\ell^\times]$-modules

$$0 \to \left(\mathrm{tors}\, KU^{-2k}(X)[P^{-1}]\right)^\wedge_\ell \to \hat{KU}_\ell^{-2k}(X)[P^{-1}] \to \left(\mathrm{torsfree}\, KU^{-2k}(X)[P^{-1}]\right)^\wedge_\ell \to 0. \tag{29}$$

Consider the long exact sequence induced in $H^s_c(\hat{\mathbb{Z}}_\ell^\times; -)$ by the short exact sequence (29). The group $H^s_c(\hat{\mathbb{Z}}_\ell^\times; \mathrm{tors}\, \hat{KU}_\ell^{-2k}(X))$ is trivial if $s > 1$, since it is the continuous cohomology of a profinite group $\hat{\mathbb{Z}}_\ell^\times \cong \mathbb{F}_\ell^\times \times \hat{\mathbb{Z}}_\ell$ of ℓ-cohomological dimension 1 (see for example Corollary 2 of section I.4 of [30] for this standard argument). We claim that $H^0_c\left(\hat{\mathbb{Z}}_\ell^\times; \mathrm{torsfree}\, \hat{KU}_\ell^{-2k}(X)\right)$ is also trivial if $k > -a$ or $k < -b$. The argument is simply that the action of the Adams operations on torsfree $\hat{KU}_\ell^{-2k}(X)$ is detected on the rationalization

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{torsfree}\, \hat{KU}_\ell^{-2k}(X) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \hat{KU}_\ell^{-2k}(X),$$

and the fact that the Adams operations act diagonally on rational K-theory (the same argument, essentially by the Chern character, as we used in the proof of Theorem 2.4) implies vanishing unless $-b \le k \le -a$.

Consequently (29) yields an isomorphism

$$H^0_c(\hat{\mathbb{Z}}_\ell^\times; \text{tors}\, \hat{KU}_\ell^{-2k}(X)) \cong H^0_c(\hat{\mathbb{Z}}_\ell^\times; \hat{KU}_\ell^{-2k}(X)) \tag{30}$$

and a short exact sequence
(31)

$$0 \to H^1_c(\hat{\mathbb{Z}}_\ell^\times; \text{tors}\, \hat{KU}_\ell^{-2k}(X)) \to H^1_c(\hat{\mathbb{Z}}_\ell^\times; \hat{KU}_\ell^{-2k}(X)) \to H^1_c(\hat{\mathbb{Z}}_\ell^\times; \text{torsfree}\, \hat{KU}_\ell^{-2k}(X)) \to 0$$

for each $k > -a$ and each $k < -b$.

Another consequence of the vanishing of $H^s_c(\hat{\mathbb{Z}}_\ell^\times; -)$ for $s \geq 2$ is that the spectral sequence (28) can have no nonzero differentials. Hence we have equalities[21]

$$\left|\pi_{2k} F\left(X, L_{K(1)} S\right)\right| = \left|H^0_c\left(\hat{\mathbb{Z}}_\ell^\times; \text{tors}\, \hat{KU}_\ell^{-2k}(X)\right)\right| \quad \text{and} \tag{32}$$

$$\left|\pi_{2k-1} F\left(X, L_{K(1)} S\right)\right| = \left|H^1_c\left(\hat{\mathbb{Z}}_\ell^\times; \text{tors}\, \hat{KU}_\ell^{-2k}(X)\right)\right| \cdot \left|H^1_c\left(\hat{\mathbb{Z}}_\ell^\times; \text{torsfree}\, \hat{KU}_\ell^{-2k}(X)\right)\right| \tag{33}$$

for each $k > -a$ and each $k < -b$.

