

# RAVENEL'S GLOBAL CONJECTURE IS TRUE.

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ABSTRACT. I prove Ravenel's 1983 "Global Conjecture" on  $\text{Ext}^1$  over the classifying Hopf algebroid of formal  $A$ -modules, equivalently, the first flat cohomology group  $H_{fl}^1$  of the moduli stack  $\mathcal{M}_{flA}$  of formal  $A$ -modules. I then show that the Hecke  $L$ -functions of certain Größencharaktere of Galois extensions  $K/\mathbb{Q}$  can be extracted from  $H_{fl}^1(\mathcal{M}_{flA})$ , and vice versa; as a consequence I show that, for a large class of Galois extensions of  $\mathbb{Q}$ , two extensions  $K, L$  are arithmetically equivalent (i.e., they have the same Dedekind zeta-function) if and only if the flat cohomology groups  $H_{fl}^1(\mathcal{M}_{flOK})$  and  $H_{fl}^1(\mathcal{M}_{flOL})$  agree.

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## 1. INTRODUCTION.

In this paper<sup>1</sup> I prove the "Global Conjecture" from Ravenel's 1983 paper [20]. This conjecture is about the cohomology groups of the moduli stack of one-dimensional formal  $A$ -modules, or equivalently,  $\text{Ext}$  over the classifying Hopf algebroid  $(L^A, L^A B)$  of formal  $A$ -modules; I will now explain a bit about what this means.

A "formal  $A$ -module" is a formal group law  $F$  over an  $A$ -algebra  $R$  which is equipped with a ring homomorphism  $\rho : A \rightarrow \text{End}(F)$  such that the endomorphism  $\rho(a) \in \text{End}(F) \subseteq R[[X]]$  is congruent to  $aX$  modulo  $X^2$ . Morally,  $F$  is a "formal group law with complex multiplication by  $A$ ." An excellent introductory reference for formal  $A$ -modules is chapter 21 of [10]. Formal  $A$ -modules arise in algebraic geometry, for example, in Lubin and Tate's

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<sup>1</sup>This paper is the sixth in a series of papers on formal groups with complex multiplication and their applications in homotopy theory, but it can be read without reference to the other papers in the series.

famous theorem (in [14]) on the abelian closure of a  $p$ -adic number field, and for another example, in Drinfeld’s  $p$ -adic symmetric domains, which are (rigid analytic) deformation spaces of certain formal modules; see [7] and [19]. Formal  $A$ -modules also arise in algebraic topology, by using the natural map from the moduli stack of formal  $A$ -modules to the moduli stack of formal group laws to detect certain classes in the cohomology of the latter, particularly in order to resolve certain differentials in spectral sequences used to compute the Adams-Novikov  $E_2$ -term and stable homotopy groups of spheres; in a paper currently in preparation, for example, I use formal modules to compute the homotopy groups of the  $K(4)$ -local Smith-Toda complex  $V(3)$  at primes  $p > 5$ .

Now I want to give a little bit of background on the moduli theory of formal modules before I describe the new results. Recall (from e.g. chapter 21 of [10] or [20]) that, for every finite extension  $K/\mathbb{Q}$  with ring of integers  $A$ , there exists a classifying Hopf algebroid  $(L^A, L^A B)$  for one-dimensional formal  $A$ -modules, equivalently, a moduli stack  $\mathcal{M}_{fmA}$  of one-dimensional formal  $A$ -modules<sup>2</sup>. In the special case  $K = \mathbb{Q}$ , we have an isomorphism of Hopf algebroids  $(L^{\mathbb{Z}}, L^{\mathbb{Z}} B) \cong (L, LB)$ , where  $(L, LB)$  is the classifying Hopf algebroid for one-dimensional formal group laws studied by Lazard (see [13]) and which is very familiar to topologists due to Quillen proving (see [18]) that it is isomorphic to the Hopf algebroid  $(MU_*, MU_* MU)$  of stable co-operations in complex cobordism, and consequently that the bigraded Ext-algebra  $\text{Ext}_{(L^{\mathbb{Z}}, L^{\mathbb{Z}} B)}^{s,t}(L^{\mathbb{Z}}, L^{\mathbb{Z}})$ , i.e., the flat cohomology  $H_{fl}^{s,t}(\mathcal{M}_{fm\mathbb{Z}}; O)$ , is the  $E_2$ -term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres.

There is also a  $p$ -adic version of the above: for every finite extension  $K/\mathbb{Q}_p$  with ring of integers  $A$ , there exists a classifying Hopf algebroid  $(L^A, L^A B)$  for one-dimensional formal  $A$ -modules, and a classifying Hopf algebroid  $(V^A, V^A T)$  for  $A$ -typical one-dimensional formal  $A$ -modules, and the two Hopf algebroids are equivalent (but not isomorphic) by an  $A$ -module version of Cartier’s “ $p$ -typicalization” operation on formal group laws. See e.g. [20] for this material. Again, the base case  $K = \mathbb{Q}_p$  is very familiar to topologists, since  $(V^{\mathbb{Z}_p}, V^{\mathbb{Z}_p} T)$  is isomorphic to  $(\hat{V}_p, \hat{V}_p T)$ , the  $p$ -adic completion of the classifying Hopf algebroid  $(V, VT)$  of  $p$ -typical one-dimensional formal group laws over  $\mathbb{Z}_{(p)}$ , which Quillen proved in [18] to be isomorphic to the Hopf algebroid  $(BP_*, BP_* BP)$  of stable co-operations in Brown-Peterson homology.

Now the following conjectures were made in Ravenel’s 1983 paper [20]:

**Conjecture 1.0.1. (Ravenel’s Local-Global Conjecture.)** *Let  $K/\mathbb{Q}$  be a finite field extension with ring of integers  $A$ . Then, for all  $s, t$  and all prime ideals  $\mathfrak{p}$  in  $A$  and all graded  $(L^A, L^A B)$ -comodules  $M$ , there exists an isomorphism of  $A_{\mathfrak{p}}$ -modules*

$$A_{\mathfrak{p}} \otimes_A \text{Ext}_{(L^A, L^A B)}^{s,t}(L^A, M) \cong \text{Ext}_{(V^{A_{\mathfrak{p}}}, V^{A_{\mathfrak{p}}} T)}^{s,t}(V^{A_{\mathfrak{p}}}, V^{A_{\mathfrak{p}}} \otimes_{L^{A_{\mathfrak{p}}}} M),$$

where  $A_{\mathfrak{p}}$  is the localization of  $A$  at the prime ideal  $\mathfrak{p}$ .

This statement of Conjecture 1.0.1 is slightly paraphrased from Ravenel’s original statement in [20], which has a small error (one of the terms  $L^A$  and one of the terms  $V^{A_{\mathfrak{p}}}$  are replaced by  $A$  and  $A_{\mathfrak{p}}$ , respectively, in Ravenel’s statement of the conjecture; in context it is clear that this is a typo).

<sup>2</sup>This is really only a prestack, in the sense of [12], unless one adopts a “coordinate-free” definition of formal modules. The moduli prestack of formal  $A$ -modules “stackifies” to a moduli stack of coordinate-free formal  $A$ -modules, as in [24], and the quasicohherent flat cohomology groups in either case agree with the Ext-groups of the Hopf algebroid  $(L^A, L^A B)$ ; see Conventions 2.0.10.

I will quote Conjecture 1.0.2 verbatim from [20], because it is loosely stated, and reasonably so: one of the tasks involved in proving Conjecture 1.0.2 is to find a way to make its statement precise in such a way that the conjecture is actually true.

**Conjecture 1.0.2. (Ravenel's Global Conjecture.)** “For global  $A$ ,  $\text{Ext}_A^{1,2m} = A/J_m^A$ , where  $J_m^A$  is, up to some small factor, the ideal generated by  $a^N(a^m - 1)$  for  $a \in A$  and  $N$  sufficiently large.”

Since the Ext groups  $\text{Ext}_{(L^A, L^A B)}^{1,i}(L^A, L^A)$  are easily shown to be trivial if  $i$  is odd, Conjecture 1.0.2 is a complete description of  $\text{Ext}_{(L^A, L^A B)}^1(L^A, L^A)$ .

**Conjecture 1.0.3. (Ravenel's Local Conjecture.)** Let  $K/\mathbb{Q}_p$  be a finite field extension with ring of integers  $A$ . Suppose  $A$  has uniformizer  $\pi$  and residue field  $\mathbb{F}_q$ . Then we have an isomorphism of  $A$ -modules

$$\text{Ext}_{(V^A, V^A T)}^{1,2n(q-1)}(V^A, V^A) \cong A/I_n,$$

where  $I_n$  is the ideal in  $A$  generated by all elements of the form  $a^n - 1$  with  $a \in A$  congruent to 1 modulo  $\pi$ .

Since the Ext groups  $\text{Ext}_{(V^A, V^A T)}^{1,m}(V^A, V^A)$  are easily shown to be trivial if  $2(q - 1)$  does not divide  $m$ , Conjecture 1.0.3 is a complete description of  $\text{Ext}_{(V^A, V^A T)}^1(V^A, V^A)$ .

There is a fourth remark in Ravenel's paper [20] which is not phrased as a conjecture (indeed, Ravenel includes it as a reason for not making a fourth conjecture!) but which I regard as being of equal importance as Conjectures 1.0.1, 1.0.2, and 1.0.3, and addressing (and solving) the problem posed by Ravenel's remark is one of the main topics of the present paper. I include this remark verbatim from Ravenel's paper [20]:

**Remark 1.0.4. (Ravenel's remark on the connection to the Dedekind  $\zeta$ -function.)** “The numbers  $j_m$  of 3.8 are also related to Bernoulli numbers and the values of the Riemann zeta function at negative integers, but these properties do not appear to generalize to other number fields. For example if the field is not totally real its Dedekind zeta function vanishes at all negative integers.”

The “numbers  $j_m$ ” refers to the order of the group  $\text{Ext}_{(L^{\mathbb{Z}}, L^{\mathbb{Z}} B)}^{1,2m}(L^{\mathbb{Z}}, L^{\mathbb{Z}})$ , which Adams proved, in [1] and [2], is equal to the denominator of the rational number  $\zeta(1 - 2m)$ , up to multiplication by a power of 2, where  $\zeta$  is the Riemann  $\zeta$ -function; since the Riemann  $\zeta$ -function is the special case  $K = \mathbb{Q}$  of the Dedekind  $\zeta$ -function of a number field  $K$ , Ravenel's Remark 1.0.4 appears to exclude the possibility of a generalization of Adams's result to other number fields.

The state of all of these conjectures, as well as the progress made toward their resolution in the present paper, is as follows:

- Conjecture 1.0.1, the Local-Global Conjecture, was known classically in the  $K = \mathbb{Q}$  case at the time that Ravenel made the conjecture. The conjecture was proven in full generality (using Cartier typicalization and a Hopf algebroid change-of-rings argument) by A. Pearlman in his (unpublished) thesis, [16], shortly after Ravenel made the conjecture (indeed, Pearlman's proof came so soon after Ravenel made the conjecture that Ravenel notes, in the published version of [20], that the conjecture had already been proven by Pearlman!).
- Conjecture 1.0.3, the Local Conjecture, was classically known only in the case  $K = \mathbb{Q}_p$  when Ravenel made the conjecture. When Ravenel posed the conjecture in [20], he also offered a proof of the conjecture for all extensions  $K/\mathbb{Q}_p$  with

ramification degree  $e$  satisfying  $e < p - 1$ . (There is a small flaw in the argument Ravel presents in that paper: he relies on a formula for  $p$ -adic valuations of binomial coefficients,  $v_p\left(\binom{j^p}{k}\right) = i - v_p(k)$  for  $j$  prime to  $p$ , a formula which is often not true unless  $j = 1$ . For example, let  $j = p + 1$ , let  $i = 1$ , and let  $k = p$  for a counterexample. An argument similar to Ravel's does prove the Local Conjecture in those cases, however.)

In K. Johnson's 1987 paper [11], Johnson proves the Local Conjecture for all extensions  $K/\mathbb{Q}_p$  which are not a totally ramified extension of  $\mathbb{Q}(\zeta_p)$  of degree a power of  $p$ , where  $\zeta_p$  is a primitive  $p$ th root of unity. In the present paper (see Corollary 3.0.17) I prove Conjecture 1.0.3 for all extensions  $K/\mathbb{Q}_p$  such that  $\log_p\left(\frac{e}{p-1}\right)$  is not an integer, where  $e$  is the ramification degree of  $K/\mathbb{Q}_p$ . The computation I provide is no more general than Johnson's, but I give the computation anyway because it is self-contained and because it may be useful to have more than one proof of the Local Conjecture in these cases in the literature.

