

HEIGHT FOUR FORMAL GROUPS WITH QUADRATIC COMPLEX MULTIPLICATION.

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ABSTRACT. We construct spectral sequences for computing the cohomology of automorphism groups of formal groups equipped with additional endomorphisms given by a p -adic number ring. We then compute the cohomology of the group of automorphisms of a height four formal group law which commute with additional endomorphisms of the group law by the ring of integers in the field $\mathbb{Q}_p(\sqrt{p})$, for primes $p > 5$. This automorphism group is a large profinite subgroup of the height four strict Morava stabilizer group. The group cohomology of this group of automorphisms turns out to have cohomological dimension 8 and total rank 80. We then run the $K(4)$ -local E_4 -Adams spectral sequence to compute the homotopy groups of the homotopy fixed-point spectrum of this group's action on the Lubin-Tate/Morava spectrum E_4 .

CONTENTS

1. Introduction.	1
2. Review of Ravenel's filtration and associated May spectral sequences.	4
3. Generalizations for formal A -modules.	6
4. Cohomology computations.	12
4.1. The cohomology of the height 2 Morava stabilizer group scheme.	12
4.2. The cohomology of the automorphism group scheme of a $\hat{\mathbb{Z}}_p[\sqrt{p}]$ -height 2 formal $\hat{\mathbb{Z}}_p[\sqrt{p}]$ -module.	14
5. Topological consequences.	18
Appendix A. Explicit Milnor-Moore theory.	22
References	24

1. INTRODUCTION.

This paper is intended as a companion and sequel to [20]. In J.P. May's thesis [10], he used the results of Milnor and Moore's paper [13] to set up spectral sequences for computing the cohomology of the Steenrod algebra, the input for the classical Adams spectral sequence; in chapter 6 of [17], Ravenel adapts May's spectral sequences for the purpose of computing the cohomology of automorphism groups of formal group laws, i.e., Morava stabilizer groups, which are the input for various spectral sequence methods for computing stable homotopy groups of spheres and Bousfield localizations of various spectra. In this paper we adapt Ravenel's tools to the task of computing the cohomology of automorphism groups of formal group laws with additional endomorphisms¹ by a p -adic number ring A , i.e., formal A -modules. We show (Theorem 5.1) that these automorphism groups are closed

¹Sometimes one says that formal A -modules are "formal group laws with complex multiplication by A ," but this terminology is ambiguous, since the theory of complex multiplication on abelian varieties really demands

subgroups of the Morava stabilizer groups, so that the machinery of [4] can be used to construct and compute the homotopy fixed-point spectra of the action of these automorphism groups on Lubin-Tate/Morava E -theory spectra; and then, most importantly, we actually use all this machinery to do some nontrivial computations: in Theorem 4.5, we compute the cohomology of the group scheme $\text{strAut}({}_1\mathbb{G}_{1/2}^{\mathbb{Z}_p}[\sqrt{p}])$ of strict automorphisms of a height four formal group law which commute with “complex multiplication” by the ring of integers in the field $\mathbb{Q}_p(\sqrt{p})$, for primes $p > 5$. This is an eight-dimensional pro- p -subgroup of the sixteen-dimensional height four strict Morava stabilizer group scheme. The group cohomology $H^*(\text{strAut}({}_1\mathbb{G}_{1/2}^{\mathbb{Z}_p}[\sqrt{p}]); \mathbb{F}_p)$ turns out to have cohomological dimension 8, total rank 80, and Poincaré series

$$(1 + s)^4(1 + 3s^2 + s^4) = 1 + 4s + 9s^2 + 16s^3 + 20s^4 + 16s^5 + 9s^6 + 4s^7 + s^8.$$

We then run the descent/ $K(4)$ -local E_4 -Adams spectral sequence to compute the homotopy groups of the homotopy fixed-point spectrum $E_4^{h\text{Aut}({}_1\mathbb{G}_{1/2}^{\mathbb{Z}_p}[\sqrt{p}]_{\otimes_{\mathbb{F}_p}\overline{\mathbb{F}_p}}) \rtimes \text{Gal}(\overline{k}/\mathbb{F}_p)}$ smashed with the Smith-Toda complex $V(3)$ (which exists, since we are still assuming that $p > 5$). The computation is Theorem 5.2; see the statement of Theorem 5.2 for a presentation of the homotopy groups of that spectrum, or Corollary 5.3 for just its Poincaré series.

Part of the appeal of Theorem 5.2, the main result of this paper, is that it is one of extremely few calculations of v_4 -periodic homotopy groups which have ever been made. Beyond that, deeper v_4 -periodic calculations can be made by building upon Theorem 5.2: in particular, see the preprint [19] for some applications of the computations in this paper as input for further, more difficult computations which eventually arrive at the cohomology of the height four Morava stabilizer group scheme at primes $p > 7$.

The computations in section 4 of this paper appeared already in the (unpublished, and not submitted for publication) announcement [19]; any version of that announcement which is ever submitted for journal publication will feature at most only an abbreviated version of these computations, with the idea that the complete versions are those provided in the present paper.

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Conventions 1.1.

- In this paper, all formal groups and formal modules are implicitly assumed to be *one-dimensional* and *commutative*.
- In this paper, “formal group” is used to mean “formal group law,” i.e., a power series satisfying appropriate associativity, commutativity, unitality, and inverse axioms, as in [5] or [17]. We do *not* mean “formal group” in the coordinate-free sense, i.e., a group structure on the formal affine line $\hat{\mathbb{A}}^1$, as in [21]. Similarly, our formal modules all have chosen coordinates, i.e., they are given by formal power series data. (The difference between the “coordinate-free” and “coordinate-chosen” definitions is, of course, simply a choice of power series generator for the ring of global sections of $\hat{\mathbb{A}}^1$. The statements of the computational results in this paper apply equally well to formal modules in the coordinate-free sense, although some of the proofs only make sense after a choice of coordinate.)

that the ring A have as large a rank as possible given the dimension of the abelian variety, a restriction whose analogue for formal group laws we do not make in this paper.

- Whenever \mathbb{G} is a formal group or a formal module over a field k , we will write $\text{Aut}(\mathbb{G})$ for the automorphism group *scheme* of \mathbb{G} , i.e., $\text{Aut}(\mathbb{G})$ is the functor from commutative k -algebras to groups which sends a k -algebra A to the automorphism group of $\mathbb{G} \otimes_k A$. For emphasis: *our notation* $\text{Aut}(\mathbb{G})$ *designates a group scheme, not a group*. Similarly, $\text{strAut}(\mathbb{G})$ is the group *scheme* of strict automorphisms of \mathbb{G} , i.e., $\text{strAut}(\mathbb{G}) : \text{Comm Alg}(k) \rightarrow \text{Groups}$ sends A to the subgroup of $\text{Aut}(\mathbb{G})(A)$ consisting of those automorphisms which are given by formal power series in $A[[X]]$ which are congruent to X modulo X^2 . In particular, when \mathbb{G} is p -typical, $\text{strAut}(\mathbb{G})$ is corepresented by the Hopf algebra $k \otimes_{BP_*} BP_* BP \otimes_{BP_*} k$, where k is a BP_* -module via the classifying map $BP_* \rightarrow k$ of \mathbb{G} . In the special case in which $BP_* \rightarrow \mathbb{F}_p$ sends v_n to 1 and sends v_i to zero if $i \neq n$, $\text{strAut}(\mathbb{G})$ is co-represented by the Morava stabilizer algebra $\Sigma(n)$, as in section 6.2 of [17].

The notation $k[\text{strAut}(\mathbb{G})]^*$ then refers to the commutative Hopf algebra of functions on the group scheme $\text{strAut}(\mathbb{G})$; see [22] for a textbook treatment.

It is well-known (see e.g. Theorem 6.2.3 of [17]) that, if k is finite, then $\text{Aut}(\mathbb{G})$ and $\text{strAut}(\mathbb{G})$ are pro-étale: that is, after a separable base change, they become pro-constant. Hence $\text{Aut}(\mathbb{G} \otimes_k \bar{k})$ and $\text{strAut}(\mathbb{G} \otimes_k \bar{k})$ are k -linear duals of honest profinite *groups*. This point is discussed more at the start of section 5. The cohomological consequences of $\text{Aut}(\mathbb{G})$ being a group scheme and not a group are rather slight: see the proof of Theorem 5.2 for an easy Galois descent argument which lets us pass between the cohomology of the group scheme $\text{Aut}(\mathbb{G})$ and the continuous group cohomology of the automorphism group of $\mathbb{G} \otimes_k \bar{k}$.

- Throughout, we will use Hazewinkel's generators for BP_* (and, more generally, for the classifying ring V^A of A -typical formal A -modules, where A is a discrete valuation ring).
- By a “ p -adic number field” we mean a finite field extension of the p -adic rationals \mathbb{Q}_p for some prime p .
- When a ground field k is understood from context, we will write $\Lambda(x_1, \dots, x_n)$ for the exterior/Grassmann k -algebra with generators x_1, \dots, x_n .
- Given a field k , we write $k\{x_1, \dots, x_n\}$ for the abelian Lie algebra over k with basis x_1, \dots, x_n .
- When L is a restricted Lie algebra over a field k and M is a module over the restricted enveloping algebra VL , we write $H_{res}^*(L, M)$ for restricted Lie algebra cohomology, i.e., $H_{res}^*(L, M) \cong \text{Ext}_{VL}^*(k, M)$, and we write $H_{unr}^*(L, M)$ for unrestricted Lie algebra cohomology, i.e., $H_{unr}^*(L, M) \cong \text{Ext}_{UL}^*(k, M)$, where UL is the universal enveloping algebra of L .
- Whenever convenient, we make use, without comment, of the well-known theorem of Milnor and Moore, from [13]: given a field k of characteristic $p > 0$, the functors P (restricted Lie algebra of primitives) and V (restricted enveloping algebra) establish an equivalence of categories between restricted Lie algebras over k and primitively generated cocommutative Hopf algebras over k , and this equivalence preserves cohomology, i.e., $H_{res}^*(L, M) \cong \text{Ext}_{VL}^*(k, M)$.
- Whenever convenient, we make use of the Chevalley-Eilenberg complex of a Lie algebra L to compute (unrestricted) Lie algebra cohomology $H_{unr}^*(L, M)$, as in [2].
- Many of the differential graded algebras in this paper have a natural action by a finite cyclic group; given an action by a finite cyclic group C_n on some DGA, we will always fix a generator for C_n and write σ for that generator.

2. REVIEW OF RAVENEL'S FILTRATION AND ASSOCIATED MAY SPECTRAL SEQUENCES.

This paper continues from [20]; for a brief introduction to formal A -modules, their moduli, A -typicality, A -height, and so on, the reader can consult that paper. A more complete account is in [16], and an even more complete account is chapter 21 of [5]. Briefly, the most important fact we will use is that, for A the ring of integers in a p -adic number field, the classifying ring of A -typical formal A -modules is $V^A \cong A[v_1^A, v_2^A, \dots]$, and the classifying ring of strict isomorphisms of A -typical formal A -modules is $V^A T \cong V^A[t_1^A, t_2^A, \dots]$, with v_n^A and t_n^A each in grading degree $2(q^n - 1)$, where q is the cardinality of the residue field of A .

Definition 2.1 and Theorem 2.2 appeared in [20]:

Definition 2.1. *Let K be a p -adic number field with ring of integers A and residue field k , and let n be a positive integer. Let k' be a field extension of k , and let $\alpha \in (k')^\times$. We write ${}_\alpha\mathbb{G}_{1/n}^A$ for the the formal A -module over k' classified by the map $V^A \rightarrow k'$ sending v_n^A to α and sending v_i^A to zero if $i \neq n$.*

We remind the reader of our convention, from Conventions 1.1, that the symbol $\text{strAut}(\mathbb{G})$ refers to the strict automorphism group *scheme* of a formal group law \mathbb{G} , and $k[\text{strAut}(\mathbb{G})]^*$ is the commutative Hopf k -algebra of functions on that group scheme.

Theorem 2.2. *Let L/K be a finite field extension of degree d , with K, L p -adic number fields with rings of integers A, B respectively. Let k, ℓ be the residue fields of A and B , let e be the ramification degree and f the residue degree of L/K , let q be the cardinality of ℓ , and let π_A, π_B be uniformizers for A, B , respectively. Let n be a positive integer. If ℓ' is a field extension of ℓ and $\beta \in (\ell')^\times$, then the underlying formal A -module of ${}_\beta\mathbb{G}_{1/n}^B$ is ${}_\alpha\mathbb{G}_{1/dn}^A$ where*

$$\alpha = \frac{\pi_A}{\pi_B^e} \beta^{\frac{q^{en}-1}{q^n-1}}.$$

Furthermore, the ring map

$$(2.1) \quad \begin{aligned} k(\alpha)[\text{strAut}({}_\alpha\mathbb{G}_{1/dn}^A)]^* &= k(\alpha)[t_1^A, t_2^A, \dots]/(t_i^A \alpha^{q^{ei}-1} - (t_i^A)^{q^{en}} \quad \forall i) \\ &\rightarrow \ell[t_1^B, t_2^B, \dots]/(t_i^B \beta^{q^i-1} - (t_i^B)^{q^n} \quad \forall i) = \ell[\text{strAut}({}_\beta\mathbb{G}_{1/n}^B)]^* \end{aligned}$$

classifying the strict formal A -module automorphism of ${}_\alpha\mathbb{G}_{1/dn}^A$ underlying the universal strict formal B -automorphism of ${}_\beta\mathbb{G}_{1/n}^B$ sends t_i^A to $t_{i/f}^A$ if i is divisible by the residue degree f of L/K , and sends t_i^A to zero if i is not divisible by the residue degree f .