Assuming that $k > -a+1$, we claim that the ℓ-adic valuation of $\left|H^1_c\left(\hat{\mathbb{Z}}_\ell^\times; \text{torsfree}\, \hat{KU}_\ell^{-2k}(X)\right)\right|$ agrees with the ℓ-adic valuation of the product

$$\prod_{w \in \mathbb{Z}} \text{denom}\left(\dot{\zeta}_{KU}^{(w)}(1-k, X)\right) \tag{34}$$

since both have the same order as the stable homotopy group $\pi_{2k-1} F(X[Q^{-1}], L_{K(1)} S)$, where Q is the set of primes

$$\{2\} \cup \left\{p : p \text{ divides } \left|\text{tors}\, KU^0(X)[P^{-1}]\right|\right\}.$$

To see this, observe that the map $X \to X[Q^{-1}]$ induces an Adams-operation-preserving isomorphism

$$\left(\text{torsfree } KU^0(X)\right)[Q^{-1}] \xrightarrow{\cong} KU^0(X)[Q^{-1}],$$

and the homotopy fixed-point spectral sequence

$$E_2^{s,t} \cong H^s_c\left(\hat{\mathbb{Z}}_\ell^\times; \hat{KU}_\ell^{-t}(X[Q^{-1}])\right) \Rightarrow \pi_{t-s} F\left(X[Q^{-1}], L_{K(1)} S\right) \tag{35}$$

collapses at the E_2-page with no nonzero differentials, yielding the isomorphism

$$H^1_c\left(\hat{\mathbb{Z}}_\ell^\times; \text{torsfree}\, \hat{KU}_\ell^{-2k}(X)\right) \cong \pi_{2k-1} F\left(X[Q^{-1}], L_{K(1)} S\right). \tag{36}$$

Isomorphism (36) relates $H^1_c\left(\hat{\mathbb{Z}}_\ell^\times; \text{torsfree}\, \hat{KU}_\ell^{-2k}(X)\right)$ to $\pi_{2k-1} F\left(X[Q^{-1}], L_{K(1)} S\right)$, but still must explain the relationship of the latter to $\pi_{2k-1} F\left(X[Q^{-1}], L_{KU} S\right)$. The relevant kind of argument is standard in stable homotopy theory, but we have chosen to present it in more detail than that audience would find necessary, since the author hopes that some number

[21] Readers with familiarity with Iwasawa theory will likely notice that this proof, particularly (32) and (33), bears a close resemblance to the kinds of manipulation of Iwasawa modules and their cohomology found in, for example, [19]. We suggest that future work of the kind appearing in this paper—i.e., establishing relationships between orders of Bousfield-localized stable homotopy groups, and special values of L-functions—would likely benefit from using the techniques of Iwasawa theory as a natural intermediary between stable homotopy groups (by expressing the input for descent spectral sequences along the lines of (28) in terms of cohomology of Iwasawa modules) and special values of L-functions.

theorists may read this paper too. Write $E(n)$ for the ℓ-primary height n Johnson-Wilson theory. The “fracture square”[22]

$$\begin{tikzcd} L_{E(1)}S \arrow[r] \arrow[d] & L_{K(1)}S \arrow[d] \\ L_{E(0)}S \arrow[r] & L_{E(0)}L_{K(1)}S \end{tikzcd} \tag{37}$$

yields a homotopy fiber sequence

$$L_{E(1)}S \to L_{K(1)}S \vee L_{E(0)}S \to L_{E(0)}L_{K(1)}S. \tag{38}$$

Bousfield localization at $E(0)$ is simply rationalization, so $L_{E(0)}S$ is simply the Eilenberg-Mac Lane spectrum $H\mathbb{Q}$ representing rational cohomology. Running the homotopy fixed-point spectral sequence (28) for the sphere to calculate $\pi_*(L_{K(1)}S)$ yields that $L_{E(0)}L_{K(1)}S$ splits as a wedge $H\mathbb{Q}_p \vee \Sigma^{-1}H\mathbb{Q}_p$, i.e., mapping into $L_{E(0)}L_{K(1)}S$ yields two copies of p-adic rational cohomology, with one copy shifted in degree by 1. Consequently the homotopy fiber of the map $L_{E(1)}S \to L_{K(1)}S$ is weakly equivalent to $\Sigma^{-1}H(\mathbb{Q}_p/\mathbb{Q}) \vee \Sigma^{-2}H\mathbb{Q}_p$. Consequently, the long exact sequence obtained by applying $[X[Q^{-1}], -]$ to (38) degenerates to an isomorphism