- Conjecture 1.0.2, the Global Conjecture, was classically known only in the case  $K = \mathbb{Q}$  when Ravel made the conjecture, and this is still the only case of the Global Conjecture which has been proven before the results of the present paper: no progress has been made on the Global Conjecture in the 32 years since [20] was written. (The "small factor" in the statement of Conjecture 1.0.2 is 2 in the case  $K = \mathbb{Q}$ .)

In the present paper I address the issue of making the statement of the Global Conjecture precise by defining an  $n$ -congruing ideal (Definition 4.1.1) to be an ideal  $I$  of  $A$  with the property that, for each  $a \in A$ , there exists  $N \in \mathbb{N}$  such that  $a^N(a^n - 1) \in I$ . I then prove (in Theorem 4.1.7) a Hasse principle for  $n$ -congruing ideals, that is, an ideal in  $A$  is  $n$ -congruing if and only if its  $\mathfrak{p}$ -adic completion is  $n$ -congruing, for all maximal ideals  $\mathfrak{p}$  of  $A$ . As a consequence of Theorem 4.1.7 we find that there exists a unique minimal  $n$ -congruing ideal in  $A$ . Consequently, a rigorous statement of the Global Conjecture is:

**Conjecture 1.0.5. (Ravel's Global Conjecture, precise form.)** *Let  $K/\mathbb{Q}$  be a finite extension with ring of integers  $A$ . Then there exists some number  $c \in \mathbb{N}$  such that, for all  $n \in \mathbb{N}$ ,*

$$\mathrm{Ext}_{(L^A, L^A B)}^{1, 2n}(L^A, L^A)[c^{-1}] \cong A/(J_n)[c^{-1}],$$

where  $J_n$  is the minimal  $n$ -congruing ideal of  $A$ .

A stronger version of the conjecture is the following:

**Conjecture 1.0.6. (Ravel's Global Conjecture, stronger precise form.)** *Let  $K/\mathbb{Q}$  be a finite extension with ring of integers  $A$ . Then there exists some "correcting factor"  $c \in A$  such that, for all  $n \in \mathbb{N}$ ,*

$$\mathrm{Ext}_{(L^A, L^A B)}^{1, 2n}(L^A, L^A) \cong A/((c_n)J_n),$$

where  $c_n$  is some factor of  $c$ , and  $J_n$  is the minimal  $n$ -congruing ideal of  $A$ .

In the present paper (see Corollary 5.0.9) I prove Conjecture 1.0.5 for all finite Galois extensions  $K/\mathbb{Q}$ . The number  $c \in \mathbb{N}$  in the statement of Conjecture 1.0.5 can be taken to be the "prime-power-ramification discriminant"  $\underline{\Delta}_{K/\mathbb{Q}}$ , defined in Definition 5.0.5, which is a certain divisor of two times the classical discriminant,  $2\Delta_{K/\mathbb{Q}}$ .

In Corollary 5.0.10, I also prove many cases of Conjecture 1.0.6. Specifically, I prove Conjecture 1.0.6, with correcting factor  $c = 1$  (i.e., no correcting factor is necessary, unlike the  $K = \mathbb{Q}$  case!), for all finite Galois extensions  $K/\mathbb{Q}$  with the property that, for all primes  $\mathfrak{p}$  of the ring of integers  $A$  of  $K$ , the number  $\log_{\mathfrak{p}}(\frac{e_{\mathfrak{p}}}{p-1})$  is not an integer, where  $p$  is the prime of  $\mathbb{Z}$  under  $\mathfrak{p}$ , and  $e_{\mathfrak{p}}$  is the ramification degree of  $\mathfrak{p}$ . (In these cases the “prime-power-ramification discriminant”  $\underline{\Delta}_{K/\mathbb{Q}}$  is 1.) This includes, for example, all Galois extensions  $K/\mathbb{Q}$  of odd prime degree in which 2 ramifies.

- Finally, as for Remark 1.0.4, Ravel is indeed correct to point out that  $\zeta_K(-m) = 0$  for all positive integers  $m$  and all non-totally-real number fields  $K$ . It is also not the case that the denominator of  $\zeta_K(1-m)$  coincides with the order of  $\text{Ext}_{(L^A, L^A B)}^{1, 2m}(L^A, L^A)$ , even up to multiplication by a constant correcting factor, for totally real number fields  $K/\mathbb{Q}$  with ring of integers  $A$  (specifically, I checked this by explicit computation for several totally real quadratic extensions  $K$  of  $\mathbb{Q}$ ).

However, there is more to say on the subject of Remark 1.0.4. In Definition 6.2.1 I define a certain “unramified straightening transform”  $\mathbb{S}$  on Euler products, with the following desirable properties:

- (1) Proposition 6.2.2:  $\mathbb{S}(\zeta(s))(2n)$  is the denominator of  $\zeta(1-2n)$  for all positive integers  $n$ .

- (2) Theorem 6.2.4: Let  $K/\mathbb{Q}$  be a finite Galois extension with ring of integers  $A$ . Let  $\chi_{2\Delta_{K/\mathbb{Q}}}$  be the trivial Größencharakter of  $K$  of modulus equal to  $2\Delta_{K/\mathbb{Q}}$ , two times the classical discriminant of  $K/\mathbb{Q}$ , and let  $L(s, \chi_{2\Delta_{K/\mathbb{Q}}})$  be its associated Hecke  $L$ -function.

Then, for all positive  $n \in \mathbb{N}$ , the following numbers are all equal:

- the order of the abelian group  $\text{Ext}_{(L^A, L^A B)}^{1, 2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$ ,
- the order of the abelian group  $H_{fl}^{1, 2n}(\mathcal{M}_{f mA}; \mathcal{O})[(2\Delta_{K/\mathbb{Q}})^{-1}]$ ,
- the order of the abelian group  $A[(2\Delta_{K/\mathbb{Q}})^{-1}]/J_n$ , where  $J_n$  is the minimal  $n$ -congruencing ideal in  $A[(2\Delta_{K/\mathbb{Q}})^{-1}]$ , and
- the number  $\mathbb{S}(L(s, \chi_{2\Delta_{K/\mathbb{Q}}})) (n)$ .

- (3) The definition of  $\mathbb{S}$  is very, very simple.

In Definition 6.3.3 I define a certain “Galois-Dedekind straightening transform”  $\mathbb{S}_{GD}$  on Euler products. The Galois-Dedekind straightening transform has a more complicated and less natural-seeming definition than the unramified straightening transform, but it has following desirable properties:

- (1) Example 6.3.4:  $\mathbb{S}_{GD}(\zeta(s))(2n)$  is the denominator of  $\zeta(1-2n)$  for all positive integers  $n$ .

- (2) Theorem 6.3.5: Let  $K/\mathbb{Q}$  be a finite Galois extension with ring of integers  $A$ . Let  $\chi_{\underline{\Delta}_{K/\mathbb{Q}}}$  be the trivial Größencharakter of  $K$  of modulus equal to the prime-power-ramification discriminant  $\underline{\Delta}_{K/\mathbb{Q}}$ , and let  $L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})$  be its associated Hecke  $L$ -function.

Then, for all positive  $n \in \mathbb{N}$ , the following numbers are all equal:

- the order of the abelian group  $\text{Ext}_{(L^A, L^A B)}^{1, 2n}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$ ,
- the order of the abelian group  $H_{fl}^{1, 2n}(\mathcal{M}_{f mA}; \mathcal{O})[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$ ,
- the order of the abelian group  $A[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]/J_n$ , where  $J_n$  is the minimal  $n$ -congruencing ideal in  $A[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$ , and
- the number  $\mathbb{S}_{GD}(L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})) (n)$ .

- (3) Corollary 6.3.6: Let  $K/\mathbb{Q}$  be a finite field extension with ring of integers  $A$ . Suppose  $K/\mathbb{Q}$  is Galois and suppose that the prime-power-ramification discriminant  $\underline{\Delta}_{K/\mathbb{Q}}$  is equal to one.

Then, for all positive  $n \in \mathbb{N}$ , the following numbers are all equal:

- the order of the abelian group  $\text{Ext}_{(L^A, L^A B)}^{1, 2n}(L^A, L^A)$ ,
  - the order of the abelian group  $H_{fl}^{1, 2n}(\mathcal{M}_{f mA}; O)$ ,
  - the order of the abelian group  $A/J_n$ , where  $J_n$  is the minimal  $n$ -congruence ideal in  $A$ , and
  - the number  $\mathbb{S}_{GD}(\zeta_K(s))(n)$ , where  $\zeta_K(s)$  is the Dedekind  $\zeta$ -function of  $K$ .
- (4) Proposition 6.4.5: there exists an “inverse” transform  $\mathbb{S}_{GD}^{-1}$  such that  $\mathbb{S}_{GD}^{-1}(\mathbb{S}_{GD}(\zeta_K(s))) = \zeta_K(s)$  for every finite Galois extension  $K/\mathbb{Q}$ .

Finally, as a consequence of these facts about  $\mathbb{S}_{GD}$ , I prove Theorem 6.4.7, which states that, if  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are finite Galois extensions with rings of integers  $A_1$  and  $A_2$ , respectively, and  $m$  is any integer which is divisible by both  $\underline{\Delta}_{K_1/\mathbb{Q}}$  and  $\underline{\Delta}_{K_2/\mathbb{Q}}$ , then the following statements are all equivalent:

- (1) The Hecke  $L$ -function of the trivial Größencharakter of  $K_1$  with modulus  $m$  is equal to the Hecke  $L$ -function of the trivial Größencharakter of  $K_2$  with modulus  $m$ .

- (2) For all  $n$ , the order of the abelian group

$$\text{Ext}_{(L^{A_1}, L^{A_1 B})}^{1, n}(L^{A_1}, L^{A_1})[m^{-1}]$$

is equal to the order of the abelian group

$$\text{Ext}_{(L^{A_2}, L^{A_2 B})}^{1, n}(L^{A_2}, L^{A_2})[m^{-1}].$$

- (3) For all positive  $n \in \mathbb{N}$ , the order of the abelian group

$$H_{fl}^{1, n}(\mathcal{M}_{f mA_1}; O)[m^{-1}]$$

is equal to the order of the abelian group

$$H_{fl}^{1, n}(\mathcal{M}_{f mA_2}; O)[m^{-1}].$$

- (4) The graded abelian group

$$\text{Ext}_{(L^{A_1}, L^{A_1 B})}^1(L^{A_1}, L^{A_1})[m^{-1}]$$

is isomorphic to the graded abelian group

$$\text{Ext}_{(L^{A_2}, L^{A_2 B})}^1(L^{A_2}, L^{A_2})[m^{-1}].$$

- (5) The graded abelian group

$$H_{fl}^{1, n}(\mathcal{M}_{f mA_1}; O)[m^{-1}]$$

is isomorphic to the graded abelian group

$$H_{fl}^{1, n}(\mathcal{M}_{f mA_2}; O)[m^{-1}].$$

- (6) For all  $n \in \mathbb{N}$ , the order of the abelian group  $A_1/(J_{n,1})[m^{-1}]$  is equal to the order of the abelian group  $A_2/(J_{n,2})[m^{-1}]$  where  $J_{n,1}$  is the minimal  $n$ -congruence ideal of  $A_1[m^{-1}]$ . and  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_2[m^{-1}]$ .

- (7) For all  $n \in \mathbb{N}$ , the abelian group  $A_1/(J_{n,1})[m^{-1}]$  is isomorphic to the abelian group  $A_2/(J_{n,2})[m^{-1}]$  where  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_1[m^{-1}]$ . and  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_2[m^{-1}]$ .

Corollary 6.4.8 follows as a consequence: if  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are finite Galois extensions with rings of integers  $A_1$  and  $A_2$ , respectively, and the prime-power-ramification discriminants  $\underline{\Delta}_{K_1/\mathbb{Q}}$  and  $\underline{\Delta}_{K_2/\mathbb{Q}}$  are both equal to 1, then the following are equivalent:

- (1) The Dedekind  $\zeta$ -functions of  $K_1$  and of  $K_2$  are equal.
- (2) For all positive  $n \in \mathbb{N}$ , the order of the abelian group

$$\mathrm{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,n}(L^{A_1}, L^{A_1})$$

is equal to the order of the abelian group

$$\mathrm{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,n}(L^{A_2}, L^{A_2}).$$

- (3) For all positive  $n \in \mathbb{N}$ , the order of the abelian group

$$H_{f_1}^{1,n}(\mathcal{M}_{f_{m_{A_1}}}; \mathcal{O})$$

is equal to the order of the abelian group

$$H_{f_1}^{1,n}(\mathcal{M}_{f_{m_{A_2}}}; \mathcal{O}).$$

- (4) The graded abelian group

$$\mathrm{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,n}(L^{A_1}, L^{A_1})$$

is isomorphic to the graded abelian group

$$\mathrm{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,n}(L^{A_2}, L^{A_2}).$$

- (5) The graded abelian group

$$H_{f_1}^{1,n}(\mathcal{M}_{f_{m_{A_1}}}; \mathcal{O})$$

is isomorphic to the graded abelian group

$$H_{f_1}^{1,n}(\mathcal{M}_{f_{m_{A_2}}}; \mathcal{O}).$$

- (6) For all  $n \in \mathbb{N}$ , the order of the abelian group  $A_1/(J_{n,1})$  is equal to the order of the abelian group  $A_2/(J_{n,2})$  where  $J_{n,1}$  is the minimal  $n$ -congruence ideal of  $A_1$ . and  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_2$ .
- (7) For all  $n \in \mathbb{N}$ , the abelian group  $A_1/(J_{n,1})$  is isomorphic to the abelian group  $A_2/(J_{n,2})$  where  $J_{n,1}$  is the minimal  $n$ -congruence ideal of  $A_1$ . and  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_2$ .