Ravenel writes $S(n)$ for the Hopf algebra $\mathbb{F}_{p^n}[\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{Z}_p})]^*$; most of his constructions and computations work equally well for the more general Hopf algebra $\mathbb{F}_p[\text{strAut}({}_\alpha\mathbb{G}_{1/dn}^A)]^*$.

Definition 2.3 and Theorems 2.4 and 2.5 were given in section 6.3 of [17].

Definition 2.3. (Ravenel's numbers.) *Fix a prime number p and a positive integer n . Let $d_{n,i}$ be the integer defined by the formula*

$$d_{n,i} = \begin{cases} 0 & \text{if } i \leq 0 \\ \max\{i, pd_{n,i-n}\} & \text{if } i > 0. \end{cases}$$

(Clearly $d_{n,i}$ depends on the prime number p , but the choice of prime p is suppressed from the notation for $d_{n,i}$.)

Now equip the continuous \mathbb{F}_p -linear dual Hopf algebra

$$\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{Z}_p})]^* \cong \mathbb{F}_p \otimes_{BP_*} BP_* \otimes_{BP_*} \mathbb{F}_p \cong \mathbb{F}_p[t_1, t_2, \dots]/(t_i^{p^n} - t_i \quad \forall i)$$

with the increasing filtration in which the element t_i^j is in filtration degree $s_p(j)d_{n,i}$, where $s_p(j)$ is the sum of the digits in the base p expansion of j . Here the BP_* -module structure of \mathbb{F}_p is given by the ring map $BP_* \rightarrow \mathbb{F}_p$ sending v_n to 1 and sending v_i to 0 for all $i \neq 0$. We call this filtration the Ravenel filtration.

Theorem 2.4. (Theorems 6.3.1-6.3.3 of [17].) *The Ravenel filtration is an increasing Hopf algebra filtration, and its associated graded Hopf algebra $E^0S(n)$ is \mathbb{F}_p -linearly dual to a primitively generated Hopf algebra. The Hopf algebra $E^0S(n)$ is isomorphic, as an \mathbb{F}_p -algebra, to a truncated polynomial algebra:*

$$E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p})]^* \cong \mathbb{F}_p[t_{i,j} : 1 \leq i, j \in \mathbb{Z}/n\mathbb{Z}]/t_{i,j}^p,$$

where the coproduct is given by

$$(2.2) \quad \Delta(t_{i,j}) = \begin{cases} \sum_{0 \leq k \leq i} t_{k,j} \otimes t_{i-k,k+j} & \text{if } i < \frac{pn}{p-1}, \\ \sum_{0 \leq k \leq i} t_{k,j} \otimes t_{i-k,k+j} + \bar{b}_{i-n,j+n-1} & \text{if } i = \frac{pn}{p-1}, \\ t_{i,j} \otimes 1 + 1 \otimes t_{i,j} + \bar{b}_{i-n,j+n-1} & \text{if } i > \frac{pn}{p-1}, \end{cases},$$

where $t_{i,j}$ is the element of $E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p})]^*$ corresponding to $t_i^j \in \mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p})]^*$, $t_{0,j} = 1$, and \bar{x} is the image in $E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p})]^*$ of an element $x \in \mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p})]^*$. The \bar{b} elements have a fairly complicated combinatorial description; see 4.3.14 of [17].

The Hopf algebra $E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p})]^*$ is the \mathbb{F}_p -linear dual of the restricted enveloping algebra of a restricted Lie algebra $L(n)$. Let $x_{i,j}$ denote the \mathbb{F}_p -linear functional on $E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p})]^*$ which is dual to $t_{i,j}$; then the set

$$\{x_{i,j} : i > 0, j \in \mathbb{Z}/n\mathbb{Z}\}$$

is a \mathbb{F}_p -linear basis for $L(n)$. We describe the bracket and the restriction ξ on $L(n)$:

$$\begin{aligned} [x_{i,j}, x_{k,l}] &= \begin{cases} \delta_{i+j}^l x_{i+k,j} - \delta_{k+l}^j x_{i+k,l} & \text{if } i+k \leq \frac{pn}{p-1}, \\ 0 & \text{if } i+k > \frac{pn}{p-1}. \end{cases}, \\ \xi(x_{i,j}) &= \begin{cases} x_{i+n,j+1} & \text{if } i > \frac{n}{p-1} \text{ or } i = \frac{n}{p-1} \text{ and } p > 2, \\ x_{2n,j} + x_{2n,j+1} & \text{if } i = n \text{ and } p = 2, \\ 0 & \text{if } i < \frac{n}{p-1}. \end{cases}, \end{aligned}$$

where $\delta_b^a = 1$ if $a \equiv b$ modulo n , and $\delta_b^a = 0$ if $a \not\equiv b$ modulo n .

The two spectral sequences of J.P. May's thesis [10], in this context, take the form:

Theorem 2.5. *We have spectral sequences*

$$(2.3) \quad E_2^{s,t,u} \cong H_{\text{unr}}^{s,t}(L(n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} P(b_{i,j} : i \geq 1, j \in \mathbb{Z}/n\mathbb{Z}) \Rightarrow H_{\text{res}}^{s,t}(L(n); \mathbb{F}_p)$$

$$d_r : E_r^{s,t,u} \rightarrow E_r^{s+1,t,u+r-1}$$

$$(2.4) \quad E_1^{s,t} \cong H_{\text{res}}^{s,t}(L(n); \mathbb{F}_p) \Rightarrow H^{s,t}(\text{strAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p}); \mathbb{F}_p)$$

$$d_r : E_r^{s,t} \rightarrow E_r^{s+1,t-r},$$

where H_{unr}^* is (unrestricted) Lie algebra cohomology and H_{res}^* is restricted Lie algebra cohomology. Furthermore, the filtered DGA which gives rise to spectral sequence 2.3 splits as a tensor product of a term with trivial E_∞ -term with a term whose E_2 -term is

$$H^*\left(L(n, \lfloor \frac{pn}{p-1} \rfloor); \mathbb{F}_p\right) \otimes_{\mathbb{F}_p} P\left(b_{i,j} : 1 \leq i \leq \frac{n}{p-1}, j \in \mathbb{Z}/n\mathbb{Z}\right),$$

where $L(n, \lfloor \frac{pn}{p-1} \rfloor)$ is the quotient restricted Lie algebra of $L(n)$ in which we quotient out by the elements $x_{i,j}$ with $i > \lfloor \frac{pn}{p-1} \rfloor$. Consequently there exists a spectral sequence

(2.5)

$$E_2^{s,t,u} \cong H_{\text{unr}}^{s,t}(L(n, \lfloor \frac{pn}{p-1} \rfloor); \mathbb{F}_p) \otimes_{\mathbb{F}_p} P(b_{i,j} : 1 \leq i \leq \frac{n}{p-1}, 0 \leq j < n) \Rightarrow H_{\text{res}}^{s,t}(L(n); \mathbb{F}_p)$$

$$d_r : E_r^{s,t,u} \rightarrow E_r^{s+1,t,u+r-1}$$

Computation of the Chevalley-Eilenberg complex of $L(n)$ and of $L(n, \lfloor \frac{pn}{p-1} \rfloor)$ is routine, and appears in Theorem 6.3.8 of [17]:

Theorem 2.6. *Let $\mathcal{X}(n, m)$ be the differential graded \mathbb{F}_p -algebra which is the exterior algebra $\Lambda(h_{i,j} : 1 \leq i \leq m, j \in \mathbb{Z}/n\mathbb{Z})$ with differential*

$$d(h_{i,j}) = \sum_{k=1}^{i-1} h_{k,j} h_{i-k,j+k},$$

with the convention that $h_{i,k+n} = h_{i,k}$. Then $H^*(\mathcal{X}(n, \lfloor \frac{pn}{p-1} \rfloor)) \cong H_{\text{unr}}^*(L(n, \lfloor \frac{pn}{p-1} \rfloor); \mathbb{F}_p)$.

3. GENERALIZATIONS FOR FORMAL A -MODULES.

Recall that a graded Hopf algebra A over a field k is said to be *finite-type* if, for each $n \in \mathbb{Z}$, the grading degree n summand A^n of A is a finite-dimensional k -vector space.

Proposition 3.1. *Let K/\mathbb{Q}_p be a field extension of degree d and ramification degree e and residue degree f . Let $q = p^f$, let A be the ring of integers of K , let π be a uniformizer for A , and let k be the residue field of A . Let n be a positive integer, and let $\omega \in \bar{k}$ be a $\frac{q^n-1}{q-1}$ th root of $\frac{\pi^e}{p}$. Then the underlying formal $\hat{\mathbb{Z}}_p$ -module of ${}_{\omega}\mathbb{G}_{1/n}^A$ is ${}_1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p}$, and the Ravenel filtration on $\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^*$ induces a compatible filtration on the Hopf algebra*

$$(3.6) \quad k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^* \cong k(\omega)[t_f, t_{2f}, \dots] / (t_{if}^{q^n} - \omega^{q^i-1} t_{if} \quad \forall i).$$

The associated graded Hopf algebra $E^0 k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^*$ is the graded k -linear dual of a primitively generated finite-type Hopf algebra, which, as a quotient of $E^0 \mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^* \otimes_{\mathbb{F}_p} k(\omega)$, is given by

$$k(\omega)[t_{i,j} : i \geq 1, j \in \mathbb{Z}/fn\mathbb{Z}] / (t_{i,j}^p \quad \forall i, j),$$

with coproduct

$$(3.7) \quad \Delta(t_{i,j}) = \begin{cases} \sum_{0 \leq k \leq i} t_{k,j} \otimes t_{i-k,k+j} & \text{if } i < \frac{pdn}{p-1}, \\ \sum_{0 \leq k \leq i} t_{k,j} \otimes t_{i-k,k+j} + \bar{b}_{i-dn,j+dn-1} & \text{if } i = \frac{pdn}{p-1}, \\ t_{i,j} \otimes 1 + 1 \otimes t_{i,j} + \bar{b}_{i-dn,j+dn-1} & \text{if } i > \frac{pdn}{p-1}. \end{cases}$$

Here $t_{i,j}$ is the element of $E^0 k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^*$ corresponding to $t_i^{p^j} \in k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^*$, and:

$$(3.8) \quad \begin{aligned} t_{0,j} &= 1, \\ t_{i,j} &= 0 \text{ if } f \nmid i > 0, \\ t_{i,j+fn} &= \omega^{p^j(q^i-1)} t_{i,j}, \text{ and} \\ \bar{b}_{i,j+fn} &= \omega^{p^{j+1}(q^i-1)} \bar{b}_{i,j}. \end{aligned}$$

Proof. The claim that the underlying formal $\hat{\mathbb{Z}}_p$ -module of ${}_{\omega}\mathbb{G}_{1/n}^A$ is ${}_1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p}$ is simply a special case of Theorem 2.2, as is the isomorphism 3.6. The fact that the Ravenel filtration on $\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^*$ induces a filtration on the Hopf algebra $k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^*$ is straightforward: the Hopf algebra $k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^*$ is the quotient of $(\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^*) \otimes_{\mathbb{F}_p} k(\omega)$ by the ideal I generated by $t_{if}^{q^n} - \omega^{q^{if}-1}t_{if}$ for all i and by t_j for all j not divisible by f ; these generators for this ideal I are all homogeneous in the Ravenel filtration.

Computation of the associated graded, including the formula for the coproduct, is routine: simply reduce the formulas of Theorem 2.4 modulo I . Deriving formula 3.8 requires consulting the definition of $b_{i,j}$ in 4.3.14 of [17] in terms of Witt polynomials; the essential observations here are that $b_{i,j+1}$, modulo p , is obtained from $b_{i,j}$ by replacing each element t_m with t_m^p , and that $t_i^{ap^{j+fn}} \otimes t_i^{(p-a)p^{j+fn}} = (t_i^{p^{fn}})^{ap^j} \otimes (t_i^{p^{fn}})^{(p-a)p^j} = \omega^{p^{j+1}(p^{fn}-1)}t_i^{ap^j} \otimes t_i^{(p-a)p^j}$.