$$\pi_{2k-1}F\left(X[Q^{-1}], L_{K(1)}S\right) \cong \pi_{2k-1}F\left(X[Q^{-1}], L_{E(1)}S\right)$$

for $k > -a$ and for $k < -b$, since the rational cohomology of X vanishes in the relevant degrees. Since the ℓ-localization of $L_{KU}S$ is $L_{E(1)}S$, the ℓ-adic valuation of $\left|\pi_{2k-1}F(X[Q^{-1}], L_{KU}S)\right|$ is equal to the ℓ-adic valuation of $\left|\pi_{2k-1}F(X[Q^{-1}], L_{E(1)}S)\right|$ Finally, if $k > -a+1$, then Theorem 2.8 gives the equality of the order of $\pi_{2k-1}F(X[Q^{-1}], L_{KU}S) \cong \pi_{2k-1}L_{KU}DX[Q^{-1}]$ with (34). Hence the factor $\left|H^1_c\left(\hat{\mathbb{Z}}_\ell^\times; \text{torsfree } \hat{KU}_\ell^{-2k}(X)\right)\right|$ in (33) is accounted for by the ℓ-adic valuation of the denominator of $\dot{\zeta}_{KU}(1-k, X)$, when $k > 1-a$.

We still need to relate the factors $\left|H^s_c\left(\hat{\mathbb{Z}}_\ell^\times; \text{tors } \hat{KU}_\ell^{-2k}(X)\right)\right|$ in (32) and (33) to $L_{\text{tors } KU[P^{-1}]}(1-k, X)$. For this we use the finite filtration

$$\left(\text{tors } KU^{-2k}(X)[P^{-1}]\right)^\wedge_\ell = F_k^a \supseteq F_k^{a+1} \supseteq F_k^{a+2} \supseteq \dots \supseteq F_k^{b-1} \supseteq F_k^b = 0 \tag{39}$$

of tors $KU^{-2k}(X)[P^{-1}]$, where F_k^t is the ℓ-adic completion of

$$\ker\left(\text{tors } KU^{-2k}(X)[P^{-1}] \to \text{tors } KU^{-2k}(X^{2t})[P^{-1}]\right).$$

That is, (39) is the ℓ-adic completion of the skeletal filtration (18) on KU^{-2k}. Applying the continuous group cohomology functor $H^*_c(\hat{\mathbb{Z}}_\ell^\times; -)$ to (39) yields the strongly convergent spectral sequence

$$E_1^{s,t} \cong H^s_c(\hat{\mathbb{Z}}_\ell^\times; F_k^t/F_k^{t+1}) \Rightarrow H^s_c(\hat{\mathbb{Z}}_\ell^\times; \text{tors } \hat{KU}_\ell^{-2k}(X)) \tag{40}$$
$$d_r : E_r^{s,t} \to E_r^{s+1,t+r}.$$

The order of the abelian group F_k^t/F_k^{t+1} is equal to the largest power of ℓ which divides n_t. Since we have assumed that n_t is square-free for all t, the abelian group F_k^t/F_k^{t+1} must be either trivial or a one-dimensional $\mathbb{F}_\ell$-vector space. In the latter case, the pro-ℓ-Sylow subgroup of $\hat{\mathbb{Z}}_\ell^\times$ must act trivially, and consequently the action of $\hat{\mathbb{Z}}_\ell^\times$ on $\mathbb{F}_\ell$ is determined by the action of the quotient $\mathbb{F}_\ell^\times$ of $\hat{\mathbb{Z}}_\ell^\times$. There are $\ell-1$ possible actions of $\mathbb{F}_\ell^\times$ on $\mathbb{F}_\ell$. Of those $\ell-1$ actions, there are $\ell-2$ which fix only the zero element, and consequently have

[22] The homotopy pullback square (37) is classical. See Bauer’s chapter [4] in the book [13] for a nice write-up.