I see Theorem 6.4.7 and Corollary 6.4.8 as a satisfying resolution to the state of affairs observed by Ravel in Remark 1.0.4: for finite Galois extensions of  $\mathbb{Q}$ , when the prime-power-ramification discriminant is trivial (i.e., equal to 1), the orders of the  $\mathrm{Ext}^1$ -groups are determined by the Dedekind  $\zeta$ -function of  $K$  (not just their special values at negative integers!), and vice versa.

It is worth mentioning that these theorems about  $\mathbb{S}(\zeta_K(s))$  do not require any assumptions about  $K/\mathbb{Q}$  being an abelian extension. In Example 5.0.6 I give an example of a non-abelian Galois extension of  $\mathbb{Q}$  with trivial prime-power-ramification discriminant, i.e., an example of a non-abelian extension of  $\mathbb{Q}$  to which Corollary 6.4.8 applies.

It is also worth mentioning that a tremendous amount of work in number theory has gone into the number-theoretic properties and applications of the deformation spaces of formal  $A$ -modules, i.e., the formal neighborhoods of the various points in the moduli stack of formal  $A$ -modules. For example, Lubin and Tate's proof, in [14], of the  $p$ -adic version of Kronecker's "jugendtraum" uses the deformation space of a formal  $O_K$ -module of

$O_K$ -height 1 to construct the abelian closure of a finite extension  $K/\mathbb{Q}_p$ , and Carayol’s non-abelian Lubin-Tate theory, as in [6], lays out a program to use the deformation spaces of height  $> 1$  formal  $O_K$ -modules to produce local Langlands and Jacquet-Langlands correspondences; this program was ultimately successful, as described in [9], where, for a finite extension  $K/\mathbb{Q}$ , a “globalization” process of passing from local data ( $\ell$ -adic representations and local Euler factors, described by appropriate cohomology groups of deformation spaces of formal  $(O_K)_{\hat{\mathfrak{p}}}$ -modules, for the various maximal ideals  $\mathfrak{p}$  of  $O_K$ ) to the global data (complex representations and their  $L$ -functions) was accomplished by using certain Shimura varieties with the property that the formal neighborhoods of the various points are the relevant deformation spaces. The results of the present paper suggest that the moduli stack  $\mathcal{M}_{fmO_K}$  of formal  $O_K$ -modules itself can play a role very much like these Shimura varieties:  $\mathcal{M}_{fmO_K}$  is also a “globalization” of the various deformation spaces of formal  $(O_K)_{\hat{\mathfrak{p}}}$ -modules, with the property that its flat cohomology captures  $L$ -function-theoretic data about the number ring  $O_K$ . The stack  $\mathcal{M}_{fmO_K}$  is less amenable to uniformization techniques (as in [5]) which are used for Shimura varieties, but  $\mathcal{M}_{fmO_K}$  has the advantage of an explicit presentation by a Hopf algebroid, with important connections to homotopy theory, already mentioned above.

I am grateful to the topology groups at University of Rochester and University of Chicago for their patience with me while I prattled on and on in their topology seminars about the contents of this paper. I am especially grateful to D. Ravenel for many useful conversations about formal modules and their moduli theory.

## 2. REVIEW OF FORMAL MODULES, THEIR MODULI THEORY, AND COMMON NOTATIONS.

In this section I give a brief review of some definitions and some results which I regard as classical in the theory of formal modules. An excellent reference is [10].

**Definition 2.0.7.** *If  $A$  is a commutative ring and  $R$  is a commutative  $A$ -algebra, then a (one-dimensional) formal  $A$ -module over  $R$  is a formal group law  $F$  over  $R$  equipped with a ring homomorphism  $\rho : A \rightarrow \text{End}(F)$  such that, for all  $a \in A$ , the power series  $\rho(a) \in \text{End}(F) \subseteq R[[X]]$  is congruent to  $aX$  modulo  $X^2$ . A homomorphism of formal  $A$ -modules is (as one would guess) just a homomorphism of formal group laws which commutes with the action of  $A$ . A strict isomorphism of formal  $A$ -modules is an isomorphism (i.e., a homomorphism with an inverse) of formal  $A$ -modules which is strict as an isomorphism of the underlying formal group laws, i.e., as a power series, the isomorphism is congruent to  $X$  modulo  $X^2$ .*

*All formal modules in this paper will be assumed one-dimensional.*

Hence a formal  $A$ -module is a “formal group law with complex multiplication by  $A$ ” (this perspective was taken, for example, already in [14]).

**Definition 2.0.8.** *Suppose  $A$  is a local commutative ring with finite residue field  $\mathbb{F}_q$  and uniformizer  $\pi$ . Suppose  $R$  is a commutative  $A$ -algebra which is  $\pi$ -torsion-free, i.e., multiplication by  $\pi$  is injective on  $R$ . Suppose further that  $R$  is characteristic zero. We say that a formal  $A$ -module  $F$  over  $R$  is  $A$ -typical if the logarithm  $\log_F$  of the underlying formal group law of  $F$  has the form*

$$\log_F(X) = X + \lambda_1 X^q + \lambda_2 X^{q^2} + \lambda_3 X^{q^3} + \dots$$

Definition 2.0.8 admits a natural, canonical extension to a notion of  $A$ -typicality for arbitrary formal  $A$ -modules, not just those defined over certain (characteristic zero, etc.)  $A$ -algebras; see chapter 21 of [10] for this.



See chapter 21 of [10] for the following results, which are phrased below as one “omnibus” theorem:

**Theorem 2.0.9. Lazard-type theorems for formal  $A$ -modules.**

- **(Global case.)** Let  $A$  be the ring of integers in a finite extension  $K/\mathbb{Q}$ .
  - Then there exists a “formal  $A$ -module Lazard ring”  $L^A$  and a “classifying ring for strict formal  $A$ -module isomorphisms”  $L^A B$ , having the property that, for any commutative  $A$ -algebra  $R$ , there is a bijection between  $A$ -algebra morphisms  $L^A \rightarrow R$  and formal  $A$ -modules over  $R$ , and a bijection between  $A$ -algebra morphisms  $L^A B \rightarrow R$  and strict isomorphisms of formal  $A$ -modules over  $R$ . These bijections are natural in  $R$ .
  - The rings  $L^A$  and  $L^A B$  assemble to form a Hopf algebroid  $(L^A, L^A B)$ , with left unit, right unit, augmentation, conjugation, and coproduct maps being the ring homomorphisms that express the operations taking the domain of a strict isomorphism, taking the codomain of a strict isomorphism, taking the identity strict isomorphism on a formal  $A$ -module, taking the inverse of a strict isomorphism, and composing a composable pair of strict isomorphisms, respectively.
  - If the class number of  $A$  is one, then  $L^A$  is isomorphic to a polynomial ring  $L^A \cong A[S_1^A, S_2^A, S_3^A, \dots]$ , and  $L^A B$  is isomorphic to a polynomial ring  $L^A B \cong L^A[b_1^A, b_2^A, b_3^A, \dots]$ . If the class number of  $A$  is not one, then  $L^A$  and  $L^A B$  are subalgebras of polynomial algebras; see [23] for explicit computations.
  - Given a commutative  $A$ -algebra  $R$ , a unit  $r \in R^\times$ , and a formal  $A$ -module  $F$  over  $R$  with logarithm  $\log_F(X)$ , the formal group law with logarithm  $r^{-1} \log_F(rX)$  also has the natural structure of a formal  $A$ -module. This establishes an action of the multiplicative group scheme  $\mathbb{G}_m$  on the functor  $\text{hom}_{\text{comm. } A\text{-alg}}(L^A, -)$ , hence a grading on  $L^A$ . By convention, we double all grading degrees of gradings arising from  $\mathbb{G}_m$ -actions, so that all our graded rings are concentrated in even degrees and so there is no question of whether we are using the graded-commutativity sign convention. With this convention, the generator  $S_n^A$  is in grading degree  $2n$ . By a similar construction,  $L^A B$  also has a natural grading, with  $b_n^A$  in grading degree  $2n$ .
- **(Local case.)** Now let  $A$  instead be the ring of integers in a finite extension  $K/\mathbb{Q}_p$ .
  - Then there exists a “formal  $A$ -module Lazard ring”  $L^A$  and a “classifying ring for strict formal  $A$ -module isomorphisms”  $L^A B$ , with the same universal properties as in the global case.
  - There also exists an “ $A$ -typical formal  $A$ -module Lazard ring”  $V^A$  and a “ $A$ -typical classifying ring for strict formal  $A$ -module isomorphisms”  $V^A T$ , having the property that, for any commutative  $A$ -algebra  $R$ , there is a bijection between  $A$ -algebra morphisms  $V^A \rightarrow R$  and  $A$ -typical formal  $A$ -modules over  $R$ , and a bijection between  $A$ -algebra morphisms  $V^A T \rightarrow R$  and strict isomorphisms of  $A$ -typical formal  $A$ -modules over  $R$ . These bijections are natural in  $R$ .
  - The rings  $L^A$  and  $L^A B$  assemble to form a Hopf algebroid  $(L^A, L^A B)$ , just as in the global case. The rings  $V^A$  and  $V^A T$  also assemble to form a Hopf algebroid  $(V^A, V^A T)$ , for the same reasons.
  - The ring  $L^A$  is isomorphic to a polynomial ring  $L^A \cong A[S_1^A, S_2^A, S_3^A, \dots]$ , and  $L^A B$  is isomorphic to a polynomial ring  $L^A B \cong L^A[b_1^A, b_2^A, b_3^A, \dots]$  (there is

no class number condition here, since the class group of  $A$  is automatically trivial!). These rings are graded just as in the global case.

- The ring  $V^A$  is isomorphic to a polynomial ring  $V^A \cong A[v_1^A, v_2^A, v_3^A, \dots]$ , and  $V^A T$  is isomorphic to a polynomial ring  $V^A T \cong V^A[t_1^A, t_2^A, t_3^A, \dots]$ . These rings have a natural grading by the same construction as in the global case, and the grading degrees of  $v_n^A$  and of  $t_n^A$  are each equal to  $2(q^n - 1)$ , where  $q$  is the cardinality of the residue field of  $A$ .

This paper is about Ext groups in certain categories of comodules over certain Hopf algebroids, specifically the Hopf algebroids classifying formal modules over various number rings. I had better say a little bit about what this means and why this is worth doing:

**Conventions 2.0.10.** Suppose that  $(A, \Gamma)$  is a Hopf algebroid with  $\Gamma$  flat over  $A$ . Suppose further that  $M, N$  are left  $\Gamma$ -comodules. Then I will write

$$\text{Ext}_{(A, \Gamma)}^*(M, N)$$

for the relative Ext groups in the category of left  $\Gamma$ -comodules, relative to the allowable class generated by the extended  $\Gamma$ -comodules. See Appendix 1 of Ravenel’s book [22] for this very standard construction. It has the following desirable properties:

- The two-sided cobar complex of  $(A, \Gamma)$  with coefficients in  $M$  and  $N$  computes  $\text{Ext}_{(A, \Gamma)}^*(M, N)$  if  $M$  is projective as an  $A$ -module (see A1.2.12 of [22]).
- When  $\Gamma$  is smooth (respectively, formally smooth) over  $A$ , then the stack associated to the groupoid scheme  $(\text{Spec } A, \text{Spec } \Gamma)$  is an Artin (respectively, “formally Artin”) stack  $\mathcal{X}$ , and the category of left  $\Gamma$ -comodules is equivalent to the category of quasicoherent modules over the structure sheaf  $\mathcal{O}$  of the fpqc site on  $\mathcal{X}$ . This equivalence preserves cohomology, in the following sense: write  $\tilde{M}$  for the quasicoherent  $\mathcal{O}$ -module associated to a left  $\Gamma$ -module  $M$ , and write  $\overline{\mathcal{F}}$  for the left  $\Gamma$ -comodule associated to a quasicoherent  $\mathcal{O}$ -module  $\mathcal{F}$ . Then for all nonnegative integers  $n$ , we have isomorphisms

$$\text{Ext}_{(A, \Gamma)}^n(A, M) \cong H_{fl}^n(\mathcal{X}; \tilde{M}) \text{ and}$$

$$\text{Ext}_{(A, \Gamma)}^n(A, \overline{\mathcal{F}}) \cong H_{fl}^n(\mathcal{X}; \mathcal{F}),$$

natural in  $\mathcal{F}$  and in  $M$ . See [17] or [24] for these results.