The fact that $E^0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^*$ is finite-type is immediate from its given presentation; and it is dual to a primitively generated Hopf algebra since its linear dual is a sub-Hopf-algebra of the linear dual of $E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^* \otimes_{\mathbb{F}_p} k(\omega)$, which is primitively generated. \square

Theorem 3.2. (Structure of $PE_0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]$.) *Let K/\mathbb{Q}_p be a field extension of degree d and ramification degree e and residue degree f . Let A be the ring of integers of K , let π be a uniformizer for A , and let k be the residue field of A . Let n be a positive integer, and let $\omega \in \bar{k}$ be a $\frac{q^{en}-1}{q^n-1}$ th root of $\frac{\pi^e}{p}$. Let $PE_0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]$ be the restricted Lie algebra of primitives in the $k(\omega)$ -linear dual Hopf algebra $(E^0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^*)^*$. Let $x_{i,j}^A$ be the element of $PE_0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]$ dual to the indecomposable $t_{i,j} \in E^0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^*$. Then $\{x_{i,j}^A : f \mid i, j \in \mathbb{Z}/fn\mathbb{Z}\}$ is a $k(\omega)$ -linear basis for $PE_0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]$, and dual to the natural surjection*

$$\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^* \otimes_{\mathbb{F}_p} k(\omega) \rightarrow k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]^*,$$

we have an inclusion of restricted Lie algebras over $k(\omega)$:

$$(3.9) \quad \begin{aligned} PE_0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)] &\xrightarrow{\iota} PE_0k(\omega)[\text{strAut}({}_1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})] \\ \iota(x_{i,j}^A) &= \sum_{\ell=0}^{e-1} \omega^{p^j(q^j-1)\frac{q^{dn}-1}{q^n-1}} x_{i,j+\ell fn}. \end{aligned}$$

When $p = \pi^e$, the bracket on $PE_0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]$ is given by

$$(3.10) \quad [x_{i,j}^A, x_{k,\ell}^A] = \begin{cases} \tilde{\delta}_{i+j}^{\ell} x_{i+k,j}^A - \tilde{\delta}_{k+\ell}^j x_{i+k,\ell}^A & \text{if } i+k \leq \frac{pdn}{p-1} \\ 0 & \text{if } i+k > \frac{pdn}{p-1} \end{cases},$$

where $\tilde{\delta}_a^b = 1$ if $a \equiv b$ modulo fn , and $\tilde{\delta}_a^b = 0$ if $a \not\equiv b$ modulo fn . The restriction on $PE_0k(\omega)[\text{strAut}({}_{\omega}\mathbb{G}_{1/n}^A)]$ is given by

$$(3.11) \quad \xi(x_{i,j}^A) = \begin{cases} x_{i+n,j+1}^A & \text{if } i > \frac{dn}{p-1} \\ x_{i+n,j+1}^A + x_{pi,j}^A & \text{if } i = \frac{dn}{p-1} \text{ and } fn \mid i \\ x_{i+n,j+1}^A & \text{if } i = \frac{dn}{p-1} \text{ and } fn \nmid i \\ x_{pi,j}^A & \text{if } i < \frac{dn}{p-1} \text{ and } fn \mid i \\ 0 & \text{if } i < \frac{dn}{p-1} \text{ and } fn \nmid i \end{cases},$$

where $x_{i+n,fn}^A = x_{i+n,0}^A$.

Proof. Formula 3.9 follows from checking where elements in $E^0\mathbb{F}_p[\text{strAut}(1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^*$ are sent to $E^0k(\omega)[\text{strAut}(\omega\mathbb{G}_{1/n}^A)]^*$ by the canonical surjection

$$E^0\mathbb{F}_p[\text{strAut}(1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^* \otimes_{\mathbb{F}_{p^n}} k(\omega) \rightarrow E^0k(\omega)[\text{strAut}(\omega\mathbb{G}_{1/n}^A)]^*,$$

using the description of the map 2.1 in Theorem 2.2 to accomplish this. The map $\mathbb{F}_p[\text{strAut}(1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^* \otimes_{\mathbb{F}_{p^n}} k(\omega) \rightarrow k(\omega)[\text{strAut}(\omega\mathbb{G}_{1/n}^A)]^*$ sends $t_i^{p^j}$ to

$$\begin{aligned} t_i^{p^{j_0+j_1fn}} &= ((\dots((t_i^{p^{fn}})^{p^{fn}})^{p^{fn}} \dots)^{p^{fn}})^{p^{j_0}} \\ &= \omega^{p^{j_0}(q^j-1)\frac{q^{j_1n}-1}{q^{j_1n}-1}} t_i^{p^{j_0}}, \end{aligned}$$

where j_0, j_1 are the unique nonnegative integers such that $j = j_0 + j_1fn$ and $j_0 < fn$; hence the map $E^0\mathbb{F}_p[\text{strAut}(1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]^* \otimes_{\mathbb{F}_{p^n}} k(\omega) \rightarrow E^0k(\omega)[\text{strAut}(\omega\mathbb{G}_{1/n}^A)]^*$ sends $t_{i,j}$ to $\omega^{p^{j_0}(q^j-1)\frac{q^{j_1n}-1}{q^{j_1n}-1}} t_{i,j_0}$. Formula 3.9 follows at once.

Now suppose that $p = \pi^e$. We compute the Lie bracket in $PE_0k(\omega)[\text{strAut}(\omega\mathbb{G}_{1/n}^A)]$:

$$\begin{aligned} [x_{i,j}^A, x_{k,\ell}^A] &= \left[\sum_{a=0}^{e-1} x_{i,j+afn}, \sum_{b=0}^{e-1} x_{k,\ell+bf n} \right] \\ &= \sum_{a=0}^{e-1} \sum_{b=0}^{e-1} [x_{i,j+afn}, x_{k,\ell+bf n}] \\ &= \begin{cases} \sum_{a=0}^{e-1} \sum_{b=0}^{e-1} (\delta_{i+j+afn}^{\ell+bf n} x_{i+k,j+afn} - \delta_{k+\ell+bf n}^{j+afn} x_{i+k,\ell+bf n}) & \text{if } i+k \leq \frac{pdn}{p-1} \\ 0 & \text{if } i+k > \frac{pdn}{p-1} \end{cases} \\ &= \begin{cases} \tilde{\delta}_{i+j}^{\ell} \sum_{a=0}^{e-1} x_{i+k,j+afn} - \tilde{\delta}_{k+\ell}^j \sum_{b=0}^{e-1} x_{i+k,\ell+bf n} & \text{if } i+k \leq \frac{pdn}{p-1} \\ 0 & \text{if } i+k > \frac{pdn}{p-1} \end{cases} \\ &= \tilde{\delta}_{i+j}^{\ell} x_{i+k,j}^A - \tilde{\delta}_{k+\ell}^j x_{i+k,\ell}^A, \end{aligned}$$

where $\delta_a^b = 1$ if $a \equiv b$ modulo fn , and $\delta_a^b = 0$ if $a \not\equiv b$ modulo fn .

For the restriction on $PE_0k(\omega)[\text{strAut}(\omega\mathbb{G}_{1/n}^A)]$, we could proceed as above, computing

$$\xi \left(\sum_{\ell=0}^{e-1} x_{i,j+\ell fn} \right)$$

in $PE_0\mathbb{F}_p[\text{strAut}(1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]$; but this immediately means contending with the non-linearity of the restriction map, which complicates the computation. Instead it is easier to compute ξ on $PE_0k(\omega)[\text{strAut}(\omega\mathbb{G}_{1/n}^A)]$ in basically the same way that Ravenel computes ξ on $PE_0\mathbb{F}_p[\text{strAut}(1\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]$ in the proof of Proposition 6.3.3 of [17]; we sketch that method here. To compute $\xi(x_{i,j}^A)$ we just need to find which elements $t_{a,b} \in E^0k(\omega)[\text{strAut}(\omega\mathbb{G}_{1/n}^A)]^*$ have the property that the $(p-1)$ st iterate $\Delta \circ \dots \circ \Delta(t_{a,b})$ of Δ , applied to $t_{a,b}$, has a monomial term which is a scalar multiple of the p -fold tensor power $t_{i,j} \otimes \dots \otimes t_{i,j}$. When $i > \frac{dn}{p-1}$,

then $i + dn > \frac{pdn}{p-1}$ and hence, by formula 3.7, we have

$$\begin{aligned} \Delta(t_{i+dn,j+1}) &= t_{i+dn,j+1} \otimes 1 + 1 \otimes t_{i+dn,j+1} + \bar{b}_{i,j} \\ &= t_{i+dn,j+1} \otimes 1 + 1 \otimes t_{i+dn,j+1} - \sum_{0 < \ell < p} \frac{1}{p} \binom{p}{\ell} t_{i,j}^\ell \otimes t_{i,j}^{p-\ell}, \end{aligned}$$

and hence after $p - 1$ iterations of Δ applied to $t_{i+dn,j+1}$, we get a copy of the monomial $t_{i,j} \otimes \dots \otimes t_{i,j}$. When $pi \leq \frac{pdn}{p-1}$ and $i = kfn$ for some positive integer k , then formula 3.7 gives us that the $(p - 1)$ st iterate of Δ , applied to $\Delta(t_{pi,j})$, contains the monomial

$$\begin{aligned} &= t_{kfn,j} \otimes t_{kfn,j+kfn} \otimes t_{kfn,j+2kfn} \otimes \dots \otimes t_{kfn,j+(p-1)kfn} \\ &= t_{kfn,j} \otimes t_{kfn,j} \otimes \dots \otimes t_{kfn,j} \end{aligned}$$

It is simple to show that no further monomials $t_{a,b}$ have the property that their $(p - 1)$ st iterated coproducts contain the p th tensor power monomial $t_{i,j} \otimes \dots \otimes t_{i,j}$. Formula 3.11 follows.

The relation $x_{i+n,fn}^A = x_{i+n,0}^A$ follows from the fact that $x_{i+n,fn}^A$ is dual to $t_{i+n,fn}$ and $x_{i+n,0}^A$ is dual to $t_{i+n,0}$. \square

Definition 3.3. Let K/\mathbb{Q}_p be a finite extension with degree d , ramification degree e , residue degree f , and let A be its ring of integers, π a uniformizer for A , and k the residue field of A . Let n be a positive integer, and let ω be a $\left(\frac{q^n-1}{q-1}\right)$ th root, in \bar{k} , of $\frac{\pi^e}{p}$. We have the restricted graded Lie algebras $PE_0k(\omega)[\text{strAut}(\omega\mathbb{G}_{1/n}^A)]$ and $PE_0k(\omega)[\text{strAut}(\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]$ over $k(\omega)$, and we write $L_\omega^A(n)$ and $L(dn)$, respectively, as shorthand notations for them. If ℓ is a positive integer, we will also write $L(dn, \ell)$ for the quotient Lie algebra of $L(dn)$ in which we quotient out all generators $x_{i,j}$ for which $i > \ell$ (this notation agrees with that of Theorem 2.5); and we will write $L_\omega^A(n, \ell)$ for the quotient Lie algebra of $L_\omega^A(n)$ in which we quotient out all generators $x_{i,j}^A$ for which $i > \ell$. We have an obvious commutative diagram of homomorphisms of Lie algebras:

$$\begin{array}{ccc} L_\omega^A(n, \ell) & \longrightarrow & L(dn, \ell) \\ \uparrow & & \uparrow \\ L_\omega^A(n) & \longrightarrow & L(dn) \end{array}$$

Now here is a very useful corollary of Theorem 3.2:

Corollary 3.4. The restricted Lie algebra $L_1^{\hat{\mathbb{Z}}_p} \langle \sqrt[p]{p} \rangle(n, m)$ is isomorphic to the restricted Lie algebra $L_1^{\hat{\mathbb{Z}}_p}(n, m) = L(n, m)$ as long as $m \leq \frac{pn}{p-1}$.

J.P. May actually constructed two different types of spectral sequence in his thesis [10]: the spectral sequence of a filtered Hopf algebra, as in 2.4, is the one most typically called a ‘‘May spectral sequence.’’ The other spectral sequence, of Corollary 9 of [9] (as in 2.3), is the one which computes restricted Lie algebra cohomology from unrestricted Lie algebra cohomology; we will call that spectral sequence the *Lie-May spectral sequence* to distinguish it from the May spectral sequence (unfortunately, there is probably no perfect choice of terminology to be made here; e.g. chapter 6 of Ravenel’s book [17] refers to both spectral sequences as May spectral sequences).