trivial group cohomology in all degrees. The unique cohomologically nontrivial action of $\mathbb{F}_\ell^\times$ on $\mathbb{F}_\ell$ is the trivial action. We will write $\mathbb{F}_\ell^{\mathrm{triv}}$ for $\mathbb{F}_\ell$ with the resulting trivial $\hat{\mathbb{Z}}_\ell^\times$-action. Then $H_c^*(\hat{\mathbb{Z}}_\ell^\times; \mathbb{F}_\ell^{\mathrm{triv}}) \cong \Lambda(h)$, an exterior $\mathbb{F}_\ell$-algebra on a single generator h in degree 1. Consequently the E_1-page of (40) is described as follows:

- $E_1^{s,t}$ is trivial if $s \neq 0, 1$,
- $E_1^{0,t} \cong E_1^{1,t}$ for all t,
- $E_1^{0,t}$ is a one-dimensional $\mathbb{F}_\ell$-vector space if the action of the Adams operations on $\mathrm{tors}^{(t)}\ \hat{KU}_\ell^{-2k}(X)$ is trivial,
- and $E_1^{0,t}$ is trivial otherwise.

Consider the direct sum of the spectral sequences (40) over all integers k. Call this the "torsion spectral sequence." The cohomology $H_c^*(\hat{\mathbb{Z}}_\ell^\times; \mathrm{tors}\, \hat{KU}_\ell^*(X))$ is periodic under the action of v_1^{-1}, i.e., the $(\ell-1)$th power of the Bott class in KU^2. Consequently, for each weight w such that F_k^w/F_k^{w+1} is nontrivial, we get a contribution to the E_1-page of the torsion spectral sequence which is isomorphic to the E_2-page of the homotopy fixed-point spectral sequence

$$E_2^{s,t} \cong H_c^s\left(\hat{\mathbb{Z}}_\ell^\times; \hat{KU}_\ell^{-t}(S^{2w-1}/\ell)\right) \Rightarrow \pi_{t-s} F\left(S^{2w-1}/\ell, L_{K(1)}S\right)$$

of the $(2w-1)$-dimensional mod ℓ Moore spectrum S^{2w-1}/ℓ. This is because $\hat{KU}_\ell^{-2t}(S^{2w-1}/\ell)$ is isomorphic to $\mathbb{F}_\ell$ with Ψ^p acting as multiplication by p^{w+t}, which is the same as the action of Ψ^p on F_t^w/F_t^{w+1}.

Since the mod ℓ Moore spectrum is rationally acyclic, the square (37) gives us the weak equivalences

$$\begin{aligned} F\left(S^{2w-1}/\ell, L_{K(1)}S\right) &\simeq F\left(S^{2w-1}/\ell, L_{KU}S\right) \\ &\simeq L_{KU} D\left(S^{2w-1}/\ell\right) \\ &\simeq L_{KU}\left(S^{-2w}/\ell\right). \end{aligned}$$

Now Proposition 3.6, together with the results of [29] summarized above in (27), yields that the group $\pi_{2k-1} L_{KU}\left(S^{-2w}/\ell\right)$, a factor of the E_1-page of the torsion spectral sequence, has order equal to the denominator of $L^{(w,\ell)}_{\mathrm{tors}\, KU[P^{-1}]}(1-k, X)$.

The conclusion here is that the orders of the groups in the E_1-page of the torsion spectral sequence, in the range of degrees described above, agree with the products of the denominators of special values of isoweight factors of $L_{\mathrm{tors}\, KU[P^{-1}]}(s, X)$ described in (26). Meanwhile, the abutment of the torsion spectral sequence is the summand coming from K-theoretic torsion in the input for the descent spectral sequence (28) which converges to $\pi_*(L_{KU}DX)[P^{-1}]$ and has no nonzero differentials. Consequently (26) is equal to the order of $\pi_{2k-1}(L_{KU}DX)$ for all $k > 1-a$, up to powers of primes in P, if and only if the torsion spectral sequence collapses at the E_1-page with no nonzero differentials involving a bidegree which contributes to $E_2^{p,q}$ in (28) in a degree which contributes to $\pi_n(L_{KU}DX)$ in the abutment, with $2n > -a$.

