OBSTRUCTIONS TO COMPATIBLE SPLITTINGS.

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Abstract. Suppose one has a map of split short exact sequences in a category of modules, or more generally, in any abelian category. Do the short exact sequences split compatibly, i.e., does there exist a splitting of each short exact sequence which commutes with the map of short exact sequences? The answer is sometimes yes and sometimes no. We define and prove basic properties of a group of obstructions to the existence of compatible splittings.

Contents

1. Introduction.

Suppose we have a map of short exact sequences (of modules over a ring, sheaves over a scheme, in general of objects in any abelian category):

(1.0.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0. \end{array}$$

Suppose the top row splits and suppose the bottom row also splits. Does there exist a *compatible* splitting of the two short exact sequences? In other words, can we find choices of splitting maps of the two rows which commute with the vertical maps in the diagram? We call this the *compatible splitting problem.*

At a glance one might hope that the answer to this question is always "yes," but in fact the answer is often "no," even for very simple short exact sequences in very simple (in fact, semisimple!) abelian categories. For example, the reader can easily verify that, if k is a field,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & k & \xrightarrow{\text{id}} & k & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \text{id}} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & k & \xrightarrow{\text{id}} & k & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

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is a map of short exact sequences of k-vector spaces in which both the top and the bottom row split, but the rows cannot be *compatibly* split.

Now one asks the natural question:

Question 1.1. What conditions on f or on g guarantee the existence of a compatible splitting?

This question is an obvious one, and one which arose for us during a concrete and practical computation (see below, at the end of this introduction), but we could not find any answer to it in the existing literature. In this paper, we answer this question by developing an obstruction group to the existence of certain splittings, with our main application being to the compatible splitting problem. (The reader who does not care to see the development of the whole obstruction theory may at this point wish to skip to Corollaries 5.5 and 5.6, where Question 1.1 gets an answer: for a fixed choice of f, the rows split compatibly in all diagrams of the form 1.0.1 if and only if f is split monic; dually, for a fixed choice of g, the rows split compatibly in all diagrams of the form 1.0.1 if and only if g is split epic. But other results in this paper are also of use when neither f nor g are split.)

As one might expect, our group of obstructions to compatible splittings is the kernel of a map of Ext-groups. Specifically, if diagram 1.0.1 is a commutative diagram with split exact rows in an abelian category C, then the whole diagram represents some element α of $\mathrm{Ext}^1_{C^{(\bullet\to\bullet)}}(M'' \stackrel{f}{\to} N'', M' \stackrel{g}{\to} N')$, where $C^{(\bullet\to\bullet)}$ is the (abelian) category of morphisms in C, i.e., functors from the category $(\bullet \to \bullet)$, the category with two objects and one morphism between them, to C. Furthermore, α is in the kernel of the natural map

$$\mathrm{Ext}^1_{C^{(\bullet\to\bullet)}}(M'' \stackrel{f}{\to} N'', M' \stackrel{g}{\to} N') \to \mathrm{Ext}^1_C(M'', M') \times \mathrm{Ext}^1_C(N'', N') \tag{1.0.2}$$

since the top and bottom rows of diagram 1.0.1 are assumed to split.

The kernel of map 1.0.2 is our obstruction group to compatible splitting. It bears a resemblance to the Tate-Shafarevich group classifying failure of the Hasse principle for an abelian variety, as in e.g. [4]. For that reason, we adopt the notation $Ш^1(f, g)$ for this obstruction group.

We actually define (in Definition 2.1) a sequence of $Ш^n$ groups and in somewhat wider generality, so that they apply to not only to the compatible splitting problem as defined above, but also to longer exact sequences, i.e., those classified by Ext^n rather than Ext^1; and so that they are also the home of the splitting obstructions arising in the following more general context. One might have an exact sequence in an abelian category, and one might know that, on forgetting some structure possessed by objects in that abelian category, the exact sequence is split. One wants to know if the original exact sequence is also split. The compatible splitting problem, described above, is the special case in which one has a short exact sequence in the category of arrows in C, and one forgets the arrows, only remembering their domains and codomains. But one could instead, for example, have an exact sequence of $k[x, y]$-modules which is split as an exact sequence of $k[x]$-modules and as an exact sequence of $k[y]$-modules, or one could have, for another example, an exact sequence of representations of a profinite group G which is split on restricting to any maximal rank elementary abelian p-subgroup of G. (In the latter case, when G has p-rank one and finitely many conjugacy classes of rank 1 elementary abelian p-subgroups, the group $Ш^n$ occurs as the kernel of Quillen's map

$$H^n_c(G; \mathbb{F}_p) \to \lim_{E \in \mathcal{A}_p(G)} H^n_c(E; \mathbb{F}_p)$$

in continuous group cohomology, from [5], where $\mathcal{A}_p$ is the category of elementary abelian p-subgroups of G.) Our obstruction groups $Ш^n$, and some of the theorems we prove about them, are general enough to apply to these situations as well.

Here is a brief synopsis of what we accomplish in each section of the paper.

- In section 2, we construct the compatible splitting obstruction groups $Ш^n$, and we give a precise formulation of the splitting problems to whose solvability these groups are the obstructions.
- Like the classical Tate-Shafarevich group, and as is clear from the case when the $Ш^n$ groups are the kernel of Quillen's map above, the groups $Ш^n$ are difficult to compute. This difficulty arises in part because they are not (co)homological, that is, they do not turn short exact sequences (other than split ones) into exact sequences. So in section 3, we produce a relative-cohomological approximation Ext^n_E to $Ш^n$, which has the advantage that it turns (certain) non-split exact sequences into long exact sequences. In Theorem 3.1 we prove a "Hurewicz theorem" for this cohomological approximation, i.e., we produce a natural transformation between this Ext^n_E group and $Ш^n$ which is an isomorphism when $n = 1$.

  This is worth repeating: the most important case of the obstruction groups $Ш^n$ is the $n = 1$ case, and in this case we have a homological description of the obstruction group $Ш^1$, as a relative-homological Ext group with respect to a certain allowable class (the reader who is not familiar with relative homological algebra can consult Chapter IX of [2] for a good treatment of the basics).
- In section 4, we produce the "compatible splitting spectral sequence," which relates the higher Ext^n_E groups and the higher $Ш^n$ groups. As a corollary, we show that, when the allowable class E defining these Ext^n_E-groups is hereditary, the groups $Ш^n$ and $Ш^{n+1}$ fit into a certain exact sequence. This relationship between $Ш^n$ and $Ш^{n+1}$ is a curious duality-like phenomenon which in fact occurs (the relative-hereditary condition is satisfied) in our most important application, the compatible splitting problem, as we demonstrate in the next section.
- Finally, in section 5 we specialize to the case of the compatible splitting problem. In Corollary 5.5, we use our cohomological approximation to $Ш$ to show that, for a fixed choice of map g, a compatible splitting exists for all diagrams of the form 1.0.1 with split exact rows if and only if g is split epic. Dually, in Corollary 5.6, we show that for a fixed choice of map f, a compatible splitting exists for all diagrams of the form 1.0.1 with split exact rows if and only if f is split monic. (These last two sentences together constitute the simplest and straightforward answer, but not the most general answer, to our Question 1.1.) In Corollary 5.7, we then demonstrate *compatible splitting duality*, a concrete special case of the duality described in the previous section: the obstruction group $Ш^n(f,g)$ is, by definition, the *kernel* of the map