Theorem 3.5. *Let K/\mathbb{Q}_p be a field extension of degree d and ramification degree e and residue degree f . Let A be the ring of integers of K , let π be a uniformizer for A , and let k be the residue field of A . Let n be a positive integer, and let $\omega \in \bar{k}$ be a $\frac{q^n-1}{q-1}$ th root of $\frac{\pi^e}{p}$.*

We have the morphism of Lie-May spectral sequences

$$\begin{array}{ccc} H_{unr}^*(L(dn); k(\omega)) \otimes_{k(\omega)} k(\omega) \left[\left\{ b_{i,j} : i \geq 1, 0 \leq j \leq dn-1 \right\} \right] & \Longrightarrow & H_{res}^*(L(dn); k(\omega)) \\ \downarrow & & \downarrow \\ H_{unr}^*(L_\omega^A(n); k(\omega)) \otimes_{k(\omega)} k(\omega) \left[\left\{ b_{i,j} : f \mid i, 0 \leq j \leq fn-1 \right\} \right] & \Longrightarrow & H_{res}^*(L_\omega^A(n); k(\omega)), \end{array}$$

with $b_{i,j}$ in bidegree $(2, 0)$ (these two gradings are, respectively, cohomological degree and Lie-May degree) and with auxiliary bidegree $(p\|t_{i,j}\|, 2p^{j+1}(p^j-1))$ (these two gradings are, respectively, the grading coming from the Ravenel filtration, and the grading coming from the topological grading on BP_*BP) in each spectral sequence, where $\|t_{i,j}\|$ is the Ravenel degree of $t_{i,j}$. The elements in $H_{unr}^t(L_\omega^A(n); k(\omega))$ are in bidegree $(0, t)$. The differential is, as is typical for the spectral sequence of a filtered cochain complex, $d_r^{s,t} : E_r^{s,t} \rightarrow E_r^{s+r,t-r+1}$.

If $p = \pi^e$, then we have a tensor splitting of each of these Lie-May spectral sequences, such that the splittings are respected by the morphism 3.5 of spectral sequences: the Lie-May spectral sequence

$$H_{unr}^*(L(dn); \mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q \left[\left\{ b_{i,j} : i \geq 1, 0 \leq j \leq dn-1 \right\} \right] \Rightarrow H_{res}^*(L(dn); \mathbb{F}_q)$$

splits into a tensor product of a spectral sequence

$$\begin{aligned} H_{unr}^*(L(dn, \frac{pdn}{p-1}); \mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q \left[\left\{ b_{i,j} : 1 \leq i \leq \frac{dn}{p-1}, 0 \leq j \leq dn-1 \right\} \right] \\ \Rightarrow H_{res}^*(L(dn, \frac{pdn}{p-1}); \mathbb{F}_q) \end{aligned}$$

with a spectral sequence with trivial E_∞ -term; and likewise, the Lie-May spectral sequence

$$H_{unr}^*(L_1^A(n); \mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q \left[\left\{ b_{i,j} : f \mid i, 0 \leq j \leq fn-1 \right\} \right] \Rightarrow H_{unr}^*(L_1^A(n); \mathbb{F}_q)$$

splits into a tensor product of a spectral sequence

$$\begin{aligned} H_{unr}^*(L^A(n, \frac{pdn}{p-1}); \mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q \left[\left\{ b_{i,j} : f \mid i, 1 \leq i \leq \frac{dn}{p-1}, 0 \leq j \leq fn-1 \right\} \right] \\ \Rightarrow H_{res}^*(L^A(n, \frac{pdn}{p-1}); \mathbb{F}_q), \end{aligned}$$

which we will call the reduced Lie-May spectral sequence, with a spectral sequence with trivial E_∞ -term.

We have a morphism of spectral sequences:

$$\begin{array}{ccc} H_{unr}^*(L(dn, \frac{pdn}{p-1}); \mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q \left[\left\{ b_{i,j} : 1 \leq i \leq \frac{dn}{p-1}, 0 \leq j \leq dn-1 \right\} \right] & \Longrightarrow & H_{res}^*(L(dn, \frac{pdn}{p-1}); \mathbb{F}_q) \\ \downarrow & & \downarrow \\ H_{unr}^*(L_1^A(n, \frac{pdn}{p-1}); \mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q \left[\left\{ b_{i,j} : f \mid i, i \leq \frac{dn}{p-1}, 0 \leq j \leq fn-1 \right\} \right] & \Longrightarrow & H_{res}^*(L_1^A(n, \frac{pdn}{p-1}); \mathbb{F}_q). \end{array}$$

Proof. That the morphism 3.5 of spectral sequences exists follows from May's construction of the Lie-May spectral sequence in [9]. The splittings occur because of formula 3.10, which tells us that the unrestricted Lie algebra underlying $L(dn)$ splits into a product of

$L(dn, \frac{pdn}{p-1})$ with an abelian Lie algebra generated by $\{x_{i,j} : i > \frac{pdn}{p-1}\}$, and the unrestricted Lie algebra underlying $L_1^A(n)$ splits into a product of $L^A(n, \frac{pdn}{p-1})$ with an abelian Lie algebra generated by $\{x_{i,j}^A : i > \frac{pdn}{p-1}\}$; and formula 3.9 tells us that the morphism $L_1^A(n) \hookrightarrow L(dn)$ respects these product splittings. By formula 3.11, the restriction on $PE_0\mathbb{F}_p[\text{strAut}(\mathbb{G}_{1/dn}^{\hat{\mathbb{Z}}_p})]$ sends $x_{i,j}$ to $x_{i+n,j+1}$ when $i > \frac{dn}{p-1}$; so the filtered chain complex (see Theorem 5 in [11] or Corollary 9 of [9]) whose associated spectral sequence is the Lie-May spectral sequence has the property that it splits into a tensor product of a cohomologically trivial filtered chain complex and one whose associated graded chain complex has cohomology $H_{\text{unr}}^*(L(dn, \frac{pdn}{p-1}); \mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q \left[\{b_{i,j} : 1 \leq i \leq \frac{dn}{p-1}, 0 \leq j \leq dn-1\} \right]$. An analogous statement holds for the bracket and restriction on $PE_0\mathbb{F}_q[\text{strAut}(\mathbb{G}_{1/n}^A)]$ and the filtered chain complex giving its Lie-May spectral sequence. \square

Definition 3.6. Let K/\mathbb{Q}_p be a field extension of degree d and ramification degree e and residue degree f . Let A be the ring of integers of K , let π be a uniformizer for A , and let k be the residue field of A . Let n be a positive integer, and let $\omega \in k$ be a $\frac{q^m-1}{q-1}$ th root of $\frac{\pi}{p}$. We write $\mathcal{X}_\omega^A(n)$ for the Chevalley-Eilenberg DGA of the Lie algebra $L_\omega^A(n)$. If m is a positive integer, we write $\mathcal{X}_\omega^A(n, m)$ for the Chevalley-Eilenberg DGA of the Lie algebra $L_\omega^A(n, m)$. (Note that the Chevalley-Eilenberg DGA depends only on the underlying unrestricted Lie algebra.)

The cyclic group C_{dn} acts on $\mathcal{X}_\omega^A(n)$ by sending $h_{i,j}$ to $h_{i,j+1}$, and when $\omega = 1$, this action reduces to an action of C_n on $\mathcal{X}_\omega^A(n)$.

The DGAs $\mathcal{X}_\omega^A(n)$ and $\mathcal{X}_\omega^A(n, m)$ are equipped with several gradings which we will need to keep track of: the cohomological grading; the topological grading (sometimes also called the ‘‘internal grading’’) inherited from BP_*BP , which is only defined modulo $2(p^{fn}-1)$; and the Ravenel grading, inherited from the Ravenel filtration. Note that the C_{dn} -action preserves the cohomological gradings and the Ravenel grading, but not the internal grading; this behavior will be typical in all of the multigraded DGAs we consider, and we adopt the convention that, whenever we speak of a ‘‘multigraded equivariant DGA,’’ we assume that the group action preserves all of the gradings except possibly the internal grading.

The presentation in Theorem 2.6 generalizes as follows:

Observation 3.7. It is easy and routine to extract a presentation for the Chevalley-Eilenberg DGA from Proposition 3.1, without using the formulas in Theorem 3.2: $\mathcal{X}_\omega^A(n)$ is the exterior algebra (over $k(\omega)$) with generators given by the set of symbols $h_{i,j}$ with i divisible by the residue degree f and satisfying $1 \leq i$, and $j \in \mathbb{Z}/fn\mathbb{Z}$; the differential is given by

$$d(h_{i,j}) = \sum_{k=1}^{i-1} h_{k,j} h_{i-k,j+k},$$

with the convention that $h_{i,k+fn} = \omega^{p^k(q^i-1)} h_{i,k}$. Similarly, $\mathcal{X}_\omega^A(n, m)$ is the sub-DGA of $\mathcal{X}_\omega^A(n)$ generated by all $h_{i,j}$ with $i \leq m$.

When $\omega = 1$ and $A = \hat{\mathbb{Z}}_p[\sqrt[p]{p}]$ and $m \leq \frac{pn}{p-1}$, we write $\mathcal{X}(n, m)$ as shorthand for $\mathcal{X}_1^{\hat{\mathbb{Z}}_p[\sqrt[p]{p}]}(n, m)$ and $\mathcal{X}_1^{\hat{\mathbb{Z}}_p}(n, m)$; this notation is unambiguous because of Corollary 3.4.

4. COHOMOLOGY COMPUTATIONS.

4.1. The cohomology of the height 2 Morava stabilizer group scheme. The material in this subsection is easy and well-known, appearing already in section 6.3 of [17]. Still, we give a relatively full account of the computation in Proposition 4.1, largely to fix notations for the elements in that computation which will play a role in our later spectral sequence calculations.

Proposition 4.1. *Suppose $p > 2$. Then we have an isomorphism of trigraded C_2 -equivariant \mathbb{F}_p -algebras*

$$H^{*,*,*}(L(2, 2)) \cong \mathbb{F}_p\{1, h_{10}, h_{11}, h_{10}\eta_2, h_{11}\eta_2, h_{10}h_{11}\eta_2\} \otimes_{\mathbb{F}_p} \Lambda(\zeta_2),$$

with tridegrees and the C_2 -action as follows (remember that the internal degree is always reduced modulo $2(p^2 - 1)$, and recall from Conventions 1.1 that we write σ for a generator of C_2):

	<i>Coh. class</i>	<i>Coh. degree</i>	<i>Int. degree</i>	<i>Rav. degree</i>	<i>Image under σ</i>
	1	0	0	0	1
	h_{10}	1	$2(p-1)$	1	h_{11}
	h_{11}	1	$2p(p-1)$	1	h_{10}
	ζ_2	1	0	2	ζ_2
(4.12)	$h_{10}\eta_2$	2	$2(p-1)$	3	$-h_{11}\eta_2$
	$h_{11}\eta_2$	2	$2p(p-1)$	3	$-h_{10}\eta_2$
	$h_{10}\zeta_2$	2	$2(p-1)$	3	$h_{11}\zeta_2$
	$h_{11}\zeta_2$	2	$2p(p-1)$	3	$h_{10}\zeta_2$
	$h_{10}h_{11}\eta_2$	3	0	4	$h_{10}h_{11}\eta_2$
	$h_{10}\eta_2\zeta_2$	3	$2(p-1)$	5	$-h_{11}\eta_2\zeta_2$
	$h_{11}\eta_2\zeta_2$	3	$2p(p-1)$	5	$-h_{10}\eta_2\zeta_2$
	$h_{10}h_{11}\eta_2\zeta_2$	4	0	6	$h_{10}h_{11}\eta_2\zeta_2$.

where the cup products in $\mathbb{F}_p\{1, h_{10}, h_{11}, h_{10}\eta_2, h_{11}\eta_2, h_{10}h_{11}\eta_2\}$ are all zero aside from the Poincaré duality cup products, i.e., each class has the obvious dual class such that the cup product of the two is $h_{10}h_{11}\eta_2$, and the remaining cup products are all zero.

Proof. We have the extension of Lie algebras

$$1 \rightarrow \mathbb{F}_p\{x_{20}, x_{21}\} \rightarrow L(2, 2) \rightarrow L(2, 1) \rightarrow 1,$$

and to compute the resulting spectral sequence in cohomology, we take the Chevalley-Eilenberg DGAs and then compute the Cartan-Eilenberg spectral sequence for the extension of C_2 -equivariant trigraded DGAs

$$1 \rightarrow \mathcal{X}(2, 1) \rightarrow \mathcal{X}(2, 2) \rightarrow \Lambda(h_{20}, h_{21}) \rightarrow 1.$$

Since the differential on $\mathcal{X}(2, 1)$ is zero (see Observation 3.7), $H^{*,*,*}(\mathcal{X}(2, 1)) \cong \mathcal{X}(2, 1) \cong \Lambda(h_{10}, h_{11})$. A change of \mathbb{F}_p -linear basis is convenient here: we will write ζ_2 for the element $h_{20} + h_{21} \in \Lambda(h_{20}, h_{21})$. (This notation for this particular element is standard. As far as I know, it began with [12].) We will write η_2 for the element $h_{20} - h_{21}$.

We have the differentials

$$\begin{aligned} d\zeta_2 &= 0, \\ d\eta_2 &= -2h_{10}h_{11} \end{aligned}$$

□

The table 4.12 has one row for each element in an \mathbb{F}_p -linear basis for the cohomology ring $H^{*,*}(\mathcal{X}(2, 2))$, but from now on in this document, for the sake of brevity, when writing out similar tables for grading degrees of elements in the cohomology of a multigraded equivariant DGA, I will just give one row for each element in a set of generators for the cohomology ring of the DGA.