If the skeletal filtration of $\mathrm{tors}\, KU^0(X)[P^{-1}]$ splits completely, then the $\hat{\mathbb{Z}}_\ell[\hat{\mathbb{Z}}_\ell^\times]$-module filtration (39) splits for all k, so the torsion spectral sequence collapses at the E_1-page with no nonzero differentials, yielding formula (26), as claimed. For the converse: suppose that the skeletal filtration of $\mathrm{tors}\, KU^0(X)[P^{-1}]$ does not split completely. Then for at least one value of t, the extension of graded $\hat{\mathbb{Z}}_\ell[\hat{\mathbb{Z}}_\ell^\times]$-modules

$$0 \to \coprod_{k\in\mathbb{Z}} F_k^{t+1} \to \coprod_{k\in\mathbb{Z}} F_k^t \to \coprod_{k\in\mathbb{Z}} F_k^t/F_k^{t+1} \to 0$$

is not split. Since the skeletal filtration on tors $KU^0(X)[P^{-1}]$ was not assumed to split additively, it is not necessarily true that each F_k^t is an $\mathbb{F}_\ell$-vector space. However, by finiteness of X, it is at least true that there is an integer N such that F_k^t is an $\mathbb{Z}/\ell^N\mathbb{Z}$-module for all k and all t. Consequently the action of the Bott element on $\coprod_{k\in\mathbb{Z}} F_k^t$ is periodic[23] of some period (e.g. period $\ell^{N-1}(\ell-1)$). Consequently, if the sequence of $\hat{\mathbb{Z}}_\ell[\hat{\mathbb{Z}}_\ell^\times]$-modules

$$0 \to F_k^{t+1} \to F_k^t \to F_k^t/F_k^{t+1} \to 0 \tag{41}$$

is nonsplit for some value of k, then it is also nonsplit for arbitrarily much higher values of k, as well as arbitrarily much lower values of k.

By analysis of the torsion spectral sequence, if (41) is nonsplit, then the identity element

$$\mathrm{id} \in \mathrm{Ext}^0_{cont.\hat{\mathbb{Z}}_\ell[\hat{\mathbb{Z}}_\ell^\times]-mod}\left(\mathbb{F}_\ell^{\mathrm{triv}}, \mathbb{F}_\ell^{\mathrm{triv}}\right) \cong E_1^{0,t}$$

supports a nonzero differential of some length. Choose some value of k such that (41) is nonsplit and such that the resulting nonzero differential hits a class which contributes to bidegree $E_2^{p,q}$ in (28) in a degree which contributes to $\pi_n(L_{KU}DX)$ in the abutment, with $n >> -a$. That differential causes formula (26) to fail to agree with the order of $\pi_{2k-1}(L_{KU}DX)$ after localization at ℓ.

Consequently conditions 1 and 2 are equivalent. An entirely analogous argument proves the equivalence of conditions 1 and 3. ◻

Remark 3.11. Suppose Assumptions 3.5 are satisfied. Theorem 3.10 then asserts that three specific conditions are equivalent. The second and third conditions assert that the orders of certain homotopy groups agree with certain special values of zeta-functions. Even when the three conditions are *not* satisfied—i.e., even when the orders of those homotopy groups *fail* to agree with those special values—we still have the inequality

$$\left|\pi_{2k-1}(L_{KU}DX)[P^{-1}]\right| \leq \prod_{w\in\mathbb{Z}}\left(\mathrm{denom}\left(\dot{\zeta}_{KU}^{(w)}(1-k,X)\right)\cdot\prod_{\ell\mid n_w}\mathrm{denom}\left(L^{(w,\ell)}_{\mathrm{tors}\,KU[P^{-1}]}(1-k,X)\right)\right) \tag{42}$$

for all odd integers $2k-1$ satisfying $2k-1 > 1-2a$. We similarly have the inequality

$$\left|\pi_{2k-1}(L_{KU}DX)[P^{-1}]\right| \leq \prod_{w\in\mathbb{Z}}\left(\mathrm{denom}\left(\dot{\zeta}_{KU}^{(w)}(k+1,X)\right)\cdot\prod_{\ell\mid n_w}\mathrm{denom}\left(L^{(w,\ell)}_{\mathrm{tors}\,KU[P^{-1}]}(k+1,X)\right)\right) \tag{43}$$

for all odd integers $2k-1$ satisfying $2k-1 < -2b-3$. Both inequalities follow from the argument given in the proof of Theorem 3.10:

- the torsion-free part of the K-theory of X contributes the same factors to the left-hand side of (42) as it contributes to the right-hand side of (42),

[23] That is, the action of some power of the Bott element on $\coprod_{k\in\mathbb{Z}} F_k^t$ is not just an isomorphism of abelian groups—which, after all, is true of multiplication by the Bott element itself—but also an isomorphism of $\hat{\mathbb{Z}}_\ell[\hat{\mathbb{Z}}_\ell^\times]$-modules.

- and the "torsion spectral sequence," constructed in the proof of Theorem 3.10, has as input the contribution of the torsion in K-theory to the right-hand side of (42), and has as output the contribution of the torsion in K-theory to the left-hand side of (42).

Hence, when the skeletal filtration of $\operatorname{tors} KU^0(X)[P^{-1}]$ fails to split completely, one or more nonzero differentials in the torsion spectral sequence cut down the size of the left-hand side of (42), so that it is smaller than the right-hand side. This is why we have the inequality (42). The same argument applies with (42) replaced by (43) throughout.

One family of special cases of Theorem 3.10 was studied in the paper [29]. Using the notation introduced in the present paper[24], [29] showed that $\zeta_{KU}(s, \Sigma^{-1}S/p) = \zeta_F(s)/\zeta(s)$ for any odd prime p, where S/p is the mod p Moore spectrum, and where F is the largest totally real subfield of the cyclotomic field $\mathbb{Q}(\zeta_{p^2})$.

Recall that Theorem 2.8 made some mention of *regular primes,* i.e., those primes p which do not divide the numerator of $\zeta(1-k)$ for any positive integer k. Corollary 3.12, below, requires the following generalization: given a number field F, we say that a prime number p is *F-irregular* if p divides the numerator of $\zeta_F(1-k)$ for some positive integer k. Here $\zeta_F(s)$ is the Dedekind zeta-function of F. The $\mathbb{Q}$-irregular primes are simply the classical irregular primes.

Corollary 3.12. *Let X, P, a, b be as in Theorem 3.10. Suppose that the the skeletal filtration of $\operatorname{tors} KU^0(X)[P^{-1}]$ splits completely. Write N for the order of the group $\operatorname{tors} KU^0(X)[P^{-1}]$. Then, for all odd integers $2k-1 \geq 1-2a$, the order of $\pi_{2k-1}(L_{KU}DX)$ is equal to the denominator of $\zeta_{KU[P^{-1}]}(1-k, X)$, up to powers of primes in P and powers of F-irregular primes, where F ranges across all the wildly ramified subfields of the cyclotomic field $\mathbb{Q}(\zeta_{N^2})$.*

Proof. The Dedekind zeta-function $\zeta_{\mathbb{Q}(\zeta_{N^2})}(s)$ factors as the product of the Dirichlet L-functions of the primitive Dirichlet characters $\tilde{\chi}$, where χ ranges across the Dirichlet characters of modulus N^2. For a prime divisor ℓ of N, the Dirichlet characters χ on $\mathbb{Z}/\ell^2\mathbb{Z}^\times$ which vanish on the complementary summand of $\mathrm{Syl}(\ell)$ are those such that the L-function of $\tilde{\chi}$ is a factor of the Dedekind zeta-function of a subfield of $\mathbb{Q}(\zeta_{N^2})$ in which ℓ ramifies wildly. This material is classical; e.g. see Corollary 3.6 and Theorem 4.3 from [35]. As a consequence, the same argument (about cancellation of factors in numerators with factors in denominators) used in the proof of Theorem 2.8 suffices here to establish the second claim. □