$$\mathrm{Ext}^n_{C^{\bullet\to\bullet}}(f,g) \to \mathrm{Ext}^n_C(\operatorname{dom} f, \operatorname{dom} g) \times \mathrm{Ext}^n_C(\operatorname{cod} f, \operatorname{cod} g), \tag{1.0.3}$$

  but "compatible splitting duality" identifies the *cokernel* of the natural map

$$\mathrm{Ext}^n_C(\operatorname{dom} f, \operatorname{dom} g) \times \mathrm{Ext}^n_C(\operatorname{cod} f, \operatorname{cod} g) \to \mathrm{Ext}^n_C(\operatorname{dom} f, \operatorname{cod} g)$$

  with the compatible splitting obstruction group in the *next dimension,* that is, $Ш^{n+1}(f,g)$. This gives a curious relation between the obstruction groups in adjacent dimensions which we think is rather surprising.

We use these results in our paper [7] in the course of an answer to the elementary question: which commutative rings admit *strong Smith normal form*, that is, which commutative rings R have the property that every morphism between R-modules is isomorphic to a coproduct of morphisms between indecomposable R-modules? (Note that classical Smith normal form implies that, when R is a principal ideal domain, then every morphism between finitely-generated free R-modules is isomorphic to a coproduct of morphisms between indecomposable R-modules.) The answer to this question, which has some homotopy-theoretic applications to the computation of model structures, turns out to be that R has strong Smith normal form if and only if R is an Artinian ring such that $\mathfrak{m}^2 = 0$ for all maximal ideals $\mathfrak{m}$ in R. That result requires results from the present paper, for reasons which boil down to the following issue: suppose $h : M \to N$ is a map of R-modules, and suppose M_0 is a direct summand of M. Suppose we can even show that the image of M_0 under h is a direct summand of N. Can we then split h as a direct sum of the map $h\mid_{M_0}: M_0 \to N_0$ with another map? This is precisely the compatible splitting problem: the map $h\mid_{M_0}$ is the map g as in diagram 1.0.1, and the quotient map $h/(h\mid_{M_0})$ is the map f as in diagram 1.0.1. So we have found that the results in this paper are quite useful even for some very elementary algebraic tasks.

We are grateful to the anonymous referee for some helpful comments.

2. The compatible splitting obstruction group, and its cohomological approximation.

Definition 2.1. *Let $\mathcal{C}$ be an abelian category, I a finite set, and $\{\mathcal{C}_i : i \in I\}$ a finite set of abelian categories. Suppose that, for each $i \in I$, we have a faithful exact functor $G_i : \mathcal{C} \to \mathcal{C}_i$ and a left adjoint F_i for G_i. We will write G for the resulting functor $G : \times_{i\in I}\mathcal{C}_i \to \mathcal{C}$ given by $G((X_i)_{i\in I}) = \oplus_{i\in I}X_i$, and F for its left adjoint $F : \mathcal{C} \to \times_{i\in I}\mathcal{C}_i$ given by letting its component in the factor $\mathcal{C}_i$ be F_i. (It is easy to show that F is indeed left adjoint to G.) Then, for any $n \in \mathbb{N}$ and any objects X, Y of $\mathcal{C}$, by the n*th compatible splitting obstruction group *$Ш^n(X, Y)$ we shall mean the kernel of the map*

$$\mathrm{Ext}^n_{\mathcal{C}}(X, Y) \to \mathrm{Ext}^n_{\mathcal{C}}(FGX, Y)$$

induced by the counit map $\epsilon_X : FGX \to X$.

Remark 2.2. We use the symbol Ш to denote the compatible splitting obstruction groups because of their similarity, both in its definition and its properties, to the higher Shafarevich-Tate groups of an abelian variety (see e.g. [4]).

Note that $Ш^n(X, Y)$ certainly depends on the choices made for $\{\mathcal{C}_i : i \in I\}$ and $\{F_i : i \in I\}$, but in order to keep the notation manageable, we suppress these choices from the notation for $Ш^n(X, Y)$.

Note also that we do *not* need to assume that $\mathcal{C}$ has enough injectives or enough projectives for Definition 2.1 to make sense: $\mathrm{Ext}^n_{\mathcal{C}}$ defined after Yoneda, as equivalence classes of length $n + 2$ exact sequences in $\mathcal{C}$, does not require injective or projective resolutions. See e.g. [2] for basic material on Ext without enough injectives or enough projectives.

Definition 2.3. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories, and let $F : \mathcal{D} \to \mathcal{C}$ be a additive functor. We say that F is* resolving *if, for every object X of $\mathcal{D}$, there exists a projective object Y of $\mathcal{D}$ such that $F(Y)$ is a projective object in $\mathcal{C}$ and such that there exists an epimorphism $Y \to X$ in $\mathcal{D}$.*

For example, if $\mathcal{C}, \mathcal{D}$ are categories of modules over rings, and F sends free modules to free modules (e.g. F could be a base-change/extension of scalars functor), then F is resolving.

Lemma 2.4. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories, $G : \mathcal{C} \to \mathcal{D}$ a functor with exact left adjoint F. Suppose F is resolving. Then, for all $n \in \mathbb{N}$, we have an isomorphism*

$$\mathrm{Ext}^n_{\mathcal{C}}(FGX, Y) \cong \mathrm{Ext}^n_{\mathcal{D}}(GX, GY)$$

natural in X and Y.

Proof. Since F is resolving, we can choose, for every object X of $\mathcal{C}$, a chain complex $P_\bullet$ in $\mathcal{D}$ such that P_n is projective in $\mathcal{D}$ for all n, such that FP_n is projective in $\mathcal{C}$ for all n, and such that the homology of $P_\bullet$ is GX, concentrated in degree zero. Since F is exact, $FP_\bullet$ is a projective resolution of FGX. So now we have (natural!) isomorphisms

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{C}}(FGX, Y) &\cong H^n(\hom_{\mathcal{C}}(FP_\bullet, Y)) \\ &\cong H^n(\hom_{\mathcal{D}}(P_\bullet, GY)) \\ &\cong \mathrm{Ext}^n_{\mathcal{D}}(GX, GY). \end{aligned}$$

□

Exactness of F is necessary in Lemma 2.4. For example, suppose $\mathcal{C} = \mathrm{Mod}(k)$ and $\mathcal{D} = \mathrm{Mod}(k[x])$, and suppose that F and G are the induction and restriction of scalars functors, respectively, induced by the ring map $k[x] \to k$ sending x to 0. Then one sees easily that F is resolving but not exact, and that the conclusion of Lemma 2.4 fails dramatically.