Now suppose further that  $(A, \Gamma)$  is a *graded* Hopf algebroid with  $\Gamma$  flat over  $A$ . (I insist here that  $A, \Gamma$  both be commutative rings equipped with a grading, not graded-commutative rings which have anticommuting elements in odd degrees!) Suppose further that  $M, N$  are graded left  $\Gamma$ -comodules. Then I make the definition

$$\text{Ext}_{(A, \Gamma)}^{s, t}(M, N) := \text{Ext}_{gr. (A, \Gamma)}^s(\Sigma^t M, N),$$

that is, I will write  $\text{Ext}_{(A, \Gamma)}^{s, t}(M, N)$  for the relative Ext-group  $\text{Ext}_{gr. (A, \Gamma)}^s(\Sigma^t M, N)$  in the category of graded left  $\Gamma$ -comodules, relative to the allowable class generated by the extended graded  $\Gamma$ -comodules; here (as is usual in topology) I am writing  $\Sigma^t M$  for the  $\Gamma$ -comodule  $M$  with all of its grading degrees increased by  $t$ . See Appendix 1 of Ravenel’s book [22] for this very standard construction. It has the following desirable properties:

- The Ext-group  $\text{Ext}_{(A, \Gamma)}^s(M, N)$ , ignoring the gradings, splits as a direct sum

$$\text{Ext}_{(A, \Gamma)}^s(M, N) \cong \bigoplus_{t \in \mathbb{Z}} \text{Ext}_{(A, \Gamma)}^{s, t}(M, N).$$

Consequently the two-sided cobar complex of  $(A, \Gamma)$  with coefficients in  $M$  and  $N$  still computes  $\text{Ext}_{(A, \Gamma)}^*(M, N)$  if  $M$  is projective as an  $A$ -module (again, see A1.2.12 of [22]).

- When  $\Gamma$  is smooth (respectively, formally smooth) over  $A$ , then the Artin (respectively, “formally Artin”) stack  $\mathcal{X}$  inherits an action of the multiplicative group scheme  $\mathbb{G}_m$ , and the category of graded left  $\Gamma$ -comodules is equivalent to the category of  $\mathbb{G}_m$ -equivariant quasicoherent modules over the structure sheaf  $\mathcal{O}_{\mathcal{X}}$  of the fpqc site on  $\mathcal{X}$ . This equivalence preserves cohomology, in the following sense: write  $\tilde{M}$  for the  $\mathbb{G}_m$ -equivariant quasicoherent  $\mathcal{O}_{\mathcal{X}}$ -module associated to a graded left  $\Gamma$ -module  $M$ , and write  $\overline{\mathcal{F}}$  for the graded left  $\Gamma$ -comodule associated to a  $\mathbb{G}_m$ -equivariant quasicoherent  $\mathcal{O}_{\mathcal{X}}$ -module  $\mathcal{F}$ . Then for all nonnegative integers  $n$ , we have isomorphisms

$$(2.0.1) \quad \mathrm{Ext}_{(A,\Gamma)}^{s,t}(A, M) \cong H_{fl, \mathbb{G}_m}^s(\mathcal{X}; \tilde{M} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(-1)^{\otimes_{\mathcal{O}_{\mathcal{X}}} t}) \text{ and}$$

$$(2.0.2) \quad \mathrm{Ext}_{(A,\Gamma)}^{s,t}(A, \overline{\mathcal{F}}) \cong H_{fl, \mathbb{G}_m}^s(\mathcal{X}; \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(-1)^{\otimes_{\mathcal{O}_{\mathcal{X}}} t}),$$

natural in  $\mathcal{F}$  and in  $M$ . Here I am writing  $H_{fl, \mathbb{G}_m}^s$  for  $\mathbb{G}_m$ -equivariant cohomology of  $\mathbb{G}_m$ -equivariant quasicoherent modules over the structure sheaf  $\mathcal{O}_{\mathcal{X}}$  of the fpqc site on  $\mathcal{X}$ , and I am writing  $\mathcal{O}_{\mathcal{X}}(-1)$  for the quasicoherent “twist” module associated to the graded left  $\Gamma$ -comodule  $\Sigma^{-1}A$ .

As far as I know, there does not seem to be a standard notation for stack cohomology equipped with the additional grading one gets from a  $\mathbb{G}_m$ -action on the stack, as in 2.0.1 and 2.0.2. I will write  $H_{fl}^{s,t}(\mathcal{X}; \mathcal{F})$  as shorthand for the cohomology group in 2.0.2.

- When  $E$  is a ring spectrum whose ring of homotopy groups  $E_*$  is commutative and whose ring of stable co-operations  $E_*E$  is commutative and flat over  $E_*$ , then  $(E_*, E_*E)$  is a graded Hopf algebroid, and for any spectrum  $X$ , the bigraded abelian group

$$\mathrm{Ext}_{(E_*, E_*E)}^{**}(E_*, E_*(X))$$

is the  $E_2$ -term of the  $E$ -Adams spectral sequence converging to the homotopy groups  $\pi_*(\hat{X}_E)$  of the  $E$ -nilpotent completion of  $X$ . See chapter 2 of [22] for this material.

Specifically, when  $E = MU$ , the complex cobordism spectrum, then the Hopf algebroid  $(MU_*, MU_*MU)$  is isomorphic to Lazard’s Hopf algebroid  $(L^{\mathbb{Z}}, L^{\mathbb{Z}}B)$  classifying one-dimensional formal  $\mathbb{Z}$ -modules, i.e., one-dimensional formal group laws. When  $p$  is any prime and  $E = BP$ , the  $p$ -primary Brown-Peterson spectrum, then the Hopf algebroid  $(BP_*, BP_*BP)$  is isomorphic to Lazard’s Hopf algebroid  $(V^{\mathbb{Z}(p)}, V^{\mathbb{Z}(p)}T)$  classifying  $p$ -typical one-dimensional formal  $\mathbb{Z}_{(p)}$ -modules, i.e.,  $p$ -typical one-dimensional formal group laws over commutative  $\mathbb{Z}_{(p)}$ -algebras. These are both theorems of Quillen: see [18].

Consequently (and using Bousfield’s theorems identifying  $MU$ -nilpotent and  $BP$ -nilpotent completions of connective spectra: see chapter 2 of [22]), writing  $\mathcal{M}_{fg}$  for the moduli stack of one-dimensional formal groups over  $\mathrm{Spec} \mathbb{Z}$ ,

$$\mathrm{Ext}_{(L^{\mathbb{Z}}, L^{\mathbb{Z}}B)}^{**}(L^{\mathbb{Z}}, L^{\mathbb{Z}}) \cong H_{fl}^{**}(\mathcal{M}_{fg}, \mathcal{O})$$

is the input for the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres  $\pi_*(S)$ , and

$$\mathrm{Ext}_{(V^{\mathbb{Z}(p)}, V^{\mathbb{Z}(p)}T)}^{**}(V^{\mathbb{Z}(p)}, V^{\mathbb{Z}(p)}) \cong H_{fl}^{**}(\mathcal{M}_{fg} \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{Z}_{(p)}; \mathcal{O})$$

is the input for the  $p$ -primary Adams-Novikov spectral sequence converging to the  $p$ -local stable homotopy groups of spheres  $\pi_*(S)_{(p)}$ .

## 3. PROOF OF (MANY CASES OF) RAVENEL'S LOCAL CONJECTURE.

The proof I offer of the local conjecture hinges the computation of the differentials in a spectral sequence, and this computation uses some lemmas consisting of really elementary number theory. Here are those lemmas:

**Lemma 3.0.11.** *Let  $p$  be a prime number, let  $i, j$  be nonnegative integers such that  $j \leq i$ , and let  $\alpha$  a positive integer prime to  $p$ .*

- *In the set of integers  $\{p^i + 1, p^i + 2, p^i + 3, \dots, \alpha p^i\}$ , there exist exactly  $(\alpha - 1)p^{i-j}$  integers divisible by  $p^j$ .*
- *If  $m$  is a nonnegative integer, then in the set of integers  $\{p^i + 1 + m, p^i + 2 + m, p^i + 3 + m, \dots, \alpha p^i + m\}$ , there exist at least  $(\alpha - 1)p^{i-j}$  integers divisible by  $p^j$ .*

*Proof.* Elementary. □

**Lemma 3.0.12.** *Let  $p$  be a prime number, let  $n, e$  be positive integers, and let  $u_{p,n,e} : \{1, \dots, n\} \rightarrow \mathbb{N}$  be the function*

$$u_{p,n,e}(k) = e \cdot v_p \left( \binom{n}{k} \right) + k.$$

*Then the least value taken by the function  $u_{p,n,e}$  is achieved exactly when:*

- *$k \in \{\frac{e}{p-1}, p \frac{e}{p-1}\}$  if  $\log_p \frac{e}{p-1}$  is an integer and  $\log_p \frac{e}{p-1} < v_p(n)$ ,*
- *$k = p^{v_p(n)}$  if  $\log_p \frac{e}{p-1}$  is an integer and  $\log_p \frac{e}{p-1} \geq v_p(n)$  or  $\log_p \frac{e}{p-1}$  is not an integer and  $\lceil \log_p \frac{e}{p-1} \rceil > v_p(n)$ ,*
- *and  $k = p^{\lceil \log_p \frac{e}{p-1} \rceil}$  if  $\log_p \frac{e}{p-1}$  is not an integer and  $\lceil \log_p \frac{e}{p-1} \rceil \leq v_p(n)$ .*

*That least value taken by  $u_{p,n,e}$  is:*

•

$$u_{p,n,e}\left(\frac{e}{p-1}\right) = u_{p,n,e}\left(\frac{pe}{p-1}\right) = e \left( v_p(n) + \frac{1}{p-1} - \log_p \frac{e}{p-1} \right)$$

*if  $\log_p \frac{e}{p-1}$  is an integer and  $\log_p \frac{e}{p-1} < v_p(n)$ ,*

•

$$u_{p,n,e}(v_p(n)) = p^{v_p(n)}$$

*if  $\log_p \frac{e}{p-1}$  is an integer and  $\log_p \frac{e}{p-1} \geq v_p(n)$  or  $\log_p \frac{e}{p-1}$  is not an integer and  $\lceil \log_p \frac{e}{p-1} \rceil > v_p(n)$ ,*

- *and*

$$u_{p,n,e}(p^{\lceil \log_p \frac{e}{p-1} \rceil}) = e \left( v_p(n) - \lceil \log_p \frac{e}{p-1} \rceil \right) + p^{\lceil \log_p \frac{e}{p-1} \rceil}$$

*if  $\log_p \frac{e}{p-1}$  is not an integer and  $\lceil \log_p \frac{e}{p-1} \rceil \leq v_p(n)$ .*

*Proof.* I claim that the minimum value of  $u_{p,n,e}$  only occurs either at  $n$  or in the first half of the domain of  $u_{p,n,e}$ , i.e., if  $k \in \{1, 2, \dots, n\}$  is such that  $u_{p,n,e}(k)$  is the minimum value taken by  $u_{p,n,e}$ , then either  $k = n$  or  $k \leq \frac{n}{2}$ . The argument here is trivially easy: if  $k > \frac{n}{2}$ ,

then:

$$\begin{aligned}
u_{p,n,e}(k) &= e \cdot v_p \left( \binom{n}{k} \right) + k \\
&= e \cdot v_p \left( \binom{n}{n-k} \right) + k \\
&> e \cdot v_p \left( \binom{n}{n-k} \right) + n - k \\
&= u_{p,n,e}(n-k) \text{ if } k < n.
\end{aligned}$$

As a consequence, if  $\alpha > 0$  is an integer prime to  $p$  and  $u_{p,n,e}$  takes its minimum value at  $\alpha p^i$ , then either  $\alpha p^i = n$  or  $\alpha p^i \leq \frac{n}{2}$ , and consequently either  $\alpha p^i = n$  or

$$(\alpha + 1)p^i \leq 2\alpha p^i \leq n,$$

hence either  $\alpha p^i = n$  or

$$(3.0.3) \quad n + 1 - \alpha p^i \geq p^i + 1,$$

an inequality we will use shortly.