Proposition 4.2. *Suppose $p > 3$. Then the cohomology $H^*(\text{strAut}(\mathbb{G}_{1/2}); \mathbb{F}_p)$ of the height 2 strict Morava stabilizer group scheme is isomorphic, as a graded \mathbb{F}_p -vector space, to*

$$H^{*,*}(\mathcal{X}(2, 2)) \cong \mathbb{F}_p\{1, h_{10}, h_{11}, h_{10}\eta_2, h_{11}\eta_2, h_{10}h_{11}\eta_2\} \otimes_{\mathbb{F}_p} \Lambda(\zeta_2)$$

from Proposition 4.1. The cohomological grading on $H^*(\text{strAut}(\mathbb{G}_{1/2}); \mathbb{F}_p)$ corresponds to the cohomological grading on $H^{*,*}(\mathcal{X}(2, 2))$, so that $h_{10}, h_{11}, \zeta_2 \in H^1(\text{strAut}(\mathbb{G}_{1/2}); \mathbb{F}_p)$, $h_{10}\eta_2, h_{11}\eta_2 \in H^2(\text{strAut}(\mathbb{G}_{1/2}); \mathbb{F}_p)$, and so on.

The multiplication on $H^*(\text{strAut}(\mathbb{G}_{1/2}); \mathbb{F}_p)$ furthermore agrees with the multiplication on $H^{*,*}(\mathcal{X}(2, 2))$, modulo the question of exotic multiplicative extensions, i.e., jumps in Ravenel filtration in the products of elements in $H^*(\text{strAut}(\mathbb{G}_{1/2}); \mathbb{F}_p)$.

Proof. Spectral sequence 2.3 collapses immediately, since $p > 3$ implies that $1 > \lfloor \frac{2}{p-1} \rfloor$. Hence $\text{Cotor}_{E^0\mathbb{F}_p[\text{strAut}(\mathbb{G}_{1/2})]^*}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) \cong H^{*,*}(\mathcal{X}(4, 2))$.

We now run spectral sequence 2.4. This is, like all May spectral sequences, the spectral sequence of the filtration (in this case, Ravenel's filtration) on the cobar complex $C^\bullet(A)$ of a coalgebra A induced by a filtration on the coalgebra itself. To compute differentials, we take an element $x \in H^*(C^\bullet(E^0A))$, lift it to a cochain $\bar{x} \in H^*(C^\bullet(A))$ whose image in the cohomology of the associated graded $H^*(E^0(C^\bullet(A))) \cong H^*(C^\bullet(E^0A))$ is x , and then evaluate the differential $d(\bar{x})$ in the cobar complex $C^\bullet(A)$. If $d(\bar{x}) = 0$, then \bar{x} is a cocycle in the cobar complex $C^\bullet(A)$ and not merely in its associated graded $E^0C^\bullet(A)$, hence \bar{x} represents a cohomology class in $H^*(C^\bullet(A))$; if $d(\bar{x}) \neq 0$, then we add correcting coboundaries of lower or higher (depending on whether the filtration is increasing or decreasing) filtration until we arrive at a cocycle which we recognize as a cohomology class in the spectral sequence's E_1 -page.

It will be convenient to use the presentation

$$\mathbb{F}_p[t_{i,j} : i \geq 1, 0 \leq j \leq 1] / (t_{i,j}^p \text{ for all } i, j)$$

for $E^0(\mathbb{F}_p[\text{strAut}(\mathbb{G}_{1/2})]^*) \cong E^0(\mathbb{F}_p[t_1, t_2, \dots] / (t_i^{p^2} - t_i \text{ for all } i))$, where $t_{i,j}$ is the image in the associated graded of t_i^j . The coproduct on $\mathbb{F}_p[t_{i,j} : i \geq 1, 0 \leq j \leq 1] / (t_{i,j}^p \text{ for all } i, j)$, inherited from that of $\mathbb{F}_p[\text{strAut}(\mathbb{G}_{1/2})]^*$, is given by

$$\Delta(t_{i,j}) = \sum_{k=0}^i t_{k,j} \otimes t_{i-k,k+j}$$

for all $i < \lfloor \frac{2p}{p-1} \rfloor$; see Theorem 6.3.2 of [17] for this formula.

h_{10}, h_{11} : The class h_{10} is represented by $t_{1,0}$ in the cobar complex $C^\bullet(E^0(\mathbb{F}_p[\text{strAut}(\mathbb{G}_{1/2})]^*))$, which lifts to t_1 in the cobar complex $C^\bullet(\mathbb{F}_p[\text{strAut}(\mathbb{G}_{1/2})]^*)$. Since t_1 is a coalgebra primitive, i.e., a cobar complex 1-cocycle, all May differentials are zero on $h_{1,0}$. The C_2 -equivariance of the spectral sequence then tells us that all May differentials also vanish on $h_{1,0}$.

ζ_2 : There is no nonzero class in cohomological degree 2 and internal degree 0 for ζ_2 to hit by a May differential of any length.

$h_{10}\eta_2, h_{11}\eta_2$: The cohomology class $h_{10}\eta_2$ in the Chevalley-Eilenberg complex of the Lie algebra of primitives in $E^0(\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2})]^*)$ (of which $\mathcal{K}(4, 2)$ is a subcomplex) is represented by the 2-cocycle $t_{1,0} \otimes t_{2,0} - t_{1,0} \otimes t_{2,1} - t_{1,0} \otimes t_{1,0}t_{1,1}$ in the cobar complex of $E^0(\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2})]^*)$. This 2-cocycle lifts to the 2-cocycle $t_1 \otimes t_2 - t_1 \otimes t_2^p - t_1 \otimes t_1^{p+1}$ in the cobar complex of $\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2})]^*$. Hence all May differentials vanish on $h_{10}\eta_2$, and by C_2 -equivariance, also $h_{11}\eta_2$.

So the May differentials of all lengths vanish on the generators of the ring $\text{Cotor}_{E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2})]^*}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p)$. So $H^*(\text{strAut}({}_1\mathbb{G}_{1/2}); \mathbb{F}_p) \cong \text{Cotor}_{E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2})]^*}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p) \cong H^{*,*,*}(\mathcal{K}(2, 2))$ as a graded \mathbb{F}_p -vector space. \square

Here is a chart of (4.12), displayed using the Adams convention (i.e., the vertical axis is cohomological degree, and the horizontal axis is internal degree minus cohomological degree), at the prime $p = 7$:



Not pictured in (4.13) is v_2 in bidegree $(0, 96)$, which generates another copy of the rest of the diagram; the rest of the diagram is repeated every 96 degrees (in homotopy, i.e., along the horizontal axis).

4.2. The cohomology of the automorphism group scheme of a $\hat{\mathbb{Z}}_p[\sqrt{p}]$ -height 2 formal $\hat{\mathbb{Z}}_p[\sqrt{p}]$ -module.

Proposition 4.3. *Suppose $p > 3$. Then we have an isomorphism of trigraded C_2 -equivariant \mathbb{F}_p -algebras*

$$H^{*,*,*}(\mathcal{K}(2, 3)) \cong \mathcal{A}_{2,3} \otimes_{\mathbb{F}_p} \Lambda(\zeta_2),$$

where

$$\begin{aligned} \mathcal{A}_{2,3} \cong \mathbb{F}_p\{ & 1, h_{10}, h_{11}, h_{10}h_{30}, h_{11}h_{31}, e_{40}, \eta_2e_{40}, h_{10}\eta_2h_{30}, h_{11}\eta_2h_{31}, \\ & h_{10}\eta_2h_{30}h_{31}, h_{11}\eta_2h_{30}h_{31}, h_{10}h_{11}\eta_2h_{30}h_{31} \}, \end{aligned}$$

with tridegrees and the C_2 -action as follows (remember that the internal degrees are still reduced modulo $2(p^2 - 1)$):

	Coh. class	Coh. degree	Int. degree	Rav. degree	Image under σ
	1	0	0	0	1
	h_{10}	1	$2(p-1)$	1	h_{11}
	h_{11}	1	$2p(p-1)$	1	h_{10}
	$h_{10}h_{30}$	2	$4(p-1)$	$1+p$	$h_{11}h_{31}$
	$h_{11}h_{31}$	2	$4p(p-1)$	$1+p$	$h_{10}h_{30}$
(4.14)	e_{40}	2	0	$1+p$	$-e_{40}$
	$\eta_2 e_{40}$	3	0	$3+p$	$\eta_2 e_{40}$
	$h_{10}\eta_2 h_{30}$	3	$4(p-1)$	$3+p$	$-h_{11}\eta_2 h_{31}$
	$h_{11}\eta_2 h_{31}$	3	$4p(p-1)$	$3+p$	$-h_{10}\eta_2 h_{30}$
	$h_{10}\eta_2 h_{30}h_{31}$	4	$2(p-1)$	$3+2p$	$h_{11}\eta_2 h_{30}h_{31}$
	$h_{11}\eta_2 h_{30}h_{31}$	4	$2p(p-1)$	$3+2p$	$h_{10}\eta_2 h_{30}h_{31}$
	$h_{10}h_{11}\eta_2 h_{30}h_{31}$	5	0	$4+2p$	$-h_{10}h_{11}\eta_2 h_{30}h_{31}$
	ζ_2	1	0	2	ζ_2

where the cup products in $\mathcal{A}_{2,3}$ are all zero aside from the Poincaré duality cup products, i.e., each class has the dual class such that the cup product of the two is $h_{10}h_{11}\eta_2 h_{30}h_{31}$, and the remaining cup products are all zero. The classes in table 4.14 listed above ζ_2 are listed in order so that the class which is n lines below 1 is, up to multiplication by a unit in \mathbb{F}_p , the Poincaré dual in $\mathcal{A}_{2,3}$ of the class which is n lines above $h_{10}h_{11}\eta_2 h_{30}h_{31}$.

Proof. We have the extension of Lie algebras

$$1 \rightarrow \mathbb{F}_p\{x_{30}, x_{31}\} \rightarrow L(2, 3) \rightarrow L(2, 2) \rightarrow 1$$

and we take their Chevalley-Eilenberg DGAs, then compute the Cartan-Eilenberg spectral sequence for the extension of C_2 -equivariant trigraded DGAs:

$$1 \rightarrow \mathcal{X}(2, 2) \rightarrow \mathcal{X}(2, 3) \rightarrow \Lambda(h_{30}, h_{31}) \rightarrow 1.$$

We have the differentials

$$\begin{aligned} dh_{30} &= -h_{10}\eta_2, \\ dh_{31} &= h_{11}\eta_2, \\ d(h_{30}h_{31}) &= -h_{10}\eta_2 h_{31} - h_{11}\eta_2 h_{30}. \end{aligned}$$

and their products with classes in $H^{*,*}(\mathcal{X}(2, 2))$. The nonzero products are

$$\begin{aligned} d(h_{11}h_{30}) &= -h_{10}h_{11}\eta_2 \\ d(h_{10}h_{31}) &= -h_{10}h_{11}\eta_2 \\ d(h_{10}h_{30}h_{31}) &= h_{10}h_{11}\eta_2 h_{30} \\ d(h_{11}h_{30}h_{31}) &= -h_{10}h_{11}\eta_2 h_{31}. \end{aligned}$$

We write e_{40} for the cocycle $h_{10}h_{31} - h_{11}h_{30}$. Extracting the output of the spectral sequence from knowledge of the differentials is routine. \square

Proposition 4.4. *Suppose $p > 3$. Then we have an isomorphism of trigraded C_2 -equivariant \mathbb{F}_p -algebras*

$$H^{*,*}(\mathcal{X}(2, 4)) \cong \mathcal{A}_{2,4} \otimes_{\mathbb{F}_p} \Lambda(\zeta_2, \zeta_4),$$

where

$$\begin{aligned} \mathcal{A}_{2,4} \cong \mathbb{F}_p\{ & 1, h_{10}, h_{11}, h_{10}h_{30}, h_{11}h_{31}, h_{10}\eta_4 - \eta_2h_{30}, h_{11}\eta_4 - \eta_2h_{31}, \eta_2e_{40}, \\ & h_{10}\eta_2h_{30}, h_{11}\eta_2h_{31}, h_{10}h_{30}\eta_4, h_{11}h_{31}\eta_4, \eta_4e_{40} + 4\eta_2h_{30}h_{31}, h_{10}\eta_2h_{30}h_{31}, h_{11}\eta_2h_{30}h_{31}, \\ & h_{10}\eta_2h_{30}\eta_4, h_{11}\eta_2h_{31}\eta_4, h_{10}\eta_2h_{30}h_{31}\eta_4, h_{11}\eta_2h_{30}h_{31}\eta_4, h_{10}h_{11}\eta_2h_{30}h_{31}\eta_4\}, \end{aligned}$$

with tridegrees and the C_2 -action as follows:

(4.15)