Definition 2.5. *Let $\mathcal{C}, I, \{\mathcal{C}_i : i \in I\}$ be as in Definition 2.1. Suppose F_i is exact and resolving for all $i \in I$. We let E denote the following allowable class in $\mathcal{C}$, in the sense of relative homological algebra: E consists of the class of all short exact sequences in $\mathcal{C}$ of the form*

$$0 \to \ker \epsilon_Y \to FGY \xrightarrow{\epsilon_Y} Y \to 0,$$

where Y ranges across all objects of $\mathcal{C}$. (The map ϵ_Y is an epimorphism since each G_i is assumed faithful, hence each $F_iG_iY \to Y$ is individually already an epimorphism, by e.g. theorem 1 of section IV.3 of [3].*)*

Clearly E depends on the choices made for $\{\mathcal{C}_i : i \in I\}$ and $\{F_i : i \in I\}$, but in order to keep the notation manageable, we suppress these choices from the notation for E.

The following (easy!) theorem makes clear why the compatible splitting obstruction group is useful.

Theorem 2.6. *Let $\mathcal{C}, I, \{\mathcal{C}_i : i \in I\}$ be as in Definition 2.1. Suppose F_i is exact and resolving for all $i \in I$. Let X, Y be objects of $\mathcal{C}$. Then the following conditions are equivalent:*

- $Ш^n(X, Y) \cong 0$.
- *For every length $n + 2$ exact sequence α of the form*

  $$0 \to Y \to E_1 \to \cdots \to E_n \to X \to 0,$$

  α is split (in $\mathcal{C}$) if and only if, for all $i \in I$, the exact sequence $G_i(\alpha)$ is split (in $\mathcal{C}_i$).

If $n = 1$, the above conditions are furthermore each equivalent to:

- *For every morphism $f : \ker \epsilon_Y \to X$ in $\mathcal{C}$, there exists a map g making the diagram*

  $$\begin{array}{ccc} \ker \epsilon_Y & \longrightarrow & FGY \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ X & & \end{array}$$

  commute.

Proof. From Lemma 2.4 we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^n_{\mathcal{C}}(Y,X) & \longrightarrow & \mathrm{Ext}^n_{\mathcal{C}}(FGY,X) \\
& \searrow & \downarrow \cong \\
& & \mathrm{Ext}^n_{\mathcal{C}}(\oplus_{i\in I} F_iG_iY, X) \\
& & \downarrow \cong \\
& & \oplus_{i\in I} \mathrm{Ext}^n_{\mathcal{C}_i}(G_iY, G_iX).
\end{array}$$

Now an element $\alpha \in \mathrm{Ext}^n_{\mathcal{C}}(Y,X)$ maps to zero in $\oplus_{i\in I}\mathrm{Ext}^n_{\mathcal{C}_i}(G_iY,G_iX)$ if and only if $G_i(\alpha) = 0 \in \mathrm{Ext}^n_{\mathcal{C}_i}(G_iY,G_iX)$ for each $i \in I$, i.e., if and only if the length $n+2$ exact sequence represented by α becomes split in $\mathcal{C}_i$ after applying G_i, for all $i \in I$.

The claim for $n = 1$ follows from the exact sequence

$$\hom_{\mathcal{C}}(FGY,X) \to \hom_{\mathcal{C}}(\ker \epsilon_Y, X) \to \mathrm{Ext}^1_{\mathcal{C}}(Y,X) \to \mathrm{Ext}^1_{\mathcal{C}}(FGY,X),$$

hence the exact sequence

$$\hom_{\mathcal{C}}(FGY,X) \to \hom_{\mathcal{C}}(\ker \epsilon_Y, X) \to \text{Ш}^1(Y,X) \to 0.$$

□

Corollary 2.7. *Let $\mathcal{C}, I, \{\mathcal{C}_i : i \in I\}$ be as in Definition 2.1, and let E be as in Definition 2.5. Then the following are equivalent, for a given object X of $\mathcal{C}$:*

- $\text{Ш}^1(Y,X) \cong 0$ *for all objects Y of $\mathcal{C}$.*
- *For every short exact sequence*

  $$\alpha = (0 \to X \to Y \to Z \to 0)$$

  in $\mathcal{C}$, the short exact sequence α splits (in $\mathcal{C}$) if and only if $G_i(\alpha)$ splits (in $\mathcal{C}_i$) for all i.
- *X is E-injective.*

Proof. That the first and second conditions are equivalent follows from Theorem 2.6. The third condition is plainly the general (for all objects Y) form of the third condition from Theorem 2.6, hence equivalent to the first two. □

Another corollary of Theorem 2.6 is provided in Corollary 5.2.

3. The Hurewicz theorem.

One knows the Hurewicz theorem from classical algebraic topology: there exists a natural transformation $\pi_* \to H_*$, that is, from the homotopy groups functor to the homology groups functor; in degree 1 it is the abelianization functor; and if $\pi_i(X)$ vanishes for all $1 \leq i \leq n-1$, then the degree n Hurewicz map $\pi_n(X) \to H_n(X)$ is an isomorphism. Hence, while homotopy is difficult to compute but of great intrinsic interest, homology is homological (turns cofiber sequences to long exact sequences) and hence much easier to compute, and the Hurewicz theorem tells us that homology is a good homological approximation to homotopy.

We now prove a Hurewicz theorem which compares the Ш groups to the relative Ext-groups $\mathrm{Ext}_{\mathcal{C}/E}$. The moral to this theorem is the following: the Ш groups are not (co)homological, they do not turn (non-split) short exact sequences to long exact sequences, making them difficult to compute. However, the $\mathrm{Ext}_{C/E}$-groups turn E-short exact sequences to long exact sequences, making them (in principle, and sometimes also in practice) easier to compute than the Ш groups: see Theorem 5.4, for example, where we prove a broad vanishing theorem for the $\mathrm{Ext}_{C/E}$-groups under circumstances where the Ш-groups do not usually vanish and are usually quite nontrivial to compute. As $\mathrm{Ext}_{C/E}$ is cohomological rather than homological (and Ш is cohomotopical rather than homotopical?), our Hurewicz natural transformation goes from $\mathrm{Ext}^n_{C/E}$ to $Ш^n$, rather than the reverse. This natural transformation is an isomorphism in degree 1, just as in the classical Hurewicz theorem, as we now prove:

Theorem 3.1. (Hurewicz theorem for Ш.) *Let $\mathcal{C}, I, \{\mathcal{C}_i : i \in I\}$ be as in Definition 2.1, and let E be the allowable class defined in Definition 2.5. Suppose F_i is exact and resolving for all $i \in I$. Then the category $\mathcal{C}$ has enough E-projectives. Furthermore, for each object X of $\mathcal{C}$ and each $n \in \mathbb{N}$, there exists a natural transformation*

$$H : \mathrm{Ext}^n_{C/E}(-, X) \to Ш^n(-, X),$$

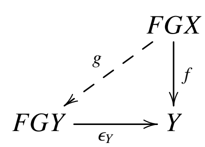
which we call the Hurewicz map for Ш. *In degree one, the Hurewicz map is a natural isomorphism:*

$$\mathrm{Ext}^1_{C/E}(-, X) \xrightarrow{\cong} Ш^1(-, X).$$

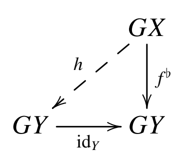
Proof.