Now I claim that the minimum value of the function  $u_{p,n,e}$  occurs at some power of  $p$  (including the possibility of  $p^0 = 1$ ). Indeed, suppose that  $u_{p,n,e}(p^i) \geq u_{p,n,e}(\alpha p^i)$ , with  $\alpha > 0$  an integer prime to  $p$ . It will be convenient to think of the binomial coefficient  $\binom{n}{k}$  as

$$\binom{n}{k} = \frac{n}{1} \frac{n-1}{2} \frac{n-2}{3} \cdots \frac{n+1-k}{k}.$$

Then:

$$\begin{aligned}
u_{p,n,e}(\alpha p^i) &= e \cdot v_p \left( \binom{n}{\alpha p^i} \right) + \alpha p^i \\
&= e \cdot \left( \sum_{j=1}^{\alpha p^i} v_p \left( \frac{n+1-j}{j} \right) \right) + \alpha p^i \\
&= e \cdot \left( \sum_{j=1}^{p^i} v_p \left( \frac{n+1-j}{j} \right) \right) + p^i + e \cdot \left( \sum_{j=p^i+1}^{\alpha p^i} v_p \left( \frac{n+1-j}{j} \right) \right) + (\alpha - 1)p^i \\
&= u_{p,n,e}(p^i) + e \cdot \left( \sum_{j=p^i+1}^{\alpha p^i} v_p \left( \frac{n+1-j}{j} \right) \right) + (\alpha - 1)p^i, \text{ so:} \\
(1 - \alpha)p^i &\geq e \cdot \left( \sum_{j=p^i+1}^{\alpha p^i} v_p \left( \frac{n+1-j}{j} \right) \right).
\end{aligned}$$

Now if  $\alpha \neq 1$ , then  $(1 - \alpha)p^i < 0$ , and consequently  $e \cdot \left( \sum_{j=p^i+1}^{\alpha p^i} v_p \left( \frac{n+1-j}{j} \right) \right)$  is negative. Hence:

$$\begin{aligned}
 0 &> \sum_{j=p^i+1}^{\alpha p^i} v_p \left( \frac{n+1-j}{j} \right) \\
 &= \left( \sum_{k=n+1-\alpha p^i}^{n-p^i} v_p(k) \right) - \left( \sum_{j=p^i+1}^{\alpha p^i} v_p(j) \right), \\
 (3.0.4) \quad &= \left( \sum_{k \in N} v_p(k) \right) - \left( \sum_{j \in N'} v_p(j) \right),
 \end{aligned}$$

where  $N$  is the set of integers  $a$  satisfying  $n+1-\alpha p^i \leq a \leq n-p^i$ , and  $N'$  is the set of integers  $a$  satisfying  $p^i+1 \leq a \leq \alpha p^i$ . If  $u_{p,n,e}$  does not take its minimum value at  $n$ , then by inequality 3.0.3, the set of integers  $N$  is equal to the set of integers  $\{p^i+1+m, p^i+2+m, p^i+3+m, \dots, \alpha p^i\}$  for some nonnegative  $m$ . By Lemma 3.0.11, the sum of the  $p$ -adic valuations of the elements of  $N$  is at least as large as the sum of the  $p$ -adic valuations of the elements of  $N'$ , i.e.,

$$\sum_{k \in N} v_p(k) \geq \sum_{j \in N'} v_p(j),$$

contradicting inequality 3.0.4. So the assumption  $\alpha \neq 1$  implies a contradiction unless  $u_{p,n,e}$  takes its minimum value at  $n$ . So  $\alpha = 1$  and hence  $u_{p,n,e}$  can only take its minimum value at  $n$  or at a power of  $p$ .

Now we need to know something about  $u_{p,n,e}(p^i)$ , for positive integers  $i$ . Write  $n$  as  $n = \beta p^m$  where  $\beta$  is an integer prime to  $p$ . First, notice that, if  $i \leq m$  and  $1 < j \leq p^i$ , then  $v_p(\beta p^m + 1 - j) = v_p(j - 1)$ . Consequently, if  $i \leq m$ , then:

$$\begin{aligned}
 u_{p,\beta p^m,e}(p^i) &= e \cdot v_p \left( \left( \frac{\beta p^m}{p^i} \right) \right) + p^i \\
 &= e \cdot v_p \left( \prod_{j=1}^{p^i} \frac{\beta p^m + 1 - j}{j} \right) + p^i \\
 &= e \cdot \sum_{j=1}^{p^i} (v_p(\beta p^m + 1 - j) - v_p(j)) + p^i \\
 &= e \cdot \left( v_p(\beta p^m) - v_p(1) + \sum_{j=2}^{p^i} (v_p(j-1) - v_p(j)) \right) + p^i \\
 &= e \cdot (v_p(\beta p^m) - v_p(p^i)) + p^i \\
 &= e \cdot (m - i) + p^i.
 \end{aligned}$$

On the other hand, suppose that  $i > m$ . Then:

$$\begin{aligned} u_{p,\beta p^m,e}(p^i) &= e \cdot v_p \left( \left( \frac{\beta p^m}{p^i} \right) \right) + p^i \\ &\geq p^i \\ &> p^m \\ &= e \cdot (m - m) + p^m \\ &= u_{p,\beta p^m,e}(p^m), \end{aligned}$$

so the least value taken by  $u_{p,\beta p^m,e}$  is either  $u_{p,\beta p^m,e}(\beta p^m)$  or it is  $u_{p,\beta p^m,e}(p^i)$  for some (possibly non-unique!) integer  $i$  satisfying  $i \leq m$ . But

$$\begin{aligned} u_{p,\beta p^m,e}(\beta p^m) &= \beta p^m \\ &\geq p^m \\ &= u_{p,\beta p^m,e}(p^m), \end{aligned}$$

so the least value taken by  $u_{p,n,e}$  is  $u_{p,n,e}(p^i)$  for some (possibly non-unique!) integer  $i$  satisfying  $i \leq v_p(n)$ .

Now we use a little bit of elementary calculus! The derivative

$$\frac{d}{di} (e(v_p(n) - i) + p^i) = -e + (\ln p)p^i$$

has a unique zero, namely, at  $i = \log_p \left( \frac{e}{\ln p} \right)$ , hence  $u_{p,n,e}(p^i)$ , as a function of a *real variable*  $i$ , has at most one local extremum in the domain  $(0, v_p(n)) \subseteq \mathbb{R}$ , namely,  $i = \log_p \frac{e}{\ln p}$ . Consequently the function  $u_{p,n,e}(p^i)$ , as a function of an *integer*  $i \in \{1, \dots, p^n\}$ , takes on its minimum value at most two times, at

$$(3.0.5) \quad \lfloor \log_p \frac{e}{\ln p} \rfloor \text{ and at } \lceil \log_p \frac{e}{\ln p} \rceil,$$

if  $u_{p,n,e}(p^i)$  does indeed take on its minimum value two times and not just once.

Now we need to decide under what circumstances  $u_{p,n,e}$  takes on its minimum value twice. We already know that, if  $u_{p,n,e}$  takes on its minimum value twice, then it takes on its minimum value at two consecutive powers of  $p$ , that is,  $u_{p,n,e}(p^i) = u_{p,n,e}(p^{i+1})$  for some  $i < n$ . Consequently:

$$\begin{aligned} 0 &= u_{p,n,e}(p^i) - u_{p,n,e}(p^{i+1}) \\ &= e(v_p(n) - i) + p^i - e(v_p(n) - (i + 1)) - p^{i+1} \\ &= e + p^i(1 - p), \text{ equivalently,} \\ i &= \log_p \frac{e}{p - 1}. \end{aligned}$$

So  $u_{p,n,e}$  takes its minimum value twice if  $\log_p \frac{e}{p-1}$  is an integer and  $p^{\log_p \frac{e}{p-1}}$  and  $p^{1+\log_p \frac{e}{p-1}}$  are both less than  $p^{v_p(n)}$ . Consequently, *the function*  $u_{p,n,e} : \{1, 2, \dots, n\} \rightarrow \mathbb{N}$  *takes on its minimum value exactly once if*  $\log_p \frac{e}{p-1}$  *is not an integer or if*  $\log_p \frac{e}{p-1} \geq v_p(n)$ , *and exactly twice if*  $\log_p \frac{e}{p-1}$  *is an integer and*  $\log_p \frac{e}{p-1} < v_p(n)$ .

In the case that  $\log_p \frac{e}{p-1}$  is an integer and  $\log_p \frac{e}{p-1} < v_p(n)$ , clearly the minimum value of  $u_{p,n,e}$  occurs as  $u_{p,n,e}(p^{\lfloor \log_p \frac{e}{p-1} \rfloor}) = u_{p,n,e}(p^{1+\lfloor \log_p \frac{e}{p-1} \rfloor})$ , by 3.0.5, but one can give a cleaner

description which does not involve the natural logarithm of  $p$ : suppose that  $u_{p,n,e}(p^i) = u_{p,n,e}(p^{i+1})$ . Then:

$$(3.0.6) \quad 0 = u_{p,n,e}(p^i) - u_{p,n,e}(p^{i+1})$$

$$(3.0.7) \quad = e(v_p(n) - i) + p^i - e(v_p(n) - i - 1) - p^{i+1}$$

$$(3.0.8) \quad = e + (1 - p)p^i, \text{ equivalently,}$$

$$(3.0.9) \quad i = \log_p \frac{e}{p-1}.$$

So: when  $\log_p \frac{e}{p-1}$  is an integer and  $\log_p \frac{e}{p-1} < v_p(n)$ , the minimum value of  $u_{p,n,e}$  occurs exactly at  $\log_p \frac{e}{p-1}$  and at  $1 + \log_p \frac{e}{p-1}$ , and this minimum value is

$$u_{p,n,e}(\log_p \frac{e}{p-1}) = e(v_p(n) - \log_p \frac{e}{p-1}) + \frac{e}{p-1},$$

as claimed.

In the case that  $\log_p \frac{e}{p-1}$  is an integer and  $\log_p \frac{e}{p-1} \geq v_p(n)$ , the function  $u_{p,n,e}(p^i)$ , as a function of  $i$ , is monotone decreasing (note that its domain is  $\{0, 1, \dots, v_p(n)\}$ ). Hence, when  $\log_p \frac{e}{p-1}$  is an integer and  $\log_p \frac{e}{p-1} \geq v_p(n)$ , the minimum value of  $u_{p,n,e}$  occurs uniquely at  $p^{v_p(n)}$ , and this minimum value is

$$u_{p,n,e}(p^{v_p(n)}) = p^{v_p(n)},$$

as claimed.

In the case that  $\log_p \frac{e}{p-1}$  is not an integer, one can ask whether the minimum value of  $u_{p,n,e}$  occurs as  $u_{p,n,e}(p^{\lfloor \log_p \frac{e}{p-1} \rfloor})$  or as  $u_{p,n,e}(p^{\lceil \log_p \frac{e}{p-1} \rceil})$  (it must be one of the other, by 3.0.5, unless both  $p^{\lfloor \log_p \frac{e}{p-1} \rfloor}$  and  $p^{\lceil \log_p \frac{e}{p-1} \rceil}$  are outside the domain of  $u_{p,n,e}$ ). It is simpler and cleaner (avoiding formulas involving  $\ln p$ ), however, to simply check under what conditions on  $i$  it is true that  $u_{p,n,e}(p^i) < u_{p,n,e}(p^{i+1})$ . By the same line of argument used in the equalities 3.0.6, 3.0.7, 3.0.8, and 3.0.9, the inequality  $u_{p,n,e}(p^i) < u_{p,n,e}(p^{i+1})$  holds if and only if  $i > \log_p \frac{e}{p-1}$ . Consequently, if  $\log_p \frac{e}{p-1}$  is not an integer and  $\lceil \log_p \frac{e}{p-1} \rceil \leq v_p(n)$ , then the minimum value of  $u_{p,n,e}$  occurs uniquely at  $p^{\lceil \log_p \frac{e}{p-1} \rceil}$ , and this minimum value is:

$$u_{p,n,e}(\lceil \log_p \frac{e}{p-1} \rceil) = e(v_p(n) - \lceil \log_p \frac{e}{p-1} \rceil) + p^{\lceil \log_p \frac{e}{p-1} \rceil},$$

as claimed.