Coh. class	Coh. degree	Int. degree	Rav. degree	Image under σ
1	0	0	0	1
h_{10}	1	$2(p-1)$	1	h_{11}
h_{11}	1	$2p(p-1)$	1	h_{10}
$h_{10}h_{30}$	2	$4(p-1)$	$1+p$	$h_{11}h_{31}$
$h_{11}h_{31}$	2	$4p(p-1)$	$1+p$	$h_{10}h_{30}$
$h_{10}\eta_4 - \eta_2h_{30}$	2	$2(p-1)$	$1+2p$	$-h_{11}\eta_4 + \eta_2h_{30}$
$h_{11}\eta_4 - \eta_2h_{31}$	2	$2p(p-1)$	$1+2p$	$-h_{10}\eta_4 + \eta_2h_{31}$
η_2e_{40}	3	0	$3+p$	η_2e_{40}
$h_{10}\eta_2h_{30}$	3	$4(p-1)$	$3+p$	$-h_{11}\eta_2h_{31}$
$h_{11}\eta_2h_{31}$	3	$4p(p-1)$	$3+p$	$-h_{10}\eta_2h_{30}$
$h_{10}h_{30}\eta_4$	3	$4(p-1)$	$1+3p$	$-h_{11}h_{31}\eta_4$
$h_{11}h_{31}\eta_4$	3	$4p(p-1)$	$1+3p$	$-h_{10}h_{30}\eta_4$
$\eta_4e_{40} + 4\eta_2h_{30}h_{31}$	3	0	$1+3p$	$\eta_4e_{40} + 4\eta_2h_{30}h_{31}$
$h_{10}\eta_2h_{30}h_{31}$	4	$2(p-1)$	$3+2p$	$h_{11}\eta_2h_{30}h_{31}$
$h_{11}\eta_2h_{30}h_{31}$	4	$2p(p-1)$	$3+2p$	$h_{10}\eta_2h_{30}h_{31}$
$h_{10}\eta_2h_{30}\eta_4$	4	$4(p-1)$	$3+3p$	$h_{11}\eta_2h_{31}\eta_4$
$h_{11}\eta_2h_{31}\eta_4$	4	$4p(p-1)$	$3+3p$	$h_{10}\eta_2h_{30}\eta_4$
$h_{10}\eta_2h_{30}h_{31}\eta_4$	5	$2(p-1)$	$3+4p$	$-h_{11}\eta_2h_{30}h_{31}\eta_4$
$h_{11}\eta_2h_{30}h_{31}\eta_4$	5	$2p(p-1)$	$3+4p$	$-h_{10}\eta_2h_{30}h_{31}\eta_4$
$h_{10}h_{11}\eta_2h_{30}h_{31}\eta_4$	6	0	$4+4p$	$h_{10}h_{11}\eta_2h_{30}h_{31}\eta_4$
ζ_2	1	0	2	ζ_2
ζ_4	1	0	$2p$	ζ_4

The classes in table 4.15 listed above ζ_2 are listed in order so that the class which is n lines below 1 is, up to multiplication by a unit in \mathbb{F}_p , the Poincaré dual in $\mathcal{A}_{2,4}$ of the class which is n lines above $h_{10}h_{11}\eta_2h_{30}h_{31}\eta_4$.

Proof. We compute the Cartan-Eilenberg spectral sequence for the extension of C_2 -equivariant trigraded DGAs

$$1 \rightarrow \mathcal{K}(2, 3) \rightarrow \mathcal{K}(2, 4) \rightarrow \Lambda(h_{40}, h_{41}) \rightarrow 1$$

arising from the extension of Lie algebras

$$1 \rightarrow \mathbb{F}_p\{x_{40}, x_{41}\} \rightarrow L(2, 4) \rightarrow L(2, 3) \rightarrow 1.$$

A change of \mathbb{F}_p -linear basis is convenient here: we will write ζ_4 for the element $h_{40} + h_{41} \in \Lambda(h_{40}, h_{41})$, and we will write η_4 for $h_{40} - h_{41}$. We have the differentials

$$\begin{aligned} d\zeta_4 &= 0, \\ d\eta_4 &= h_{10}h_{31} + h_{30}h_{11} = e_{40}, \end{aligned}$$

and a nonzero product with a class in $H^{*,*}(\mathcal{K}(2, 3))$,

$$d(\eta_2e_{40}\eta_4) = h_{10}h_{11}\eta_2h_{30}h_{31}.$$

Extracting the output of the spectral sequence from knowledge of the differentials is routine. The three classes $h_{10}\eta_4, h_{11}\eta_4, \eta_4e_{40}$ in the E_∞ -term are not cocycles in $H^{*,*,*}(\mathcal{X}(2, 4))$; adding terms of lower Cartan-Eilenberg filtration to get cocycles yields the cohomology classes $h_{10}\eta_4 - \eta_2h_{30}, h_{11}\eta_4 - \eta_2h_{31}, \eta_4e_{40} + 4\eta_2h_{30}h_{31}$. Note that this implies that there are nonzero multiplications in $\mathcal{A}_{2,4}$ other than those between each class and its Poincaré dual; for example, $h_{10}(h_{10}\eta_4 - \eta_2h_{30}) = -h_{10}\eta_2h_{30}$. \square

Theorem 4.5. *Suppose $p > 5$. Then the cohomology $H^*(\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]}); \mathbb{F}_p)$ of the strict automorphism of the $\hat{\mathbb{Z}}_p[\sqrt{p}]$ -height 2 formal $\hat{\mathbb{Z}}_p[\sqrt{p}]$ -module ${}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]}$ is isomorphic, as a graded \mathbb{F}_p -vector space, to*

$$H^{*,*,*}(\mathcal{X}(2, 4)) \cong \mathcal{A}_{2,4} \otimes_{\mathbb{F}_p} \Lambda(\zeta_2, \zeta_4),$$

from Proposition 4.4. The cohomological grading on $H^*(\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]}); \mathbb{F}_p)$ corresponds to the cohomological grading on $H^{*,*,*}(\mathcal{X}(2, 4))$.

The multiplication on $H^*(\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]}); \mathbb{F}_p)$ furthermore agrees with the multiplication on $H^{*,*,*}(\mathcal{X}(2, 4))$, modulo the question of exotic multiplicative extensions, i.e., jumps in Ravenel filtration in the products of elements in $H^*(\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]}); \mathbb{F}_p)$.

In particular, the Poincaré series expressing the \mathbb{F}_p -vector space dimensions of the grading degrees in $H^*(\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]}); \mathbb{F}_p)$ is

$$(1 + s)^2 (1 + 2s + 4s^2 + 6s^3 + 4s^4 + 2s^5 + s^6).$$

Proof. The reduced Lie-May spectral sequence of Theorem 3.5 collapses immediately, since $p > 5$ implies that $1 > \lfloor \frac{4}{p-1} \rfloor$. Hence $\text{Cotor}_{E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]})]^*}(\mathbb{F}_p, \mathbb{F}_p) \cong H^{*,*,*}(\mathcal{X}(2, 4))$.

We now run the May spectral sequence

$$(4.16) \quad E_1^{*,*,*} \cong \text{Cotor}_{E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]})]^*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H^*(\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]}); \mathbb{F}_p).$$

See appendix A and the proof of Proposition 4.2 for the general method we use. It will be convenient to use the presentation

$$\mathbb{F}_p[t_{i,j} : i \geq 1, 0 \leq j \leq 1] / (t_{i,j}^p \text{ for all } i, j)$$

for $E^0(\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]})]^*) \cong E^0(\mathbb{F}_p[t_1, t_2, \dots] / (t_i^{p^2} - t_i \text{ for all } i))$, where $t_{i,j}$ is the image of $t_i^{p^j}$ in the associated graded. The coproduct on $\mathbb{F}_p[t_{i,j} : i \geq 1, 0 \leq j \leq 1] / (t_{i,j}^p \text{ for all } i, j)$, inherited from that of $\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]})]^*$, is given by

$$\Delta(t_{i,j}) = \sum_{k=0}^i t_{k,j} \otimes t_{i-k,k+j}$$

for all $i < \lfloor \frac{4p}{p-1} \rfloor$; reduce the $n = 4$ case of Theorem 6.3.2 of [17] modulo the ideal generated by $t_i^{p^2} - t_i$, for all i , to arrive at this formula.

h_{10}, h_{11}, ζ_2 : There are no nonzero May differentials of any length on these classes, by the same computation as in the proof of Proposition 4.2.

$h_{10}h_{30}, h_{11}h_{31}$: The class $h_{10}h_{30}$ is represented by the 2-cocycle

$$(4.17) \quad t_{1,0} \otimes t_{3,0} - t_{1,0} \otimes t_{1,0}t_{2,0} - \frac{1}{2}t_{1,0}^2 \otimes t_{2,0} + \frac{1}{2}t_{1,0}^2 \otimes t_{2,1} - \frac{1}{2}t_{1,0}^2 \otimes t_{1,0}t_{1,1} - \frac{1}{3}t_{1,0}^3 \otimes t_{1,1}$$

in the cobar complex $C^\bullet \left(E^0 \left(\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2}^{\mathbb{Z}_p}[\sqrt{p}])]^* \right) \right)$, which lifts to the 2-cochain

$$t_1 \otimes t_3 - t_1 \otimes t_1 t_2 - \frac{1}{2} t_1^2 \otimes t_2 + \frac{1}{2} t_1^2 \otimes t_2^p - \frac{1}{2} t_1^2 \otimes t_1^{p+1} - \frac{1}{3} t_1^3 \otimes t_1^p$$

in the cobar complex $C^\bullet \left(\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2})]^* \right)$. Since this 2-cochain is also a 2-cocycle, all May differentials vanish on $h_{1,0}h_{3,0}$. The C_2 -equivariance of the spectral sequence then tells us that all May differentials also vanish on $h_{11}h_{31}$. (Also, the anonymous referee made the useful observation that the vanishing of the the May differentials on $h_{10}h_{30}$ and on $h_{11}h_{31}$ can also be deduced by expressing those two classes as Massey products.)

$h_{10}\eta_4, h_{11}\eta_4$: The only elements of internal degree $2(p-1)$ and cohomological degree 3 are scalar multiples of $h_{10}\zeta_2\zeta_4$, but $h_{10}\zeta_2\zeta_4$ is of higher Ravenel degree than $h_{10}\eta_4$. Hence $h_{10}\eta_4$ cannot support a May differential of any length. By C_2 -equivariance, the same is true of $h_{11}\eta_4$.

η_2e_{40} : The only elements of internal degree 0 and cohomological degree 4 are \mathbb{F}_p -linear combinations of $\zeta_2\eta_2e_{40}$, $\zeta_4\eta_2e_{40}$, $\zeta_2\eta_4e_{40}$, and $\zeta_4\eta_4e_{40}$, but all four of these elements have higher Ravenel degree than η_2e_{40} , so again η_2e_{40} cannot support a May differential of any length.

$h_{10}\eta_2h_{30}, h_{11}\eta_2h_{31}, \eta_4e_{40}$: Similar degree considerations eliminate the possibility of nonzero May differentials on these classes.

ζ_4 : The class ζ_4 is represented by the 1-cocycle

$$(4.18) \quad t_{4,0} + t_{4,1} - t_{1,0}t_{3,1} - t_{1,1}t_{3,0} - \frac{1}{2}t_{2,0}^2 - \frac{1}{2}t_{2,1}^2 + t_{1,0}t_{1,1}t_{2,0} + t_{1,0}t_{1,1}t_{2,1} - \frac{1}{2}t_{1,0}^2t_{1,1}^2,$$

in the cobar complex $C^\bullet \left(E^0 \left(\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2}^{\mathbb{Z}_p}[\sqrt{p}])]^* \right) \right)$, which lifts to the 1-cochain

$$t_4 + t_4^p - t_1t_3^p - t_1^p t_3 - \frac{1}{2}t_2^2 - \frac{1}{2}t_2^{2p} + t_1^{p+1}t_2 + t_1^{p+1}t_2^p - \frac{1}{2}t_1^{2p+2},$$

in the cobar complex $C^\bullet \left(\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2})]^* \right)$. Since this 1-cochain is also a 1-cocycle, all May differentials vanish on ζ_4 . (The anonymous referee observed that it is perhaps also possible to deduce the vanishing of the May differential on ζ_4 by finding an expression of ζ_4 in terms of a norm map, along the lines of Theorem 6.2.7 and Proposition 6.3.13 of [17], obviating the need to use a cocycle representative for ζ_4 ; we do not pursue that alternate proof here.)