Remark 3.13. When the skeletal filtration of $\operatorname{tors} KU^0(X[P^{-1}])$ is *not* completely split, the most that the author can say about the special values of $\zeta_{KU[P^{-1}]}(s, X)$ is that their denominators recover the orders of the KU-local stable homotopy groups arising from the torsion-free part of $KU^0(X)$ and from the associated graded of the skeletal filtration on $\operatorname{tors} KU^0(X)[P^{-1}]$. Put more clearly, when $KU^0(X)[P^{-1}]$ is a torsion group: the denominators of the special values of $\zeta_{KU[P^{-1}]}(s, X)$ count the orders of the KU-local homotopy groups of the wedge of Moore spectra whose cohomology, with $\mathbb{Z}[P^{-1}]$ coefficients, agrees with the cohomology $H^*(X; \mathbb{Z}[P^{-1}])$.

[24] The notation $L_{KU}(s, S/p)$ from [29] corresponds to the notation $\zeta_{KU}(s, \Sigma^{-1}S/p)$ in the present paper. The reason for the desuspension is that the approach in [29] was based around formulating KU-local L-functions whose special values recover orders of KU-local stable homotopy groups, while in the more general and natural framework of the present paper, the special values of KU-local zeta-functions instead recover the KU-local stable *cohomotopy* groups of a finite CW-complex X, or what comes to the same thing, the KU-local stable homotopy groups of the Spanier-Whitehead dual DX of X. We have $D(\Sigma^{-1}S/p) \simeq S/p$.

It would also be nice to say something more when the order of $\mathrm{tors}^{(w)} KU^0(X[P^{-1}])$ is not square-free. If the p-local component of $\mathrm{tors}^{(w)} KU^0(X[P^{-1}])$ is elementary abelian, then an argument similar to that of Theorem 3.10 still works, if the definition of the KU-local zeta-function $\zeta_{KU[P^{-1}]}(s, X)$ is amended appropriately. But the author only knows how to make this amendment in a clumsy way, by replacing the product over prime representations in (20) with a much less intuitive product. Perhaps some better approach is possible.

If the p-local component of $\mathrm{tors}^{(w)} KU^0(X[P^{-1}])$ is not elementary abelian, then at present it seems totally unclear how to proceed. In even the simplest example, where X is the mod 9 Moore spectrum $\Sigma^{-1}S/9$, the author does not at present know how to write down a natural-looking zeta-function whose special values recover the orders of the KU-local stable homotopy groups of X. The author is skeptical that an elegant one exists. One could settle for something contrived instead: let χ be the principal Dirichlet character of modulus 19. Then the denominator of $L(1-n, \chi)$, divided by the denominator of $\zeta(1-n)$, is equal to 2 times the order of $\pi_{2n-1}(L_{KU}(S/9))$. The point is that 3^2 (and also 2) divides $19-1$, so the extra Euler factor at $p = 19$ in the L-function of the imprimitive character χ makes some contributions to the factors of 3 in the special values.

Remark 3.14. In (26), it was important that we take the product, over w and ℓ, of the denominators of the special values of the weight w conductor ℓ KU-local zeta-factor. We do *not* get the same result if we simply take the denominator of $\zeta_{KU[P^{-1}]}(1-k, X)$. In Remark 2.10, we already saw that a prime factor of the denominator of the weight w_1 factor of a KU-local provisional zeta-function can cancel with a prime factor of the *numerator* of the weight w_2 factor, for $w_1 \neq w_2$. The same phenomenon occurs with the (non-provisional) KU-local zeta-function.