Finally, the last case is the one in which  $\log_p \frac{e}{p-1}$  is not an integer but  $\lceil \log_p \frac{e}{p-1} \rceil > v_p(n)$ . Then it is again true that the function  $u_{p,n,e}(p^i)$ , as a function of  $i$ , is monotone decreasing. Hence, when  $\log_p \frac{e}{p-1}$  is not an integer and  $\lceil \log_p \frac{e}{p-1} \rceil > v_p(n)$ , the minimum value of  $u_{p,n,e}$  occurs uniquely at  $p^{v_p(n)}$ , and this minimum value is again

$$u_{p,n,e}(p^{v_p(n)}) = p^{v_p(n)},$$

as claimed.  $\square$

**Lemma 3.0.13.** *Suppose that  $K/\mathbb{Q}_p$  is a finite field extension of ramification degree  $e$  and whose ring of integers  $A$  has uniformizer  $\pi$ , and suppose that  $\log_p(\frac{e}{p-1})$  is not an integer, i.e.,  $\frac{e}{p-1}$  is not a power of  $p$ . Let  $n$  be a positive integer, and let  $I_n$  be the ideal in  $A$  generated by all elements of the form  $a^n - 1$  for elements  $a \in A$  congruent to 1 modulo  $\pi$ . Then  $A/I_n \cong A/\pi^n$ , where*

$$i = \begin{cases} e(v_p(n) - \lceil \log_p \frac{e}{p-1} \rceil) + p^{\lceil \log_p \frac{e}{p-1} \rceil} & \text{if } \lceil \log_p \frac{e}{p-1} \rceil \leq v_p(n) \\ p^{v_p(n)} & \text{if } \lceil \log_p \frac{e}{p-1} \rceil > v_p(n). \end{cases}$$



*Proof.* Let  $a \in A$  be an element congruent to 1 modulo  $\pi$ , i.e.,  $a \equiv 1 + a_1\pi$  modulo  $\pi^2$ , for some  $a_1 \in A$ . Then the monomial term of  $a^n - 1$  of smallest  $\pi$ -adic valuation is also the monomial term of  $\sum_{k=1}^n \binom{n}{k} \pi^k (a_1)^k$  of smallest  $\pi$ -adic valuation, i.e., it is  $\binom{n}{k} \pi^k (a_1)^k$  for some unique  $k \in \{1, \dots, \nu_p(n)\}$ , by Lemma 3.0.12 and the assumption that  $\log_p \frac{e}{p-1}$  is not an integer. Now observe that

$$\nu_\pi \left( \binom{n}{k} \pi^k (a_1)^k \right) \geq u_{p,n,e}(k)$$

with equality if  $a_1 \in A^\times$ , with  $u_{p,n,e}$  the function defined in Lemma 3.0.12, and then the claim in the present lemma then follows from Lemma 3.0.12.  $\square$

**Lemma 3.0.14.** *Let  $R$  be a commutative ring, and let  $(A, \Gamma)$  be a commutative graded Hopf algebroid over  $R$ . Suppose that  $\Gamma$  is flat over  $A$ . Let  $F$  be the forgetful functor  $F : \text{gr Comod}(A, \Gamma) \rightarrow \text{gr Mod}(A)$  from the category of graded left  $\Gamma$ -comodules to the category of graded  $A$ -modules. Let  $e : \text{gr Mod}(A) \rightarrow \text{gr Comod}(A, \Gamma)$  be the extended comodule functor, i.e.,  $e(M) = \Gamma \otimes_A M$ , with left  $\Gamma$ -coaction given by the map*

$$\Delta \otimes_A M : \Gamma \otimes_A M \rightarrow \Gamma \otimes_A \Gamma \otimes_A M,$$

where  $\Delta : \Gamma \rightarrow \Gamma \otimes_A \Gamma$  is the coproduct map on  $\Gamma$ .

Then both  $e$  and  $F$  preserve colimits.

*Proof.* It is an easy exercise to check that  $e$  is right adjoint to  $F$ . Hence  $F$  is a left adjoint, hence  $F$  preserves colimits. Since colimits in  $\text{gr Comod}(A, \Gamma)$  are consequently computed in graded  $A$ -modules, and since the functor  $M \mapsto \Gamma \otimes_A M$  preserves colimits in graded  $A$ -modules,  $e$  also preserves colimits.  $\square$

**Lemma 3.0.15.** *Let  $R$  be a commutative ring, and let  $(A, \Gamma)$  be a commutative graded Hopf algebroid over  $R$ . Suppose that  $\Gamma$  is flat over  $A$ . Suppose  $M$  is a graded  $\Gamma$ -comodule whose underlying  $A$ -module is finitely generated and projective, and suppose that*

$$(3.0.10) \quad L_0 \rightarrow L_1 \rightarrow L_2 \rightarrow \dots$$

is a sequence of one-to-one graded left  $\Gamma$ -comodule homomorphisms. Then, for any non-negative integer  $s$  and any integer  $t$  the canonical comparison map

$$(3.0.11) \quad \text{colim}_i \text{Ext}_{(A, \Gamma)}^{s,t}(M, L_i) \rightarrow \text{Ext}_{(A, \Gamma)}^{s,t}(M, \text{colim}_i L_i)$$

is an isomorphism of  $R$ -modules.

*Proof.* Since the underlying  $A$ -module of  $M$  is assumed to be finitely generated, any given morphism of  $A$ -modules from  $M$  to  $\text{colim}_i L_i$  is determined by the image of finitely many  $A$ -module generators of  $M$ , and each of these generators  $g$  is in the image of the inclusion  $L_{i_g} \hookrightarrow \text{colim}_i L_i$  for some nonnegative integer  $i_g$ . Consequently the maximum of the nonnegative integers  $\{i_g\}$ , taken over a given finite set of generators  $g$  for  $M$  as a graded  $A$ -module, is some nonnegative integer  $j$ , and the given map  $M \rightarrow \text{colim}_i L_i$  factors, as a map of  $A$ -modules, through the inclusion map  $L_j \hookrightarrow \text{colim}_i L_i$ . Consequently the canonical map

$$\text{colim}_i \text{hom}_{\text{gr Mod}(A)}(FM, FL_i) \rightarrow \text{hom}_{\text{gr Mod}(A)}(FM, \text{colim}_i FL_i)$$

is an isomorphism of  $A$ -modules, where  $F$  is the forgetful functor defined in Lemma 3.0.14.

Now we have the commutative square of morphisms of  $A$ -modules

$$(3.0.12) \quad \begin{array}{ccc} \operatorname{colim}_i \operatorname{hom}_{\operatorname{gr} \operatorname{Comod}(A, \Gamma)}(M, L_i) & \longrightarrow & \operatorname{hom}_{\operatorname{gr} \operatorname{Comod}(A, \Gamma)}(M, \operatorname{colim}_i L_i) \\ \downarrow & & \downarrow \\ \operatorname{colim}_i \operatorname{hom}_{\operatorname{gr} \operatorname{Mod}(A)}(FM, FL_i) & \longrightarrow & \operatorname{hom}_{\operatorname{gr} \operatorname{Mod}(A)}(FM, \operatorname{colim}_i FL_i), \end{array}$$

in which the vertical map on the right is a monomorphism by definition, the vertical map on the left is a monomorphism because it is a directed union of monomorphisms and the category of graded  $A$ -modules satisfies Grothendieck's axiom AB5, and we just showed that the horizontal map on the bottom is an isomorphism, hence also a monomorphism. Hence the top map is also a monomorphism.

I claim that the square 3.0.12 is a pullback square in the category of abelian groups. Since the maps involved are all monomorphisms, this amounts to the claim that the inclusion

$$\begin{aligned} & \operatorname{colim}_i \operatorname{hom}_{\operatorname{gr} \operatorname{Comod}(A, \Gamma)}(M, L_i) \\ & \subseteq \left( \operatorname{colim}_i \operatorname{hom}_{\operatorname{gr} \operatorname{Mod}(A)}(FM, FL_i) \right) \cap \left( \operatorname{hom}_{\operatorname{gr} \operatorname{Comod}(A, \Gamma)}(M, \operatorname{colim}_i L_i) \right). \end{aligned}$$

is an equality; the intersection here is an intersection of subgroups of  $\operatorname{hom}_{\operatorname{gr} \operatorname{Mod}(A)}(FM, \operatorname{colim}_i FL_i)$ . Suppose that

$$f \in \left( \operatorname{colim}_i \operatorname{hom}_{\operatorname{gr} \operatorname{Mod}(A)}(FM, FL_i) \right) \cap \left( \operatorname{hom}_{\operatorname{gr} \operatorname{Comod}(A, \Gamma)}(M, \operatorname{colim}_i L_i) \right).$$

Then  $f$  is a graded left  $\Gamma$ -comodule morphism  $f : M \rightarrow \operatorname{colim}_i L_i$ , and there exists a graded  $A$ -module morphism  $\tilde{f} : M \rightarrow L_i$  such that  $I \circ \tilde{f} = f$ , where  $I : L_i \hookrightarrow \operatorname{colim}_i L_i$  is the canonical inclusion.

Consequently we have the (not yet known to be commutative!) diagram:

$$\begin{array}{ccccc} & & f & & \\ & \curvearrowright & & \curvearrowleft & \\ M & \xrightarrow{\tilde{f}} & L_i & \xrightarrow{I} & \operatorname{colim}_i L_i \\ & \downarrow \Psi_M & \downarrow \Psi_{L_i} & & \downarrow \Psi \\ \Gamma \otimes_A M & \xrightarrow{\Gamma \otimes_A \tilde{f}} & \Gamma \otimes_A L_i & \xrightarrow{\Gamma \otimes_A I} & \Gamma \otimes_A \operatorname{colim}_i L_i \\ & \curvearrowleft & & \curvearrowright & \\ & & \Gamma \otimes_A f & & \end{array}$$

in which the vertical maps are the comodule structure maps, and we know the following four equalities:

$$\begin{aligned} I \circ \tilde{f} &= f, \\ (\Gamma \otimes_A I) \circ (\Gamma \otimes_A \tilde{f}) &= \Gamma \otimes_A f, \\ (\Gamma \otimes_A f) \circ \Psi_M &= \Psi \circ f, \text{ and} \\ \Psi \circ I &= (\Gamma \otimes_A I) \circ \Psi_{L_i}. \end{aligned}$$

We use those four equalities to get the following equalities:

$$\begin{aligned} (\Gamma \otimes_A I) \circ (\Gamma \otimes_A \tilde{f}) \circ \Psi_M &= (\Gamma_A f) \circ \Psi_M \\ &= \Psi \circ f \\ &= \Psi \circ I \circ \tilde{f} \\ &= (\Gamma \otimes_A I) \circ \Psi_{L_i} \circ \tilde{f}. \end{aligned}$$

Now since  $I$  is one-to-one and  $\Gamma$  is assumed flat over  $A$ , the map  $\Gamma \otimes_A I$  is injective, hence left-cancellable, so

$$(\Gamma \otimes_A I) \circ (\Gamma \otimes_A \tilde{f}) \circ \Psi_M = (\Gamma \otimes_A I) \circ \Psi_{L_i} \circ \tilde{f}$$

implies  $(\Gamma \otimes_A \tilde{f}) \circ \Psi_M = \Psi_{L_i} \circ \tilde{f}$ , which is exactly the statement that  $\tilde{f}$  is a morphism of left  $\Gamma$ -comodules, and hence that  $f \in \text{colim}_i \text{hom}_{\text{gr Comod}(A, \Gamma)}(M, L_i)$ , and consequently the square 3.0.12 is a pullback square in  $A$ -modules, as claimed!

Consequently the map

$$\text{colim}_i \text{hom}_{\text{gr Comod}(A, \Gamma)}(M, L_i) \rightarrow \text{hom}_{\text{gr Comod}(A, \Gamma)}(M, \text{colim}_i L_i)$$

is a pullback of an isomorphism, hence itself an isomorphism. So the map 3.0.11 is an isomorphism for  $s = 0$ .

Now we handle  $s > 0$ . Given a graded left  $\Gamma$ -comodule  $N$ , let  $D_\Gamma(N)^\bullet$  be the cobar resolution of  $N$  (see Appendix 1 of [22] for this construction and its basic properties, along with all the other fundamentals of homological algebra of comodules over a Hopf algebra). Then  $N \mapsto D_\Gamma(N)^\bullet$  is in fact a functor from graded left  $\Gamma$ -comodules to cochain complexes of relatively injective graded left  $\Gamma$ -comodules. In particular, for each nonnegative integer  $n$ , the  $n$ -cochains functor  $N \mapsto D_\Gamma(N)^n$  is a functor from graded left  $\Gamma$ -comodules to relatively injective graded left  $\Gamma$ -comodules. Furthermore, for each  $n$ , the functor  $D_\Gamma(N)^n$  is a composite of the functors  $e$  and  $F$  (in fact,  $D_\Gamma(N)^\bullet$ , the cobar complex functor, is the cobar construction associated to the adjunction  $F \dashv e$ ), and consequently Lemma 3.0.14 implies that the canonical map

$$(3.0.13) \quad \text{colim}_i D_\Gamma(L_i) \rightarrow D_\Gamma(\text{colim}_i L_i)$$

is an isomorphism.