Now suppose that $q \geq 1$ is some integer and that we have already shown that d_r vanishes on all classes, for all $r < q$. Then $d_r(\eta_2e_{40} \cdot \eta_4e_{40}) = 0$, i.e., d_r vanishes on the duality class in the algebra $\mathcal{A}_{2,4}$. For each element in that algebra, we have shown that d_r vanishes on either that element, or on its Poincaré dual. Since d_r also vanishes on the duality class, d_r vanishes on all elements in that algebra. Since d_r also vanishes on ζ_2 and ζ_4 , d_r vanishes on all classes. By induction, the spectral sequence collapses with no nonzero differentials. So

$$H^*(\text{strAut}({}_1\mathbb{G}_{1/2}^{\mathbb{Z}_p}[\sqrt{p}]); \mathbb{F}_p) \cong \text{Cotor}_{E^0\mathbb{F}_p[\text{strAut}({}_1\mathbb{G}_{1/2}^{\mathbb{Z}_p}[\sqrt{p}])]^*}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p) \cong H^{*,*,*}(\mathcal{X}(2, 4)) \text{ as a graded } \mathbb{F}_p\text{-vector space.} \quad \square$$

5. TOPOLOGICAL CONSEQUENCES.

It is well-known, e.g. from the Barsotti-Tate module generalization of the Dieudonné-Manin classification of p -divisible groups over \bar{k} (see [8]; also see [15] for a nice treatment of the theory of Barsotti-Tate modules), that the automorphism group scheme of a formal

A -module of positive, finite height over a finite field is pro-étale; in more down-to-earth terms, the Hopf algebra corepresenting the group scheme $\text{Aut}(\omega\mathbb{G}_{1/n}^A \otimes_k \bar{k})$ is the continuous \bar{k} -linear dual of the \bar{k} -linear group ring of some profinite group, namely, the automorphism group (honestly a group, not just a group scheme!) of $\omega\mathbb{G}_{1/n}^A \otimes_k \bar{k}$. In this section we will cease to work with group schemes and we will simply write $\text{Aut}(\omega\mathbb{G}_{1/n}^A \otimes_k \bar{k})$ for that profinite group.

The following is a generalization of a result in [20], and the argument is almost word-for-word the same:

Theorem 5.1. *Let K/\mathbb{Q}_p be a field extension of degree d . Let A denote the ring of integers of K , and let π denote a uniformizer for A and k the residue field of A . Let q be the cardinality of k , and let ω denote a $\frac{q^n-1}{q^n-1}$ th root of $\frac{\pi}{p}$ in \bar{k} . Then $\text{Aut}(\omega\mathbb{G}_{1/n}^A \otimes_k \bar{k})$ is a closed subgroup of the height dn Morava stabilizer group $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$.*

Proof. By Theorem 2.2, the underlying formal $\hat{\mathbb{Z}}_p$ -module of $\omega\mathbb{G}_{1/n}^A$ is ${}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p}$. Hence the automorphisms of $\omega\mathbb{G}_{1/n}^A \otimes_k \bar{k}$ are the automorphisms of ${}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k}$ which commute with the complex multiplication by A , and hence $\text{Aut}(\omega\mathbb{G}_{1/n}^A \otimes_k \bar{k})$ is a subgroup of $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$.

Now let G_a denote the automorphism group of the underlying formal $\hat{\mathbb{Z}}_p$ -module a -bud of ${}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k}$, so that $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$ is, as a profinite group, the limit of the sequence of finite groups $\dots \rightarrow G_3 \rightarrow G_2 \rightarrow G_1$. Let H_a denote the subgroup of $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$ consisting of those automorphisms whose underlying formal $\hat{\mathbb{Z}}_p$ -module a -bud automorphism commutes with the complex multiplication by A , i.e., those whose underlying formal $\hat{\mathbb{Z}}_p$ -module a -bud automorphism is an automorphism of the underlying formal A -module a -bud of $\omega\mathbb{G}_{1/n}^A$. The index of H_a in $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$ is at most the cardinality of G_a , hence is finite. Now we use the theorem of Nikolov-Segal, from [14]: every finite-index subgroup of a topologically finitely generated profinite group is an open subgroup. The group $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$ is topologically finitely generated since

- its pro- p -subgroup $\text{strAut}({}_1\hat{\mathbb{G}}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is a p -adic analytic Lie group, hence topologically finitely generated (see [7], or Theorem 5.11 of [6] for an English-language summary of the relevant result, and
- $\text{Aut}({}_1\hat{\mathbb{G}}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is a split extension of the finite group $\mathbb{F}_{p^n}^\times$ by the topologically finitely generated group $\text{strAut}({}_1\hat{\mathbb{G}}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$, hence $\text{Aut}({}_1\hat{\mathbb{G}}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is topologically finitely generated.

So H_a is an open subgroup of $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$. Every open subgroup of a profinite group is also closed; consequently each H_a is a closed subgroup of $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$, and consequently the intersection $\cap_a H_a$ is a closed subgroup of $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$. But $\cap_a H_a$ is the group of all formal power series which are automorphisms of ${}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k}$ and whose polynomial truncations, of any length, commute with the complex multiplication by A . Consequently $\cap_a H_a = \text{Aut}(\omega\mathbb{G}_{1/n}^A \otimes_k \bar{k})$ is a closed subgroup of $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$. \square

Since $\text{Aut}(\omega\mathbb{G}_{1/n}^A \otimes_k \bar{k})$ is a closed subgroup of the height dn Morava stabilizer group $\text{Aut}({}_1\hat{\mathbb{G}}_{1/dn}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \bar{k})$, we can use the methods of [4] to construct and compute the homotopy

fixed-point spectra

$$E_4^{h \text{Aut}(\omega \mathbb{G}_{1/n}^A \otimes_{\mathbb{k}} \bar{k})} \quad \text{and} \quad E_4^{h \text{Aut}(\omega \mathbb{G}_{1/n}^A \otimes_{\mathbb{k}} \bar{k}) \rtimes \text{Gal}(\bar{k}/k)}.$$

The homotopy fixed-point spectrum $E_4^{h \text{Aut}(\omega \mathbb{G}_{1/dn}^{\mathbb{Z}_p} \otimes_{\mathbb{F}_p} \bar{k}) \rtimes \text{Gal}(\bar{k}/\mathbb{F}_p)}$ $\simeq L_{K(4)}S$ has a natural map to $E_4^{h \text{Aut}(\omega \mathbb{G}_{1/n}^A \otimes_{\mathbb{k}} \bar{k}) \rtimes \text{Gal}(\bar{k}/\mathbb{F}_p)}$, but this map is far from being an equivalence; still, very few computations of homotopy groups of $K(4)$ -local spectra exist in the literature, so Theorem 5.2 is perhaps of some interest.

Theorem 5.2. *Let p be a prime number such that the Smith-Toda complex $V(3)$ exists, i.e., $p > 5$. Then the $V(3)$ -homotopy groups of $E_4^{h \text{Aut}(\omega \mathbb{G}_{1/2}^{\mathbb{Z}_p} \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p) \rtimes \text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)}$ are isomorphic to*

$$\mathcal{A}_{2,4} \otimes_{\mathbb{F}_p} \Lambda(\zeta_2, \zeta_4) \otimes_{\mathbb{F}_p} \mathbb{F}_p[v^{\pm 1}],$$

where $v^{p^2+1} = v_4$, where $\mathcal{A}_{2,4}$ is as in Proposition 4.4, and the topological degrees and E_4 -Adams filtrations are as follows:

Htpy. class	Top. degree	E_4 – Adams filt.
1	0	0
h_{10}	$2p - 3$	1
h_{11}	$2p^2 - 2p - 1$	1
$h_{10}h_{30}$	$4p - 6$	2
$h_{11}h_{31}$	$4p^2 - 4p - 2$	2
$h_{10}\eta_4 - \eta_2h_{30}$	$2p - 4$	2
$h_{11}\eta_4 - \eta_2h_{31}$	$2p^2 - 2p - 2$	2
η_2e_{40}	-3	3
$h_{10}\eta_2h_{30}$	$4p - 7$	3
$h_{11}\eta_2h_{31}$	$4p^2 - 4p - 3$	3
$h_{10}h_{30}\eta_4$	$4p - 7$	3
$h_{11}h_{31}\eta_4$	$4p^2 - 4p - 3$	3
$\eta_4e_{40} + 4\eta_2h_{30}h_{31}$	-3	3
$h_{10}\eta_2h_{30}h_{31}$	$2p - 6$	4
$h_{11}\eta_2h_{30}h_{31}$	$2p^2 - 2p - 4$	4
$h_{10}\eta_2h_{30}\eta_4$	$4p - 8$	4
$h_{11}\eta_2h_{31}\eta_4$	$4p^2 - 4p - 4$	4
$h_{10}\eta_2h_{30}h_{31}\eta_4$	$2p - 7$	5
$h_{11}\eta_2h_{30}h_{31}\eta_4$	$2p^2 - 2p - 5$	5
$h_{10}h_{11}\eta_2h_{30}h_{31}\eta_4$	-6	6
ζ_2	-1	1
ζ_4	-1	1
v	$2p^2 - 2$	0

Proof. See [3] and [4] for the equivalence

$$L_{K(n)}S \simeq E_n^{h \text{Aut}(\omega \mathbb{G}_{1/n}^{\mathbb{Z}_p} \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p) \rtimes \text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)}.$$

Since $V(3)$ is $E(3)$ -acyclic, $L_{K(4)}V(3)$ is weakly equivalent to $L_{E(4)}V(3)$, so $L_{K(4)}V(3) \simeq L_{E(4)}V(3) \simeq V(3) \wedge L_{E(4)}S$ since $E(4)$ -localization is smashing; see [18] for the proof of Ravenel's smashing conjecture. Since $V(3)$ is finite, $(E_4 \wedge V(3))^{hG} \simeq E_4^{hG} \wedge V(3)$, and now

we use the $X = V(3)$ case of the conditionally convergent descent spectral sequence (see e.g. 4.6 of [1], or [4])

$$E_2^{s,t} \cong H_c^s(G; (E_n)_t(X)) \Rightarrow \pi_{t-s}((E_n \wedge X)^{hG})$$

$$d_r : E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}.$$

The agreement of this spectral sequence with the $K(4)$ -local E_4 -Adams spectral sequence is given by Proposition 6.6 of [4].

In the case $n = 4$ and $X = V(3)$, we have $(E_4)_* \cong W(\overline{\mathbb{F}}_p)[[u_1, u_2, u_3]][u^{\pm 1}]$ with v_i acting by $u_i u^{1-p^i}$ for $i = 1, 2, 3$, and consequently $(E_4)_*(V(3)) \cong \overline{\mathbb{F}}_p[u^{\pm 1}]$. One needs to know the action of $\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \rtimes \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on $\overline{\mathbb{F}}_p[u^{\pm 1}]$ to compute the E_2 -term of the spectral sequence; but $\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$ has the finite-index pro- p -subgroup $\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$. We will use the fact that a pro- p -group admits no nontrivial continuous action on a one-dimensional $\overline{\mathbb{F}}_p$ -vector space; this is not a new observation, but a proof goes as follows. We have $\overline{\mathbb{F}}_p^\times = \cup_j (\mathbb{F}_{p^j}^\times)$, so $GL_1(\overline{\mathbb{F}}_p)$ contains no elements of order p , but if G is a pro- p -group acting continuously on $\overline{\mathbb{F}}_p$, then since $\overline{\mathbb{F}}_p$ is discrete, the kernel of the action map $G \rightarrow GL_1(\overline{\mathbb{F}}_p)$ must be open, i.e., closed and finite-index. So the action of G on $\overline{\mathbb{F}}_p$ must factor through a finite quotient of G , i.e., a finite p -group which embeds in a group $\overline{\mathbb{F}}_p$ with no elements of order p ; so the action must be trivial.

So $\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$ acts trivially on $\overline{\mathbb{F}}_p[u^{\pm 1}]$, and we only need to know the action of $\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)/\text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \cong \mathbb{F}_p^\times$ on $\overline{\mathbb{F}}_p[u^{\pm 1}]$; from section 1 of [3] we get that an element $x \in \mathbb{F}_p^\times$ acts on $\overline{\mathbb{F}}_p\{u^j\}$ by multiplication by x^j . Consequently the (collapsing at E_2) Lyndon-Hochschild-Serre spectral sequence of the extension

$$1 \rightarrow \text{strAut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \rightarrow \text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \rightarrow \mathbb{F}_p^\times \rightarrow 1$$

gives us that $H_c^*(\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p); V(3)_t(E_4^{h\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)}))$ vanishes if t is not divisible by $2(p^2 - 1)$, and is given by Proposition 4.5 if t is divisible by $2(p^2 - 1)$. So there is a horizontal vanishing line of finite height already at the E_2 -page of the spectral sequence, hence the spectral sequence converges strongly.