- *Existence of enough E-projectives:* We claim that, for every object X of $\mathcal{C}$, the object FGX is E-projective. Indeed, every E-epimorphism is of the form $FGY \xrightarrow{\epsilon_Y} Y$ for some object Y of $\mathcal{C}$, so suppose we have a diagram

(3.0.4)



and want to produce a morphism as in the dotted arrow making the diagram commute, so that FGX satisfies the universal property for an E-projective object. Using the adjunction $F \dashv G$, diagram 3.0.4 is equivalent to the diagram



in the product category $\times_{i \in I} \mathcal{C}_i$, and now the map h exists: it is simply $h = f^\flat$. Now the desired map g in diagram 3.0.4 is simply $g = Fh = Ff^\flat$. Hence FGX is E-projective. Hence the short exact sequence

$$0 \to \ker \epsilon_X \to FGX \xrightarrow{\epsilon_X} X \to 0$$

shows that there exists an E-epimorphism from an E-projective to X. So $\mathcal{C}$ has enough E-projectives.
- *The Hurewicz map in degree* 1*:* Now, for any objects X, Y in $\mathcal{C}$, we have the exact sequence

$$\hom_{\mathcal{C}}(FGY, X) \to \hom_{\mathcal{C}}(\ker \epsilon_Y, X) \to \mathrm{Ext}^1_{C/E}(Y, X) \to \mathrm{Ext}^1_{C/E}(FGY, X)$$

and $\mathrm{Ext}^1_{\mathcal{C}}(FGY, X) \cong 0$ since FGY is E-projective. So $\mathrm{Ext}^1_{\mathcal{C}/E}(Y, X)$ is the cokernel of the map $\hom_{\mathcal{C}}(FGY, X) \rightarrow \hom_{\mathcal{C}}(\ker \epsilon_Y, X)$. Meanwhile, we have the exact sequence

$$\hom_{\mathcal{C}}(FGY, X) \rightarrow \hom_{\mathcal{C}}(\ker \epsilon_Y, X) \rightarrow \mathrm{Ext}^1_{\mathcal{C}}(Y, X) \rightarrow \mathrm{Ext}^1_{\mathcal{C}}(FGY, X),$$

in which the cokernel of the left-hand map is $\mathrm{Ext}^1_{\mathcal{C}/E}(Y, X)$, and the kernel of the right-hand map is, by definition, $Ш^1(Y, X)$. Hence the natural isomorphism $\mathrm{Ext}^1_{\mathcal{C}/E}(Y, X) \cong Ш^1(Y, X)$.

- *Construction of the Hurewicz map in degrees* > 1*:* Suppose $n > 1$. Since the composite map

$$\mathrm{Ext}^{n-1}_{\mathcal{C}}(\ker \epsilon_Y, X) \rightarrow \mathrm{Ext}^n_{\mathcal{C}}(Y, X) \rightarrow \mathrm{Ext}^n_{\mathcal{C}}(FGY, X)$$

is zero, the map $\mathrm{Ext}^{n-1}_{\mathcal{C}}(\ker \epsilon_Y, X) \rightarrow \mathrm{Ext}^n_{\mathcal{C}}(Y, X)$ factors through the inclusion of the kernel $Ш^n(Y, X) \hookrightarrow \mathrm{Ext}^n_{\mathcal{C}}(Y, X)$ of the right-hand map. So we have a factor map $f : \mathrm{Ext}^{n-1}_{\mathcal{C}}(\ker \epsilon_Y, X) \rightarrow Ш^n(Y, X)$. Now the Hurewicz map, when applied to an object Y, is simply the composite

$$\mathrm{Ext}^n_{\mathcal{C}/E}(Y, X) \xrightarrow{\cong} \mathrm{Ext}^n_{\mathcal{C}/E}(\ker \epsilon_Y, X) \xrightarrow{f} Ш^n(Y, X).$$

□

Remark 3.2. One would like to know, by analogy with the classical Hurewicz theorem in topology, whether vanishing of $Ш^i(Y, X)$ and $\mathrm{Ext}^i_{\mathcal{C}/E}(Y, X)$ for all $i < n$ implies that the degree n Hurewicz map $\mathrm{Ext}^n_{\mathcal{C}/E}(Y, X) \rightarrow Ш^n(Y, X)$ is an isomorphism. Clearly, this is true if one assumes that $Ш^i(Y, X)$ and $\mathrm{Ext}^i_{\mathcal{C}/E}(Y, X)$ vanish for $i < n$ *for all* Y, since as long as $n \geq 2$ this implies that X is E-injective and hence that $Ш^i(Y, X) \cong \mathrm{Ext}^i_{\mathcal{C}/E}(Y, X) \cong 0$ for all i and all Y. But it is natural to ask instead if the degree n Hurewicz map is an isomorphism if all lower $\mathrm{Ext}_{\mathcal{C}/E}$ vanish, "one object at a time." We do not know the answer to this question.

4. The spectral sequence.

Here is a natural spectral sequence which is *not* the one we will use in this paper! We describe it because its construction is slightly more obvious than the one we *will* use, and we want to avoid the reader mistaking one spectral sequence for the other.

Proposition 4.1. (The change-of-allowable-class spectral sequence.) *Let $\mathcal{C}$ be an abelian category, and let D, E be allowable classes in $\mathcal{C}$. Suppose $D \subseteq E$, suppose that $\mathcal{C}$ has enough D-projectives and enough E-projectives. Suppose X, Y are objects of $\mathcal{C}$, and choose an E-projective E-resolution*

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$$

of Y. Then there exists a spectral sequence

$$\begin{aligned} E_1^{s,t} \cong \mathrm{Ext}^t_{\mathcal{C}/D}(P_s, X) \quad &\Rightarrow \quad \mathrm{Ext}^{s+t}_{\mathcal{C}/E}(Y, X) \\ d_r : E_r^{s,t} \quad &\rightarrow \quad E_r^{s+r,t-r+1}. \end{aligned}$$

Proof. Special case of the usual resolution spectral sequence, as in e.g. Thm A1.3.2 of [6], arising from applying $\mathrm{Ext}_{\mathcal{C}/D}$ to

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0.$$

□

By contrast, the following spectral sequence is the one more relevant to the compatible splitting obstruction groups. We will write "absolute projectives" to mean the ordinary, usual projective objects in a category, because we shall need to refer to both relative projectives, that is, E-projectives, and the absolute projectives, and we want our terminology to be as unambiguous as possible.