Since the map 3.0.11 is a natural bijection for  $s = 0$  and since the map 3.0.13 is a natural bijection as well, we have the isomorphism of cochain complexes of  $R$ -modules

$$(3.0.14) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{colim}_i \text{hom}_{\text{gr Comod}(A, \Gamma)}(\Sigma^t M, D_\Gamma(L_i)^0) & \longrightarrow & \text{colim}_i \text{hom}_{\text{gr Comod}(A, \Gamma)}(\Sigma^t M, D_\Gamma(L_i)^1) & \longrightarrow & \dots \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \text{hom}_{\text{gr Comod}(A, \Gamma)}(\Sigma^t M, D_\Gamma(\text{colim}_i L_i)^0) & \longrightarrow & \text{hom}_{\text{gr Comod}(A, \Gamma)}(\Sigma^t M, D_\Gamma(\text{colim}_i L_i)^1) & \longrightarrow & \dots \end{array}$$

The top row in diagram 3.0.14 has cohomology  $\text{colim}_i \text{Ext}_{(A, \Gamma)}^{*, t}(M, L_i)$  since cohomology of cochain complexes of modules over a commutative ring commutes with filtered colimits (this is Grothendieck's axiom AB5 at work), in particular sequential colimits. The bottom row in diagram 3.0.14 has cohomology  $\text{Ext}_{(A, \Gamma)}^{*, t}(M, \text{colim}_i L_i)$ , and the isomorphism induced by 3.0.14 is the map 3.0.11.  $\square$

**Theorem 3.0.16. ((Many cases of) Ravenel's Local Conjecture.)** *Suppose that  $K/\mathbb{Q}_p$  is a finite field extension of ramification degree  $e$  and residue degree  $f$  and total degree  $d = ef$ , with uniformizer  $\pi$ , and suppose that  $\log_p(\frac{e}{p-1})$  is not an integer, i.e.,  $\frac{e}{p-1}$  is not a power of*

*p.* Let  $A$  be the ring of integers in  $K$ . Then Ravenel's Local Conjecture, Conjecture 1.0.3, holds for  $A$ . That is, for each  $n \in \mathbb{N}$ , we have an isomorphism of  $A$ -modules

$$\mathrm{Ext}_{(V^A, V^A T)}^{1, 2(p^f - 1)n}(V^A, V^A) \cong A/I_n,$$

where  $I_n$  is the ideal in  $A$  generated by all elements of the form  $(a^n - 1)$  with  $a$  an element in  $A$  congruent to 1 modulo  $\pi$ .

Equivalently, in terms of the moduli stack  $\mathcal{M}_{fmA}$  of one-dimensional formal  $A$ -modules over  $\mathrm{Spec} A$ , and with notation as in Conventions 2.0.10: for each  $n \in \mathbb{N}$ , we have an isomorphism of  $A$ -modules

$$H_{fl}^{1, 2(p^f - 1)n}(\mathcal{M}_{fmA}; O) \cong A/I_n,$$

where  $I_m$  is the ideal in  $A$  generated by all elements of the form  $(a^m - 1)$  with  $a$  an element in  $A$  congruent to 1 modulo  $\pi$ .

*Proof.* Recall that Ravenel constructs, in [20], a ‘‘formal  $A$ -module chromatic spectral sequence’:

$$\begin{aligned} E_1^{s,t,u} &\cong \mathrm{Ext}_{(V^A, V^A T)}^{s,u}(V^A, (v_t^A)^{-1}V^A/(\pi^\infty, (v_1^A)^\infty, \dots, (v_{t-1}^A)^\infty)) \Rightarrow \mathrm{Ext}_{(V^A, V^A T)}^{s+t,u}(V^A, V^A) \\ d_r &: E_r^{s,t,u} \rightarrow E_r^{s+1-t, t+r, u}. \end{aligned}$$

For straightforward degree reasons, in the formal  $A$ -module chromatic spectral sequence, the only terms that can contribute to  $\mathrm{Ext}_{(V^A, V^A T)}^{1,*}(V^A, V^A)$  are those on the lines  $t = 0$  and  $t = 1$ , and as the  $t = 0$  line is concentrated in tridegree  $s = 0, t = 0, u = 0$ , we have an isomorphism

$$\begin{aligned} &\mathrm{Ext}_{(V^A, V^A T)}^{1,u}(V^A, V^A) \cong \\ &(\ker d_1^{0,1,u} : (\mathrm{Ext}_{(V^A, V^A T)}^{0,u}(V^A, (v_1^A)^{-1}V^A/\pi^\infty) \rightarrow \mathrm{Ext}_{(V^A, V^A T)}^{0,u}(V^A, (v_2^A)^{-1}V^A/\pi^\infty, (v_1^A)^\infty))) \end{aligned}$$

for all  $u > 0$ .

So we just need to compute the kernel of  $d_1$  on the Ext-group

$$\mathrm{Ext}_{(V^A, V^A T)}^{0,u}(V^A, (v_1^A)^{-1}V^A/\pi^\infty) \cong (V^A \square_{V^A T} (v_1^A)^{-1}V^A/\pi^\infty)^u.$$

By symmetry of the cotensor product and the identification of Ext in a comodule category with the derived functors  $\mathrm{Cotor}$  of the cotensor product (see Appendix 1 of [22]), this Ext-group is isomorphic to

$$(3.0.15) \quad \left( \left( (v_1^A)^{-1}V^A/\pi^\infty \right) \square_{V^A T} V^A \right)^u,$$

which is slightly more convenient to work with, as it is just the degree  $u$  summand of the equalizer of the two maps

$$\Psi, \left( \mathrm{id}_{(v_1^A)^{-1}V^A/\pi^\infty} \otimes \eta_L \right) : (v_1^A)^{-1}V^A/\pi^\infty \rightarrow (v_1^A)^{-1}V^A/\pi^\infty \otimes_{V^A} V^A T \xrightarrow{\cong} (v_1^A)^{-1}V^A T/\pi^\infty$$

where  $\Psi$  is the left  $V^A T$ -comodule structure map on  $(v_1^A)^{-1}V^A/\pi^\infty$ , i.e.,

$$\begin{aligned} \Psi((v_1^A)^n/\pi^i) &= \eta_R((v_1^A)^n)/\pi^i \\ &= (v_1^A + (\pi - \pi^q))^n/\pi^i \end{aligned}$$

for  $n \geq 0$ . (The reason we swap the left coactions for the right coactions in 3.0.15 is so that we can work with the isomorphism

$$(v_1^A)^{-1}V^A/\pi^\infty \otimes_{V^A} V^A T \cong (\eta_L(v_1^A))^{-1}V^A T/\pi^\infty = (v_1^A)^{-1}V^A T/\pi^\infty$$

rather than the isomorphism

$$V^A T \otimes_{V^A} (v_1^A)^{-1}V^A/\pi^\infty \cong (\eta_R(v_1^A))^{-1}V^A T/\pi^\infty$$

which is less convenient.)

To compute  $\text{Ext}_{(V^A, V^A T)}^*(V^A, (v_1^A)^{-1}V^A/\pi^\infty)$ , we use the short exact sequence of graded  $V^A T$ -comodules

$$0 \rightarrow (v_1^A)^{-1}V^A/\pi \rightarrow (v_1^A)^{-1}V^A/\pi^\infty \xrightarrow{\pi} (v_1^A)^{-1}V^A/\pi^\infty \rightarrow 0$$

and its induced long exact sequence of Ext groups

$$(3.0.16) \quad 0 \rightarrow \text{Ext}_{(V^A, V^A T)}^0(V^A, (v_1^A)^{-1}V^A/\pi) \xrightarrow{i} \text{Ext}_{(V^A, V^A T)}^0(V^A, (v_1^A)^{-1}V^A/\pi^\infty) \\ \xrightarrow{\pi} \text{Ext}_{(V^A, V^A T)}^0(V^A, (v_1^A)^{-1}V^A/\pi^\infty) \xrightarrow{\delta} \text{Ext}_{(V^A, V^A T)}^1(V^A, (v_1^A)^{-1}V^A/\pi) \rightarrow \dots$$

Suppose  $x \in \text{Ext}_{(V^A, V^A T)}^0(V^A, (v_1^A)^{-1}V^A/\pi^\infty)$  has the property that  $\delta(x) \neq 0$ . Then I claim that there exists some positive integer  $j$  such that  $\pi^j x \in \text{im } i$ . The proof of this claim is as follows: first, observe that we have a sequence of one-to-one morphisms of graded left  $V^A T$ -comodules

$$(v_1^A)^{-1}V^A/\pi \hookrightarrow (v_1^A)^{-1}V^A/\pi^2 \hookrightarrow (v_1^A)^{-1}V^A/\pi^3 \hookrightarrow \dots$$

and Lemma 3.0.15 tells us that the map

$$\text{colim}_m \text{Ext}_{(V^A, V^A T)}^{s,t}(V^A, (v_1^A)^{-1}V^A/\pi^m) \rightarrow \text{Ext}_{(V^A, V^A T)}^{s,t}(V^A, \text{colim}_m (v_1^A)^{-1}V^A/\pi^m) \cong \text{Ext}_{(V^A, V^A T)}^{s,t}(V^A, (v_1^A)^{-1}V^A/\pi^\infty)$$

is an isomorphism for all  $s$  and  $t$ . Each of the  $A$ -modules  $\text{Ext}_{(V^A, V^A T)}^{s,t}(V^A, (v_1^A)^{-1}V^A/\pi^m)$  is a  $\pi$ -power-torsion  $A$ -module, that is, each element in  $\text{Ext}_{(V^A, V^A T)}^{s,t}(V^A, (v_1^A)^{-1}V^A/\pi^m)$  is killed by multiplication by some sufficiently large power of  $\pi$ . Consequently the colimit  $\text{colim}_m \text{Ext}_{(V^A, V^A T)}^{s,t}(V^A, (v_1^A)^{-1}V^A/\pi^m)$  is also  $\pi$ -power-torsion. So there exists some  $j$  such that  $\pi^j x = 0$ , and now exactness of sequence 3.0.16 implies that  $\pi^j x \in \text{im } i$ , as claimed.

As a consequence, if we give a description of  $\text{im } i$  as well as a description of the elements in  $\text{Ext}_{(V^A, V^A T)}^0(V^A, (v_1^A)^{-1}V^A/\pi^\infty)$  whose multiples by  $\pi^j$  are nonzero elements of  $\text{im } i$  for sufficiently large  $i$ , then we will have a complete description of  $\text{Ext}_{(V^A, V^A T)}^0(V^A, (v_1^A)^{-1}V^A/\pi^\infty)$ . (The above argument, which is the simplest way that I know of to reduce the computation of  $\text{Ext}_{(V^A, V^A T)}^0(V^A, (v_1^A)^{-1}V^A/\pi^\infty)$  to the computation of  $\text{im } i$ , was not given in Ravenel's paper [20]; Ravenel simply computes  $\text{im } i$ . Presumably something like the above line of argument was clear to Ravenel, though not stated in his paper. The argument is involved enough, requiring the two Lemmas 3.0.14 and 3.0.15, that I think it is worth spelling out in the present paper.) As Ravenel observes in [20], the image of  $i$  is generated by the cohomology class of the 0-cocycles  $v_1^i/\pi$  for  $i \in \mathbb{Z}$ , and to determine  $\text{Ext}_{(V^A, V^A T)}^0(V^A, (v_1^A)^{-1}V^A/\pi^\infty)$  we just need to determine how many times each of these cohomology classes are divisible by  $\pi$ . An element in  $\text{Ext}_{(V^A, V^A T)}^0(V^A, (v_1^A)^{-1}V^A/\pi^\infty)$  is not divisible by  $\pi$  if and only if its image under  $\delta$  is nonzero. Consequently, to compute the order of  $\text{Ext}_{(V^A, V^A T)}^{0, 2i(q-1)}(V^A, (v_1^A)^{-1}V^A/\pi^\infty)$ , we just need to divide  $v_1^i/\pi \in (v_1^A)^{-1}V^A/\pi^\infty$  by the largest possible power of  $\pi$  such that the resulting quotient  $x$  is still a 0-cocycle and such that  $\delta(x) \neq 0$ .