More specifically, the cohomology computed in Proposition 4.5 is a $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ -form of

$$H_c^*(\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p); V(3)_{2(p^2-1)j}(E_4^{h\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)})),$$

and

$$H_c^*(\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p); V(3)_{2(p^2-1)j}(E_4^{h\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)})) \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

is also a $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ -form of

$$H_c^*(\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p); V(3)_{2(p^2-1)j}(E_4^{h\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)})).$$

Since the nonabelian Galois cohomology group $H^1(\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p); GL_n(\overline{\mathbb{F}}_p))$ classifying $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ -forms of n -dimensional $\overline{\mathbb{F}}_p$ -vector spaces vanishes (this is a well-known generalization of Hilbert's Theorem 90), the invariants of the $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ -action on

$$H_c^*(\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p); V(3)_{2(p^2-1)j}(E_4^{h\text{Aut}({}_1\mathbb{G}_{1/2}^{\hat{Z}_p[\sqrt{p}]} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)}))$$

agree, up to isomorphism of graded \mathbb{F}_p -vector spaces, with the results of Proposition 4.5 (this Galois descent argument was suggested to me by T. Lawson). There is no room for differentials in the descent spectral sequence, so $E_2 \cong E_\infty$ in the spectral sequence. \square

Corollary 5.3. *The Poincaré series for the homotopy groups of $V(3)$ smashed with the*

fixed-point spectrum $E_4^{h\text{Aut}(\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]}) \rtimes \text{Gal}(\bar{k}/\mathbb{F}_p)}$ is:

$$\begin{aligned} & (s^{-6} + 2s^{-3} + 1 + s^{2p-7} + s^{2p-6} + s^{2p-4} + s^{2p-3} + s^{4p-8} + 2s^{4p-7} + s^{4p-6} + s^{2p^2-2p-5} \\ & + s^{2p^2-2p-4} + s^{2p^2-2p-1} + s^{4p^2-4p-4} + 2s^{4p^2-4p-3} + s^{4p^2-4p-2}) (1 + s^{-1})^2 \sum_{n=-\infty}^{\infty} s^{2(p^2-2)n}. \end{aligned}$$

APPENDIX A. EXPLICIT MILNOR-MOORE THEORY.

Throughout this appendix, we assume \mathfrak{g} is a Lie algebra over a field k of characteristic not equal to 2, and that either \mathfrak{g} is finite-dimensional or that it is equipped with an \mathbb{N} -grading such that the Lie bracket is of degree 0 (i.e., if $x \in \mathfrak{g}^m$ and $y \in \mathfrak{g}^n$, then $[x, y] \in \mathfrak{g}^{m+n}$) and such that \mathfrak{g}^n is a finite-dimensional k -vector-space for each $n \in \mathbb{N}$.

Several times in this paper (e.g. (4.17) and (4.18)), we have given cocycle representatives, in the cobar complex for the linear dual of the universal enveloping algebra $U\mathfrak{g}$ of \mathfrak{g} , for elements in the cohomology of \mathfrak{g} . In this appendix we explain how these cocycle representatives are obtained. The author of this paper is doubtful that the mathematics in this appendix is actually new, but a search for this material in the literature did not turn up anything, so even if these ideas are folklore, we think it is of some value to write the ideas out in this appendix.

One has an equivalence between the category of $U\mathfrak{g}$ -modules and the category of (unrestricted) representations of \mathfrak{g} , and as a consequence one gets an isomorphism

$$(A.19) \quad \text{Ext}_{U\mathfrak{g}}^*(k, k) \cong H_{\text{unr}}^*(\mathfrak{g}, k).$$

However, if one wants to run a May spectral sequence, like (4.16), whose input is $\text{Ext}_{U\mathfrak{g}}^*(k, k)$, then to compute May d_r -differentials for $r > 1$, one typically needs to know how to represent elements in the cohomology of \mathfrak{g} as cocycles in the cobar complex of $(U\mathfrak{g})^*$. (Let us be careful about the relationship between unrestricted Lie algebra cohomology and the input of a May spectral sequence like (4.16): the input for (4.16) is, a priori, the cohomology of the *restricted* Lie algebra of primitives in the Hopf algebra $E^0\mathbb{F}_p[\text{strAut}(\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]})]^*$. However, when $p > 5$, the immediate collapse of the reduced Lie-May spectral sequence of Theorem 3.5 implies that the restricted Lie algebra cohomology of the primitives in $E^0\mathbb{F}_p[\text{strAut}(\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]})]^*$ agrees with the *unrestricted* Lie algebra cohomology of the Lie subalgebra $L^{\hat{\mathbb{Z}}_p[\sqrt{p}]}(2, 4)$ of the primitives in $E^0\mathbb{F}_p[\text{strAut}(\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p[\sqrt{p}]})]^*$. Of course this $\hat{\mathbb{Z}}_p[\sqrt{p}]$ -height 2 statement generalizes to other heights and endomorphism rings other than $\hat{\mathbb{Z}}_p[\sqrt{p}]$, at appropriately large primes, as given by Theorem 3.5.)

So, after one has used (convenient, algebraically tractable) Lie algebra cohomology methods to calculate $H_{\text{unr}}^*(\mathfrak{g}, k)$, one needs to translate the names of elements in $H_{\text{unr}}^*(\mathfrak{g}, k)$ that one has from Lie-algebra-theoretic methods—e.g., from the Chevalley-Eilenberg complex of \mathfrak{g} —into cocycle representatives in the cobar complex of $(U\mathfrak{g})^*$. Here is how one can do this: let $C^\bullet(U\mathfrak{g}^*)$ denote the cobar complex (as in Definition A1.2.11 of [17]) of the k -linear dual Hopf algebra of $U\mathfrak{g}$, and let $CE^\bullet(\mathfrak{g})$ denote the Chevalley-Eilenberg complex of \mathfrak{g} , in the sense of [2]: that is, $CE^n(\mathfrak{g}) = \Lambda^n(\mathfrak{g}^*)$ with differential given on 1-cochains by

the dual of the map

$$\Lambda^2(\mathfrak{g}^*)^* \hookrightarrow \mathfrak{g} \otimes_k \mathfrak{g} \xrightarrow{[-,-]} \mathfrak{g},$$

and with the differential given on higher cochains by the Leibniz rule and the fact that every higher cochain in $CE^\bullet(\mathfrak{g})$ is a linear combination of products of 1-cochains.

We have a natural inclusion of \mathfrak{g} as a vector subspace of $U\mathfrak{g}$; let $j : \mathfrak{g} \rightarrow U\mathfrak{g}$ be 2 times this natural inclusion map. (The factor of 2 will be necessary to get diagram (A.20) to commute, below; remember that at the start of this appendix, we stipulated that the characteristic of the ground field k is not 2.) Dualizing, we get a map $(U\mathfrak{g})^* \rightarrow \mathfrak{g}^*$, and since the underlying k -algebra of the cobar complex $C^\bullet((U\mathfrak{g})^*)$ is the free associative k -algebra on $C^1((U\mathfrak{g})^*) = (U\mathfrak{g})^*$, the universal property of the free associative k -algebra gives us a canonical map of graded k -algebras $\rho : C^\bullet((U\mathfrak{g})^*) \rightarrow CE^\bullet(\mathfrak{g})$.

We claim that ρ is not only a map of graded k -algebras, but is also a chain map, i.e., ρ is a morphism of DGAs over k . One checks easily that the diagram

$$(A.20) \quad \begin{array}{ccccc} U\mathfrak{g} & \xleftarrow{\nabla} & U\mathfrak{g} \otimes_k U\mathfrak{g} & & \\ \uparrow j & & \swarrow (j \otimes j) \circ i & & \\ \mathfrak{g} & \xleftarrow{\text{bracket}} & \mathfrak{g} \otimes_k \mathfrak{g} & \xleftarrow{\quad} & \Lambda^2(\mathfrak{g}^*)^* \end{array}$$

commutes, where ∇ is the multiplication on $U\mathfrak{g}$, and i is the inclusion of $\Lambda^2(\mathfrak{g}^*)^*$ into $\mathfrak{g} \otimes_k \mathfrak{g}$ as the vector space of antisymmetric tensors. (But it is worth noticing that, if one tries to include $j \otimes j$ as a vertical arrow in (A.20), the resulting diagram *wouldn't* commute: in particular the left-hand square would fail to commute.) Taking the linear dual of diagram (A.20) yields the commutative square

$$\begin{array}{ccc} (U\mathfrak{g})^* & \longrightarrow & (U\mathfrak{g})^* \otimes_k (U\mathfrak{g})^* \\ \downarrow & & \downarrow \\ \mathfrak{g}^* & \longrightarrow & \Lambda^2(\mathfrak{g}^*), \end{array}$$

in which the vertical maps are given by ρ , the top horizontal map is the differential $C^1((U\mathfrak{g})^*) \rightarrow C^2((U\mathfrak{g})^*)$ in the cobar complex, and the bottom horizontal map $CE^1(\mathfrak{g}) = \mathfrak{g}^* \rightarrow \Lambda^2(\mathfrak{g}^*) = CE^2(\mathfrak{g})$ is the differential in the Chevalley-Eilenberg complex of \mathfrak{g} . So ρ commutes with the differentials on the 1-cochains. An analogous argument for higher cochains gives us that ρ is a chain map.

Clearly ρ is surjective (since j is surjective, and so $j^{\otimes n}$ is surjective for all n), so if we write I for the kernel of ρ , then I is a differential graded ideal of $C^\bullet((U\mathfrak{g})^*)$ with the property that the DGA $C^\bullet((U\mathfrak{g})^*)/I$ is isomorphic to $CE^\bullet(\mathfrak{g})$. With all that said, the process for representing elements in $H_{\text{unr}}^*(\mathfrak{g}, k)$ by cocycles in $C^\bullet((U\mathfrak{g})^*)$ is very simple: given an element x in $H_{\text{unr}}^n(\mathfrak{g}, k)$, one represents x by an n -cocycle in $CE^\bullet(\mathfrak{g}) \cong C^\bullet((U\mathfrak{g})^*)/I$, lifts that n -cocycle to an n -cochain in $C^\bullet((U\mathfrak{g})^*)$, and then adds ‘‘correcting terms’’ in I to get an n -cocycle in $C^\bullet((U\mathfrak{g})^*)$.

We demonstrate this process ‘‘in action’’ to show how the cocycle representative (4.17) for $h_{10}h_{30} \in H_{\text{unr}}^2(L^{\hat{z}_p[\sqrt{p}]}(2, 4))$ was obtained: the notation $h_{10}h_{30}$ denotes the element of $H_{\text{unr}}^2(L^{\hat{z}_p[\sqrt{p}]}(2, 4))$ which arose as the product of $h_{10} \in H_{\text{unr}}^1(L^{\hat{z}_p[\sqrt{p}]}(2, 1))$ and $h_{30} \in H_{\text{unr}}^1(L^{\hat{z}_p[\sqrt{p}]}(2, 3)/L^{\hat{z}_p[\sqrt{p}]}(2, 2))$, and which survived the Cartan-Eilenberg spectral sequences of Propositions 4.3 and 4.4. Since h_{10} is represented in the Chevalley-Eilenberg DGA (and the cobar complex) of $L^{\hat{z}_p[\sqrt{p}]}(2, 1)$ by the 1-cocycle $t_{1,0}$, and since h_{30} is represented in the Chevalley-Eilenberg DGA (and the cobar complex) of $L^{\hat{z}_p[\sqrt{p}]}(2, 3)/L^{\hat{z}_p[\sqrt{p}]}(2, 2)$ by the

1-cocycle $t_{3,0}$, $h_{10}h_{30}$ is represented in $CE^2(L^{\hat{\mathbb{Z}}_p[\sqrt{p}]}(2, 4)) \cong \Lambda^2(L^{\hat{\mathbb{Z}}_p[\sqrt{p}]}(2, 4)^*)$ by $t_{1,0}t_{3,0}$, which we see is a 2-cocycle in $CE^\bullet(L^{\hat{\mathbb{Z}}_p[\sqrt{p}]}(2, 4))$:

$$d_{CE}(t_{1,0}t_{3,0}) = -t_{1,0}(t_{1,0}t_{2,1} + t_{2,0}t_{1,0}) = 0.$$

The element $t_{1,0}t_{3,0}$ lifts to the 2-cochain $t_{1,0} \otimes t_{3,0}$ in $C^\bullet((UL^{\hat{\mathbb{Z}}_p[\sqrt{p}]}(2, 4))^*)$, but there it is not a 2-cocycle:

$$\begin{aligned} d_{cobar}(t_{1,0} \otimes t_{3,0}) &= -t_{1,0} \otimes (t_{1,0} \otimes t_{2,1} + t_{2,0} \otimes t_{1,0}) \\ &= d_{cobar} \left(t_{1,0} \otimes t_{1,0}t_{2,0} + \frac{1}{2}t_{1,0}^2 \otimes t_{2,0} - \frac{1}{2}t_{1,0}^2 \otimes t_{2,1} + \frac{1}{2}t_{1,0}^2 \otimes t_{1,0}t_{1,1} + \frac{1}{3}t_{1,0}^3 \otimes t_{1,1} \right). \end{aligned}$$

Since $t_{1,0} \otimes t_{1,0}t_{2,0} + \frac{1}{2}t_{1,0}^2 \otimes t_{2,0} - \frac{1}{2}t_{1,0}^2 \otimes t_{2,1} + \frac{1}{2}t_{1,0}^2 \otimes t_{1,0}t_{1,1} + \frac{1}{3}t_{1,0}^3 \otimes t_{1,1} \in I$, we have our cocycle representative (4.17) for $h_{10}h_{30}$ in the cobar complex.

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