Theorem 4.2. (The compatible splitting spectral sequence.) *Let $\mathcal{C}, I, \{\mathcal{C}_i : i \in I\}$ be as in Definition 2.1, and let E be the allowable class defined in Definition 2.5. Suppose F_i is exact and resolving for all $i \in I$, and suppose $\mathcal{C}$ has enough absolute projectives. Let X, Y be objects in $\mathcal{C}$, and define sequences of objects U_i, V_i in $\mathcal{C}$ inductively as follows: let $U_0 = V_0 = Y$, and for all $i \geq 1$, let $U_i = FGV_i$ and let $V_i = \ker(\epsilon_{V_{i-1}} : U_{i-1} \to V_{i-1})$. Then there exists a spectral sequence*

$$\begin{array}{rcl} E_1^{s,t} \cong \mathrm{Ext}^t_{\mathcal{C}}(U_s, X) & \Rightarrow & 0 \\ d_r : E_r^{s,t} & \to & E_r^{s+r,t-r+1} \end{array}$$

with the following properties:

- *As stated above, this spectral sequence converges to the zero bigraded abelian group.*
- *We have an identification of the E_2-page of the spectral sequence:*

$$E_2^{s,t} \cong \begin{cases} \left(R^{s-1}_E \mathrm{Ext}^t_{\mathcal{C}}(-,X)\right)(Y) & \text{if} \quad s \geq 2 \\ \left(\left(R^0_E \mathrm{Ext}^t_{\mathcal{C}}(-,X)\right)(Y)\right) / \left(\mathrm{Ext}^t_{\mathcal{C}}(Y,X)\right) & \text{if} \quad s = 1 \\ \text{Ш}^t(Y,X) & \text{if} \quad s = 0. \end{cases}$$

- *In particular, $E_2^{s,0} \cong \mathrm{Ext}^{s-1}_{\mathcal{C}/E}(Y,X)$ if $s \geq 2$, and $E_2^{0,0} \cong E_2^{1,0} \cong 0$.*

Proof. Special case of the usual resolution spectral sequence, as in e.g. Thm A1.3.2 of [6], arising from applying $\mathrm{Ext}_{\mathcal{C}}$ to the long exact sequence

$$\cdots \to U_3 \to U_2 \to U_1 \to U_0 \to 0 \qquad (4.0.5)$$

obtained by splicing the short exact sequences

$$0 \to V_{i+1} \to U_i \to V_i \to 0.$$

In more detail: long exact sequence 4.0.5 is an E-projective E-resolution for $U_0 = Y$. We choose an absolute projective resolution for each U_i and obtain a double complex:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \to & P_{2,2} & \to & P_{2,1} & \to & P_{2,0} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \to & P_{1,2} & \to & P_{1,1} & \to & P_{1,0} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \to & P_{0,2} & \to & P_{0,1} & \to & P_{0,0} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \to & 0 & \to & 0 & \to & 0 & \to & 0 \end{array}$$

such that each $P_{i,j}$ is an absolute projective in $\mathcal{C}$, the rows are exact, the homology of the column $P_{\bullet,i}$ is U_i concentrated in degree 0, and the maps induced in degree 0 homology by the horizontal differentials in the double complex are the maps in resolution 4.0.5. Then we apply $\hom_{\mathcal{C}}(-, X)$ to the entire double complex to yield a new double complex $\hom_{\mathcal{C}}(P_{\bullet,\bullet}, Y)$. Now we have the two spectral sequences of the double complex $\hom_{\mathcal{C}}(P_{\bullet,\bullet}, Y)$, as in [1]: in the first spectral sequence, the E_1-term is given by the cohomology of the rows in $\hom_{\mathcal{C}}(P_{\bullet,\bullet}, Y)$, each of which is $\mathrm{Ext}^*_{\mathcal{C}}(0, Y)$, since each row is $\hom_{\mathcal{C}}(-, Y)$ applied to a projective resolution of the zero object. Hence the spectral sequence is zero in the E_1-term, hence zero in the E_∞-term. The two spectral sequences converge to the same object, hence the second spectral sequence converges to zero. In the second spectral sequence, the E_1-term is given by the cohomology of the columns in $\hom_{\mathcal{C}}(P_{\bullet,\bullet}, Y)$, i.e., $\mathrm{Ext}^*_{\mathcal{C}}(U_*, Y)$.

To prove our claims about E_2 we need to examine the d_1 differential in this second spectral sequence. Along the rows of the spectral sequence, the d_1 differential is the differential of a cochain complex

$$\dots \xrightarrow{d_1} \mathrm{Ext}^t_{\mathcal{C}}(U_s, X) \xrightarrow{d_1} \mathrm{Ext}^t_{\mathcal{C}}(U_{s+1}, X) \xrightarrow{d_1} \mathrm{Ext}^t_{\mathcal{C}}(U_{s+2}, X) \xrightarrow{d_1} \dots,$$

and since 4.0.5 is an E-projective E-resolution of Y, if $s \geq 2$, the cohomology of this cochain complex is $(R^{s-1}_E \mathrm{Ext}^t_{\mathcal{C}}(-, X))(Y)$, the $(s-1)$th E-right-derived functor of $\mathrm{Ext}^t_{\mathcal{C}}(-, X)$. The degree shift as well as the $s = 0$ and $s = 1$ special cases are because we did not truncate the degree 0 part of resolution 4.0.5 before applying $\hom_{\mathcal{C}}(-, X)$ as one typically does when computing a derived functor; instead we left Y in its place in the long exact sequence when we applied $\hom_{\mathcal{C}}(-, X)$. Since $(R^0_E \mathrm{Ext}^t_{\mathcal{C}}(-, X))(Y)$ is the kernel of the map

$$\mathrm{Ext}^t_{\mathcal{C}}(U_1, X) \xrightarrow{d_1} \mathrm{Ext}^t_{\mathcal{C}}(U_2, X),$$

we have that $E_2^{0,t}$ is isomorphic to the cokernel of the map

$$\mathrm{Ext}^t_{\mathcal{C}}(Y, X) \to (R^0_E \mathrm{Ext}^t_{\mathcal{C}}(-, X))(Y),$$

as claimed in the statement of the theorem. □

Note that the transgression map on the E_{n+1}-page of the compatible splitting spectral sequence goes from $Ш^n(Y, X)$ to $\mathrm{Ext}^n_{\mathcal{C}/E}(Y, X)$. Curiously, this is the reverse direction of the Hurewicz map of Theorem 3.1. The transgression, however, does *not* yield a natural transformation $Ш^n(-, X) \to \mathrm{Ext}^n_{\mathcal{C}/E}(-, X)$, since for $n > 1$, $Ш^n(Y, X)$ might support a differential before the E_{n+1}-page, and $\mathrm{Ext}^n_{\mathcal{C}/E}(-, X)$ might be hit by a differential before the E_{n+1}-page. For $n = 1$ note that the above transgression is a d_2-differential which must be an isomorphism in order for the spectral sequence to converge to zero. This gives another proof that $Ш^1$ agrees with $\mathrm{Ext}^1_{\mathcal{C}/E}$, as in Theorem 3.1.