Even within a single weight w, it is possible for a prime factor of the denominator of the weight w conductor ℓ_1 factor of a KU-local zeta-function to cancel with a prime factor of the numerator of the weight w conductor ℓ_2 factor, for $\ell_1 \neq \ell_2$. An example occurs already for the desuspended mod 21 Moore spectrum $\Sigma^{-1}S/21$. We have $KU^0(\Sigma^{-1}S/21) = \mathrm{tors}\, KU^0(\Sigma^{-1}S/21) \cong \mathbb{Z}/21\mathbb{Z}$, all in weight zero. Let R_3 (respectively, R_7) denote the set of prime representations of $\mathrm{tors}\, KU^0(\Sigma^{-1}S/21)$ with image of order 3 (respectively, order 7). Similarly, write D_3 (respectively, D_7) for the set of Dirichlet characters of conductor 9 (respectively, conductor 49) and modulus 21^2 which vanish on the complementary summand of $\mathrm{Syl}_3(\mathbb{Z}/9\mathbb{Z}^\times)$ (respectively, $\mathrm{Syl}_7(\mathbb{Z}/49\mathbb{Z}^\times)$) in $\mathbb{Z}/21^2\mathbb{Z}^\times$. Unwinding the equalities from Theorem 3.10, we have[25]

$$\begin{aligned}
\zeta_{KU}(-5, \Sigma^{-1}S/21) &= \prod_p \prod_{\rho \in R_3 \cup R_7} \frac{1}{\det\left(\mathrm{id} - p^{w-s}\Psi_{\rho_w, p, i_w}(p)\right)} \\
&= L^{(0,3)}_{\mathrm{tors}\, KU}(-5, \Sigma^{-1}S/21) \cdot L^{(0,7)}_{\mathrm{tors}\, KU}(-5, \Sigma^{-1}S/21) \\
&= \left(\prod_{\chi \in D_3} L(-5, \tilde{\chi})\right) \cdot \left(\prod_{\chi \in D_7} L(-5, \tilde{\chi})\right) \\
&= \frac{2^2 \cdot 7 \cdot 43 \cdot 1171}{3} \cdot \frac{2^6 \cdot 138054547 \cdot 163933047708171216095114393777711}{7} \qquad (44) \\
&= \frac{2^8 \cdot 43 \cdot 1171 \cdot 138054547 \cdot 163933047708171216095114393777711}{3},
\end{aligned}$$

[25]The calculation (44) appeared already in a slightly different context in [29]. Computer calculation of these special values of Dirichlet L-functions, via generalized Bernoulli numbers (see for example section 1.2 of [18]), is straightforward in SageMath [33] or in Magma [6].

whose denominator is 3. Meanwhile, we have

$$\begin{aligned}
&\mathrm{denom}\left(L^{(0,3)}_{\mathrm{tors}\,KU}(-5,\Sigma^{-1}S/21)\right)\\
&\cdot\,\mathrm{denom}\left(L^{(0,7)}_{\mathrm{tors}\,KU}(-5,\Sigma^{-1}S/21)\right) = \mathrm{denom}\left(\frac{2^2\cdot 7\cdot 43\cdot 1171}{3}\right)\\
&\qquad\qquad\cdot\,\mathrm{denom}\left(\frac{2^6\cdot 138054547\cdot 163933047708171216095114393777711}{7}\right)\\
&\qquad = 21\\
&\qquad = \left|\pi_{-13}(L_{KU}S/21)\right|.
\end{aligned}$$

The trouble is that, while 7 is a regular prime, it is also F-irregular, where F is the minimal subfield of $\mathbb{Q}(\zeta_9)$ in which 3 ramifies wildly.

This example demonstrates that Corollary 3.12 is perhaps not of great practical use, except in very special cases: the trouble is that there are simply many more F-irregular primes than classical irregular primes. Indeed, the prime 2 is already F-irregular, for the same number field F described in the previous paragraph.

The example $X = \Sigma^{-1}S/21$ is minimal in the sense that it minimizes the prime factors (3 and 7) of the order of the torsion in $KU^0(X)$. Nevertheless, the relatively large prime factors 138054547 and 163933047708171216095114393777711 occured in the numerator of (44). Similarly large (or, in fact, much larger) prime factors are often found in numerators of special values of $\zeta_{KU}(s,X)$ for CW-complexes X that have nontrivial torsion in K-theory.

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