The $s = 0$ line in the E_2-term of the compatible splitting spectral sequence measures the failure of the functors $\{G_i\}_{i \in I}$ to detect splitting of finite-length exact sequences, in the sense made precise in Theorem 2.6. We also have a conceptual interpretation of the $s = 0$ and $s = 1$ lines of the E_2-term, taken together, in the compatible splitting spectral sequence: these two lines measures the failure of $\mathrm{Ext}_{\mathcal{C}}$ to be *left E-exact.* More precisely:

Corollary 4.3. *Let $\mathcal{C}, I, \{\mathcal{C}_i : i \in I\}$ be as in Definition 2.1, and let E be the allowable class defined in Definition 2.5. Suppose F_i is exact and resolving for all $i \in I$, and suppose further that $\mathcal{C}$ has enough E-injectives. Let X be an object of $\mathcal{C}$, and let t be a nonnegative integer. Then the functor* $\mathrm{Ext}^t_{\mathcal{C}}(-, X)$ *is left E-exact if and only if the groups $E_2^{0,t}$ and $E_2^{1,t}$ vanish, for all objects Y, in the spectral sequence of Theorem 4.2.*

Proof. If $\mathcal{C}$ has enough E-injectives, then the natural map to the 0th right satellite $\mathrm{Ext}^t_{\mathcal{C}}(-,X) \to R^0_E \mathrm{Ext}^t_{\mathcal{C}}(-,X)$ is an isomorphism if and only if $\mathrm{Ext}^t_{\mathcal{C}}(-,X)$ is left E-exact. However, $(R^0_E \mathrm{Ext}^t_{\mathcal{C}}(-,X))(Y)$ is the kernel of the map $d_1 : \mathrm{Ext}^t_{\mathcal{C}}(U_1,X) \to \mathrm{Ext}^t_{\mathcal{C}}(U_2,X)$, so the natural map to the 0th right satellite fits into the exact sequence

$$0 \to E_2^{0,t} \to \mathrm{Ext}^t_{\mathcal{C}}(Y,X) \to (R^0_E \mathrm{Ext}^t_{\mathcal{C}}(-,X))(Y) \to E_2^{1,t} \to 0.$$

Hence this natural map is an isomorphism if and only if both $E_2^{0,t}$ and $E_2^{1,t}$ vanish. □

Corollary 4.4. *Let $\mathcal{C}, I, \{\mathcal{C}_i : i \in I\}$ be as in Definition 2.1, and let E be the allowable class defined in Definition 2.5. Suppose F_i is exact and resolving for all $i \in I$, and suppose that Y is an object such that $\mathrm{Ext}^n_{\mathcal{C}/E}(Y,X)$ is trivial for all $n > 1$ and all X. Then, for all $t \geq 1$ and all objects X, Y of $\mathcal{C}$, we have natural isomorphisms*

$$\begin{aligned} Ш^t(Y,X) &\cong (R^1_E \mathrm{Ext}^{t-1}_{\mathcal{C}}(-,X))(Y) \\ &\cong \mathrm{Ext}^{t-1}_{\mathcal{C}}(FG\ker\epsilon_Y, X)/\mathrm{Ext}^{t-1}_{\mathcal{C}}(FGY,X). \end{aligned}$$

Furthermore, if $\mathcal{C}$ has enough E-injectives, then for all $t \geq 1$, the functor $Ш^t(-,X)$ vanishes if and only if $\mathrm{Ext}^{t-1}_{\mathcal{C}}(-,X)$ is left E-exact.

Proof. The assumption implies that Y has an E-projective E-resolution of length 1. So E-right derived functors of contravariant functors on $\mathcal{C}$ vanish above degree 1 when applied to Y. (We are taking projective, not injective, resolutions in order to take right derived functors, because we are taking derived functors of *contravariant* functors.)

Consequently, by the identification of the E_2-term in Theorem 4.2, the compatible splitting spectral sequence is concentrated in the $s = 0, s = 1$, and $s = 2$ columns. Since the spectral sequence must converge to the zero bigraded abelian group, this implies that the $s = 1$ line vanishes, and that the d_2-differential is an isomorphism. This implies the isomorphisms

$$Ш^t(Y,X) \cong (R^1_E \mathrm{Ext}^{t-1}_{\mathcal{C}}(-,X))(Y) \cong \mathrm{Ext}^{t-1}_{\mathcal{C}}(FG\ker\epsilon_Y, X)/\mathrm{Ext}^{t-1}_{\mathcal{C}}(FGY,X).$$

Since the $s = 1$ line vanishes, by Corollary 4.3 the functor $\mathrm{Ext}^{t-1}_{\mathcal{C}}(-,X)$ is left E-exact if and only if $E_2^{0,t-1}$ vanishes for all Y, i.e., if and only if $Ш^t(Y,X)$ vanishes for all Y. □

The following duality corollary is the one we use in our most important application, in Corollary 5.7.

Corollary 4.5. *Let $\mathcal{C}, I, \{\mathcal{C}_i : i \in I\}$ be as in Definition 2.1, and let E be the allowable class defined in Definition 2.5. Suppose F_i is exact and resolving for all $i \in I$, and suppose further that E is* hereditary*, that is, the relative* Ext*-groups* $\mathrm{Ext}^n_{\mathcal{C}/E}(-,-)$ *are trivial for all* $n > 1$*. Then, for all $t \geq 1$ and all objects X, Y of $\mathcal{C}$, we have natural isomorphisms*

$$\begin{aligned} Ш^t(Y,X) &\cong (R^1_E \mathrm{Ext}^{t-1}_{\mathcal{C}}(-,X))(Y) \\ &\cong \mathrm{Ext}^{t-1}_{\mathcal{C}}(FG\ker\epsilon_Y, X)/\mathrm{Ext}^{t-1}_{\mathcal{C}}(FGY,X). \end{aligned}$$

Furthermore, if $\mathcal{C}$ has enough E-injectives, then for all $t \geq 1$, the functor $Ш^t(-,X)$ vanishes if and only if $\mathrm{Ext}^{t-1}_{\mathcal{C}}(-,X)$ is left E-exact.

5. Main application: splitting morphisms of morphisms.

In this section, we study the special case of Ш which occurs in the following way: we begin with an abelian category $\mathcal{A}$, and we consider the category $\mathcal{A}^{(\bullet\to\bullet)}$ of arrows in $\mathcal{A}$. Clearly there is a forgetful functor $\mathcal{A}^{(\bullet\to\bullet)} \to \mathcal{A}\times\mathcal{A}$ sending a morphism f in $\mathcal{A}$ to the pair

$(\mathrm{dom}\, f, \mathrm{cod}\, f)$ consisting of the domain of f and the codomain of f. For this section we will let $Ш^n(f, g)$ be the kernel of the map

$$\mathrm{Ext}^n_{\mathcal{A}^{(\bullet\to\bullet)}}(f, g) \to \mathrm{Ext}^n_{\mathcal{A}}(\mathrm{dom}\, f, \mathrm{dom}\, g) \times \mathrm{Ext}^n_{\mathcal{A}}(\mathrm{cod}\, f, \mathrm{cod}\, g).$$

In other words, $Ш^n(f, g)$ is the group of equivalence classes of diagrams

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathrm{dom}\, g & \longrightarrow & X_1 & \longrightarrow & \dots & \longrightarrow & X_n & \longrightarrow & \mathrm{dom}\, f & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & \mathrm{cod}\, g & \longrightarrow & Y_1 & \longrightarrow & \dots & \longrightarrow & Y_n & \longrightarrow & \mathrm{cod}\, f & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$, where the top row and the bottom row are each *split* exact sequences. Such a diagram represents zero in $Ш^n(f, g)$ if and only if there exists splittings of the top and bottom rows which are *compatible* with the vertical maps. This is an important special case of Definition 2.1.

Proposition 5.1. *Let* $(\bullet\bullet)$ *denote the category with two objects and no non-identity morphisms, and let* $(\bullet \to \bullet)$ *denote the category with two objects and a single non-identity morphism from one object to the other. Then we have the abelian category* $\mathcal{A}^{(\bullet\bullet)} \cong \mathcal{A} \times \mathcal{A}$ *and the abelian category* $\mathcal{A}^{(\bullet\to\bullet)}$ *of morphisms in* $\mathcal{A}$*, and the exact faithful functor* $G : \mathcal{A}^{(\bullet\to\bullet)} \to \mathcal{A}^{(\bullet\bullet)}$*, that is, the functor induced by the inclusion of the subcategory* $(\bullet\bullet) \hookrightarrow (\bullet \to \bullet)$*. The functor G has a resolving left adjoint* $F : \mathcal{A}^{(\bullet\bullet)} \to \mathcal{A}^{(\bullet\to\bullet)}$ *given by:*

$$F(X, Y) = \left(X \stackrel{[\mathrm{id}_X\ 0]}{\longrightarrow} X \oplus Y \right).$$

Consequently, $FG(X \stackrel{f}{\longrightarrow} Y) = (X \stackrel{[\mathrm{id}_X\ 0]}{\longrightarrow} X \oplus Y)$*, and the counit map* $\epsilon_f : FGf \to f$ *of the adjunction* $F \dashv G$ *consists of the horizontal maps in the commutative diagram in* $\mathcal{A}$*:*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle [\mathrm{id}_X\ 0]}\downarrow & & \downarrow{\scriptstyle f} \\ X \oplus Y & \xrightarrow[{[f\ \mathrm{id}_Y]^\perp}]{} & Y. \end{array}$$

Proof. Elementary. ◻

The following is now a corollary of Theorem 2.6:

Corollary 5.2. *Let* $\mathcal{A}$ *be an abelian category. Let* $f : X' \to X$ *and* $g : Y' \to Y$ *be morphisms in* $\mathcal{A}$*. Then the following are equivalent:*

- $Ш^n(f, g) \cong 0$.
- *Each length* $n + 2$ *diagram in* $\mathcal{A}$ *with exact rows*

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & Y' & \longrightarrow & E_1' & \longrightarrow & \dots & \longrightarrow & E_n' & \longrightarrow & X' & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & Y & \longrightarrow & E_1 & \longrightarrow & \dots & \longrightarrow & E_n & \longrightarrow & X & \longrightarrow & 0 \end{array} \qquad (5.0.6)$$

  is compatibly split if and only if its top row and its bottom row are both split.

If $n = 1$*, the above conditions are furthermore each equivalent to:*

- *For each pair of maps $i : Y' \to Z$ and $h : Z' \to Z$ in $\mathcal{A}$, there exist maps j, k making the following diagram commute:*

$$\begin{array}{ccccc} Y' & \xrightarrow{[-g \ \ \mathrm{id}_{Y'}]} & Y \oplus Y' & \xleftarrow{[0 \ \ \mathrm{id}_{Y'}]} & Y' \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} \\ & & Z & \xleftarrow[h]{} & Z'. \end{array}$$

Proof. This is the case of Theorem 2.6 in which $\mathcal{C} = \mathcal{A}^{(\bullet\to\bullet)}$; in which I consists of only a single element, which we shall write $I = \{i\}$; in which $\mathcal{C}_i = \mathcal{A}^{(\bullet\bullet)} \cong \mathcal{A} \times \mathcal{A}$; and in which the functors F_i, G_i are the functors F, G of Proposition 5.1. A length $n+2$ exact sequence in $\mathcal{C}$ is then precisely a diagram of the form 5.0.6 with exact rows, and this diagram is split (in $\mathcal{A}^{(\bullet\bullet)}$) after applying G if and only if its top row and bottom row are each split (in $\mathcal{A}$).

The third condition of Theorem 2.6 is equivalent to the third condition given above, using Proposition 5.1 to identify $\ker \epsilon_f$ and FGf. ◻

Lemma 5.3. *Let $\mathcal{A}$ be an abelian category and let $f : X \to Y$ be a morphism in $\mathcal{A}$. Then the kernel* $\ker \epsilon_f$ *of the counit of the adjunction $F \dashv G$ on $\mathcal{A}^{(\bullet\to\bullet)}$ is isomorphic to the map $0 \to X$.*

Proof. The short exact sequence in $\mathcal{A}^{(\bullet\to\bullet)}$

$$0 \to \ker \epsilon_f \to FGf \to f \to 0 \tag{5.0.7}$$

is the commutative diagram with exact rows in $\mathcal{A}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle [\mathrm{id}_X \ \ 0]} & & \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & \ker [f \ \ \mathrm{id}_Y] & \longrightarrow & X \oplus Y & \xrightarrow[{[f \ \ \mathrm{id}_Y]^{\perp}}]{} & Y & \longrightarrow & 0. \end{array} \tag{5.0.8}$$

All we need is to produce an isomorphism $\ker [f \ \ \mathrm{id}_Y] \cong X$. Consider the commutative diagram with exact rows in $\mathcal{A}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker [f \ \ \mathrm{id}_Y] & \longrightarrow & X \oplus Y & \xrightarrow{[f \ \ \mathrm{id}_Y]^{\perp}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \mathrm{id}_Y} & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & X \oplus Y & \xrightarrow[{[0 \ \ \mathrm{id}_Y]^{\perp}}]{} & X & \longrightarrow & 0 \end{array} \tag{5.0.9}$$

where m is given by the matrix of maps

$$m = \begin{bmatrix} \mathrm{id}_X & f \\ 0 & \mathrm{id}_Y \end{bmatrix}.$$

The map m is invertible, with inverse given by

$$m^{-1} = \begin{bmatrix} \mathrm{id}_X & -f \\ 0 & \mathrm{id}_Y \end{bmatrix},$$

so the vertical maps m and id_Y in diagram 5.0.9 are isomorphisms. Hence the map $\ker [f \ \ \mathrm{id}_Y] \to X$ of kernels is also an isomorphism. ◻

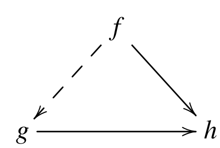
Theorem 5.4. *Let $\mathcal{A}$ be an abelian category and let E be the allowable class on the abelian category $\mathcal{A}^{(\bullet\to\bullet)}$ of morphisms in $\mathcal{A}$ consisting of all short exact sequences of the form*

$$0 \to \ker \epsilon_f \to FGf \to f \to 0$$

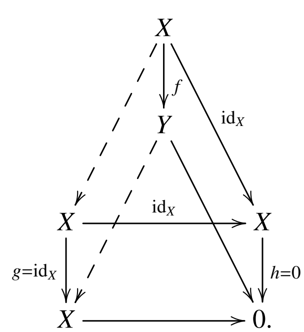
for all objects f in $\mathcal{A}^{(\bullet\to\bullet)}$, i.e., for all morphisms f in $\mathcal{A}$. (This is a special case of Definition 2.5.) Here F, G are as in Proposition 5.1. Then each of the following statements is true:

- *A morphism f of $\mathcal{A}$ is E-projective if and only if f is a split monomorphism.*
- *The category $\mathcal{A}^{(\bullet\to\bullet)}$ has enough E-projectives, that is, every object in $\mathcal{A}^{(\bullet\to\bullet)}$ is the codomain of some E-epimorphism with E-projective domain.*
- *The allowable class E is* hereditary*, i.e., the relative* Ext*-groups* $\mathrm{Ext}^n_{C/E}(f, g)$ *vanish for all f, g if $n > 1$.*
- *A morphism f of $\mathcal{A}$ is E-injective if and only if f is a split epimorphism.*

Proof. First, suppose $f : X \to Y$ is E-projective. Then consider the diagram in $\mathcal{A}^{(\bullet\to\bullet)}$:

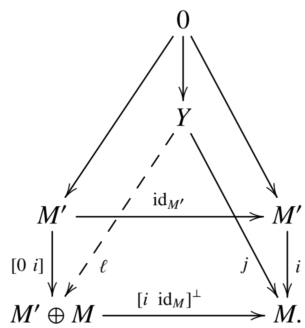


given by the following commutative diagram in $\mathcal{A}$:



Since f is assumed E-projective, maps exist as in the dotted lines which make the diagram commute. The top dotted map must then be the identity map on X. This in turn forces the bottom dotted map to be a retraction of f. So f must be split monic.

Now suppose $f : X \to Y$ is split monic. We want to show that it is E-projective. Since f is split monic, it can be written as the direct sum $f = f' \oplus f''$ with f' an isomorphism and f'' having zero domain. So it suffices to show that every isomorphism in $\mathcal{A}$ is E-projective and every map with zero domain in $\mathcal{A}$ is E-projective. For the isomorphisms this is trivial. For the maps f'' with zero domain, we must simply produce a map ℓ to fill in the dotted line in each commutative diagram of the form



The desired map ℓ is $\ell = [0 \ \ \mathrm{id}_M]^\perp \circ j$.

So we have proven that a morphism in $\mathcal{A}$ is E-projective if and only if the morphism is split monic. Now for any morphism $f : X \to Y$ in $\mathcal{A}$ we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle [\mathrm{id}_X \ 0]} & & \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & X \oplus Y & \xrightarrow[{[f \ \ \mathrm{id}_Y]}]{} & Y & \longrightarrow & 0 \end{array}$$

in which the lower-right copy of X is (isomorphic to) X by virtue of Lemma 5.3. So in the equivalent short exact sequence in $\mathcal{A}^{(\bullet\to\bullet)}$,

$$0 \to \ker \epsilon_f \to FGf \xrightarrow{\epsilon_f} f \to 0,$$

both the maps $\ker \epsilon_f$ and FGf in $\mathcal{A}$ are split monomorphisms, hence they are each E-projective objects of $\mathcal{A}^{(\bullet\to\bullet)}$. Hence every object of $\mathcal{A}^{(\bullet\to\bullet)}$ has a length 1 E-resolution by E-projective objects. Hence E is hereditary and $\mathcal{A}^{(\bullet\to\bullet)}$ has enough E-projectives.

The proof that an object f of $\mathcal{A}^{(\bullet\to\bullet)}$ is E-injective if and only if f is split epic is dual to the proof that f is E-projective if and only if it is split monic, given above. □

Corollary 5.5. *Let $\mathcal{A}$ be an abelian category and let E be as in Theorem 5.4. Let $h : Z' \to Z$ be a morphism in $\mathcal{A}$. Then the following are equivalent:*

- *If*

(5.0.10)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{i'} & Y' & \xrightarrow{\pi'} & Z' & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow \\ 0 & \longrightarrow & X & \xrightarrow{i} & Y & \xrightarrow{\pi} & Z & \longrightarrow & 0 \end{array}$$

  is a commutative diagram in $\mathcal{C}$ with split exact rows, then there exists a splitting $r' : Y' \to Y$ of i' and a splitting $r : Y \to X$ of i such that $f \circ r' = r \circ g$.
- *If diagram 5.0.10 is a commutative diagram in $\mathcal{C}$ with split exact rows, then there exists a splitting $s' : Y' \to Y$ of π' and a splitting $s : Y \to X$ of π such that $g \circ s' = s \circ h$.*
- *The first compatible splitting obstruction group $\text{Ш}^1(h, f) \cong 0$ is trivial for all morphisms f in $\mathcal{A}$.*
- *The morphism h is E-projective in $\mathcal{A}^{(\bullet\to\bullet)}$.*
- *The morphism h is split monic.*

Proof. Follows immediately from Theorem 2.6 and Theorem 5.4. □

Corollary 5.6. *Let $\mathcal{A}$ be an abelian category and let E be as in Theorem 5.4. Let $f : X' \to X$ be a morphism in $\mathcal{A}$. Then the following are equivalent:*

- *If diagram 5.0.10 is a commutative diagram in C with split exact rows, then there exists a splitting $r' : Y' \to Y$ of i' and a splitting $r : Y \to X$ of i such that $f \circ r' = r \circ g$.*
- *If diagram 5.0.10 is a commutative diagram in C with split exact rows, then there exists a splitting $s' : Y' \to Y$ of π' and a splitting $s : Y \to X$ of π such that $g \circ s' = s \circ h$.*
- *The first compatible splitting obstruction group $Ш^1(h, f) \cong 0$ is trivial for all morphisms h in $\mathcal{A}$.*
- *The morphism f is E-injective in $\mathcal{A}^{(\bullet\to\bullet)}$.*
- *The morphism f is split epic.*

Proof. Follows immediately from Theorem 2.6 and Theorem 5.4. □

Corollary 5.7. (Compatible splitting duality.) *Let $\mathcal{A}$ be an abelian category with enough projectives and let E be as in Theorem 5.4. Let $f : X \to Y$ and $g : V \to W$ be morphisms in $\mathcal{A}$. Then, for all $t \geq 1$, we have isomorphisms*

$$\begin{aligned} Ш^t(f, g) &\cong (R^1_E \operatorname{Ext}^{t-1}_{\mathcal{A}^{(\bullet\to\bullet)}}(-, g))(f) \\ &\cong \operatorname{Ext}^{t-1}_{\mathcal{A}^{(\bullet\to\bullet)}}(FG \ker \epsilon_f, g) / \operatorname{Ext}^{t-1}_{\mathcal{A}^{(\bullet\to\bullet)}}(FGf, g) \end{aligned}$$

and an exact sequence

$$0 \to Ш^t(f, g) \to \operatorname{Ext}^t_{\mathcal{A}^{(\bullet\to\bullet)}}(f, g) \to \operatorname{Ext}^t_{\mathcal{A}}(X, V) \times \operatorname{Ext}^t_{\mathcal{A}}(Y, W) \to \operatorname{Ext}^t_{\mathcal{A}}(X, W) \to Ш^{t+1}(f, g) \to 0.$$

Proof. The isomorphisms are corollaries of Theorem 5.4 and Corollary 4.5. The exact sequence follows from the given isomorphisms, together with the observation (in Lemma 5.3) that $\ker \epsilon_f$ is the map $0 \to X$, hence $\operatorname{Ext}^t_{\mathcal{A}^{(\bullet\to\bullet)}}(\ker \epsilon_f, g) \cong \operatorname{Ext}^t_{\mathcal{A}}(X, W)$. □

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