Our next task is to determine how large that largest possible power of  $\pi$  is. Now let  $n$  be a positive integer, and suppose that  $\log_p \frac{p-1}{e}$  is not an integer, as in the statement of the theorem. Then the  $\pi$ -adic valuation of

$$(3.0.17) \quad (\eta_R - \eta_L)((v_1^A)^n) = \sum_{i=1}^n \binom{n}{i} (v_1^A)^{n-i} (\pi - \pi^q)^i (t_1^A)^i$$

is equal to

$$\begin{aligned} \min \left\{ v_\pi \left( \binom{n}{i} (v_1^A)^{n-i} (\pi - \pi^q)^i (t_1^A)^i \right) : i = 1, \dots, n \right\} &= \min \left\{ v_\pi \left( \binom{n}{i} \right) + i v_\pi (\pi - \pi^q) : i = 1, \dots, n \right\} \\ &= \min \left\{ e \cdot v_p \left( \binom{n}{i} \right) + i : i = 1, \dots, n \right\}. \end{aligned}$$

(Here I am using the Araki-type generators  $v_1^A, v_2^A, \dots$  for  $V^A$ , and consequently  $\eta_R(v_1^A) = v_1^A + (\pi - \pi^q)t_1^A$ . It works just as well to use the Hazewinkel-type generators  $V_1^A, V_2^A, \dots$  for  $V^A$ , in which case  $\eta_R(V_1^A) = V_1^A + \pi t_1^A$ , and the rest of the argument goes through without significant changes. See [10] for these formulas for  $\eta_R$ .) Now by Lemma 3.0.12, when  $\log_p(\frac{e}{p-1})$  is not an integer, the quantity  $e \cdot v_p \left( \binom{n}{i} \right) + i$ , as a function of  $i$ , takes its unique minimum value when  $i = p^{\lceil \log_p \frac{e}{p-1} \rceil}$  if  $\log_p \frac{e}{p-1} < v_p(n)$  and when  $i = p^{v_p(n)}$  if  $v_p(n) < \log_p \frac{e}{p-1}$ . Consequently there is a unique monomial of least  $\pi$ -adic valuation in the sum 3.0.17, and it is

$$\left( p^{\lceil \log_p \frac{e}{p-1} \rceil} \binom{n}{p^{\lceil \log_p \frac{e}{p-1} \rceil}} (v_1^A)^{n-p^{\lceil \log_p \frac{e}{p-1} \rceil}} (\pi - \pi^q)^{p^{\lceil \log_p \frac{e}{p-1} \rceil}} (t_1^A)^{p^{\lceil \log_p \frac{e}{p-1} \rceil}}, \right.$$

with  $\pi$ -adic valuation

$$e(v_p(n) - \lceil \log_p \frac{e}{p-1} \rceil) + p^{\lceil \log_p \frac{e}{p-1} \rceil}$$

if  $v_p(n) > \log_p \frac{e}{p-1}$ ; and, on the other hand, it is

$$\left( p^{v_p(n)} \binom{n}{p^{v_p(n)}} (v_1^A)^{n-p^{v_p(n)}} (\pi - \pi^q)^{p^{v_p(n)}} (t_1^A)^{p^{v_p(n)}}, \right.$$

with  $\pi$ -adic valuation  $p^{v_p(n)}$ , if  $v_p(n) < \log_p \frac{e}{p-1}$ . For brevity, we write  $a(n)$  for the  $\pi$ -adic valuation determined above, that is,

$$a(n) = \begin{cases} e(v_p(n) - \lceil \log_p \frac{e}{p-1} \rceil) + p^{\lceil \log_p \frac{e}{p-1} \rceil} & \text{if } v_p(n) > \log_p \frac{e}{p-1} \\ p^{v_p(n)} & \text{if } v_p(n) < \log_p \frac{e}{p-1}, \end{cases}$$

and we write  $b(n)$  for the exponent of  $t_1^A$  appearing in the monomial term of  $(\eta_R - \eta_L)((v_1^A)^n)$  of lowest  $\pi$ -adic valuation, that is,

$$b(n) = \begin{cases} p^{\lceil \log_p \frac{e}{p-1} \rceil} & \text{if } \log_p \frac{e}{p-1} < v_p(n), \\ p^{v_p(n)} & \text{if } v_p(n) < \log_p \frac{e}{p-1}, \end{cases}$$

$$(\eta_R - \eta_L)((v_1^A)^n) \equiv \binom{n}{b(n)} (v_1^A)^{n-b(n)} \pi^{b(n)} (t_1^A)^{b(n)} \pmod{\pi^{a(n)+1}}.$$

Hence the 0-cocycle  $(v_1^A)^n / \pi \in (v_1^A)^{-1} V^A / \pi^\infty$  in the cobar complex computing  $\text{Ext}_{(V^A, V^A T)}^*(V^A, (v_1^A)^{-1} V^A / \pi^\infty)$  is divisible by  $\pi^{a(n)-1}$ , but not divisible by  $\pi^{a(n)}$ .

Finally, all we need to do is to check that the 1-cocycle

$$(3.0.18) \quad (\eta_R - \eta_L) \left( (v_1^A)^n / \pi^{a(n)} \right) = \binom{n}{b(n)} (v_1^A)^{n-b(n)} \pi^{b(n)-a(n)} (t_1^A)^{b(n)} \\ \in (v_1^A)^{-1} V^A / \pi \otimes_{V^A} V^A T,$$

in the cobar complex for  $(V^A, V^A T)$  with coefficients in  $(v_1^A)^{-1} V^A / \pi$ , is not a 1-coboundary. From naturality of the cobar complex, if the 1-cocycle 3.0.18 is a 1-coboundary in the cobar complex for  $(V^A, V^A T)$  with coefficients in  $(v_1^A)^{-1} V^A / \pi$ , then its image in the cobar

complex for  $(\mathbb{F}_q[(v_1^A)^{\pm 1}], \Sigma^A(1))$  with coefficients in  $\mathbb{F}_q[(v_1^A)^{\pm 1}]$  is a 1-coboundary. (In fact, the converse is also true, due to the Morava-Miller-Ravenel-type isomorphism

$$\mathrm{Ext}_{(V^A, V^A T)}^{*,*}(V^A, (v_1^A)^{-1} V^A / \pi) \cong \mathrm{Ext}_{(\mathbb{F}_q[(v_1^A)^{\pm 1}], \Sigma^A(1))}^{*,*}(\mathbb{F}_q[(v_1^A)^{\pm 1}], \mathbb{F}_q[(v_1^A)^{\pm 1}]),$$

proved in [20].)

I claim that this 1-cocycle  $\binom{n}{b(n)}(v_1^A)^{n-b(n)}\pi^{b(n)-a(n)}(t_1^A)^{b(n)}$  cannot be a coboundary. The proof is trivially easy: in the cobar complex for  $(\mathbb{F}_q[(v_1^A)^{\pm 1}], \Sigma^A(1))$ , the 0-cochains are simply elements of  $\mathbb{F}_q[(v_1^A)^{\pm 1}]$ , and given some element  $x \in \mathbb{F}_q[(v_1^A)^{\pm 1}]$ , its coboundary  $\delta^0(x)$  is simply  $\delta^0(x) = \eta_R(x) - \eta_L(x)$ . Since  $\eta_R(v_1^A) = v_1^A - (\pi - \pi^q)t_1^A$ , we have that  $\eta_R(v_1^A) = 0$  modulo  $\pi$ . So  $\delta^0((v_1^A)^m) = \eta_R((v_1^A)^m) - \eta_L((v_1^A)^m) = 0$  for all  $m$ , so  $\binom{n}{b(n)}(v_1^A)^{n-b(n)}\pi^{b(n)-a(n)}(t_1^A)^{b(n)}$  is not a coboundary.

Consequently, for all  $n$ , we have the isomorphism of  $A$ -modules

$$\mathrm{Ext}_{(V^A, V^A T)}^{0, 2n(q-1)}(V^A, (v_1^A)^{-1} V^A / \pi^\infty) \cong A / \pi^{a(n)},$$

generated as an  $A$ -module by the 0-cocycle  $(v_1^A)^n / \pi^{a(n)}$ .

Now all that remains is to check that none of these 0-cocycles supports a nonzero  $d_1$ -differential in the formal  $A$ -module chromatic spectral sequence, since we already observed that, for dimensional reasons, no other differentials can interact with the height 1 layer in this spectral sequence. Computing  $d_1$  differentials in this spectral sequence is straightforward: if  $x$  is a class in

$$\mathrm{Ext}_{(V^A, V^A T)}^{s,t}(V^A, (v_h^A)^{-1} V^A / (\pi^\infty, (v_1^A)^\infty, \dots, (v_{h-1}^A)^\infty))$$

represented by a cobar complex  $s$ -cocycle

$$\bar{x} \in (v_h^A)^{-1} V^A / (\pi^\infty, (v_1^A)^\infty, \dots, (v_{h-1}^A)^\infty) \otimes_{V^A} V^A T \otimes_{V^A} \dots \otimes_{V^A} V^A T,$$

then applying the chromatic complex coboundary

$$\delta : (v_h^A)^{-1} V^A / (\pi^\infty, (v_1^A)^\infty, \dots, (v_{h-1}^A)^\infty) \rightarrow (v_{h+1}^A)^{-1} V^A / (\pi^\infty, (v_1^A)^\infty, \dots, (v_{h-1}^A)^\infty, (v_h^A)^\infty)$$

to the leftmost tensor factor in  $\bar{x}$  yields a cobar complex  $s$ -cocycle

$$d(\bar{x}) \in (v_{h+1}^A)^{-1} V^A / (\pi^\infty, (v_1^A)^\infty, \dots, (v_{h-1}^A)^\infty, (v_h^A)^\infty) \otimes_{V^A} V^A T \otimes_{V^A} \dots \otimes_{V^A} V^A T,$$

and the cobar complex  $d_1$ -differential  $d_1(x)$  is the cohomology class of this cocycle  $d(\bar{x})$ .

This process is very easy for our 0-cocycles  $(v_1^A)^n / \pi^{a(n)}$ : the relevant chromatic complex coboundary is the composite

$$\delta : (v_1^A)^{-1} V^A / \pi^\infty \rightarrow (v_2^A)^{-1} V^A / (\pi^\infty, (v_1^A)^\infty)$$

of the projection

$$(3.0.19) \quad (v_1^A)^{-1} V^A / \pi^\infty \twoheadrightarrow V^A / (\pi^\infty, (v_1^A)^\infty)$$

with the localization map

$$V^A / (\pi^\infty, (v_1^A)^\infty) \hookrightarrow (v_2^A)^{-1} V^A / (\pi^\infty, (v_1^A)^\infty).$$

If  $n \geq 0$ , then  $(v_1^A)^n / \pi^{a(n)}$  maps to zero under the projection map 3.0.19. Consequently, the  $d_1$ -differential is zero on  $\mathrm{Ext}_{(V^A, V^A T)}^{0, 2n(q-1)}(V^A, (v_1^A)^{-1} V^A / \pi^\infty)$  as long as  $n \geq 0$ .

If  $n < 0$ , the situation is even easier to understand: since  $V^A$  and  $V^A T$  are both concentrated in nonnegative grading degrees,  $\mathrm{Ext}_{(V^A, V^A T)}^{s,t}(V^A, V^A) \cong 0$  for  $t < 0$ , since a nonzero class in  $\mathrm{Ext}_{(V^A, V^A T)}^{s,t}(V^A, V^A)$  for  $t < 0$  would have to be represented by a cocycle in the cobar complex of negative total grading degree. Consequently, if  $n < 0$ , then any nonzero class

in  $\text{Ext}_{(V^A, V^A T)}^{0, 2n(q-1)}(V^A, (v_1^A)^{-1} V^A / \pi^\infty)$  must support a differential in the formal  $A$ -module chromatic spectral sequence. As we already observed, the only possible differentials interacting with the height 1 layer are possible  $d_1$ -differentials from the height 1 layer to the height 2 layer; hence every one of the classes  $(v_1^A)^n / \pi^{a(n)}$  for  $n < 0$  supports a  $d_1$ -differential, and these classes are all “gone” by the time we reach the formal  $A$ -module chromatic  $E_2$ -term.

Consequently,

$$\text{Ext}_{(V^A, V^A T)}^{0, 2n(q-1)}(V^A, (v_1^A)^{-1} V^A / \pi^\infty) \cong A / \pi^{a(n)}$$

for all  $n > 0$ .

Now by Lemma 3.0.13, the Ext-groups predicted by Ravenel’s Local Conjecture are the Ext-groups we have just computed.  $\square$

The next result, Corollary 3.0.17, is just a slightly different formulation of Theorem 3.0.16 which will be convenient to refer to when we begin proving the Global Conjecture.

**Corollary 3.0.17.** *Suppose that  $K/\mathbb{Q}_p$  is a finite field extension of ramification degree  $e$ , and suppose that  $\log_p(\frac{e}{p-1})$  is not an integer; i.e.,  $\frac{e}{p-1}$  is not a power of  $p$ . Let  $A$  be the ring of integers in  $K$ , then, for each  $n \in \mathbb{N}$ , we have isomorphisms of  $A$ -modules  $\text{Ext}_{(V^A, V^A T)}^{1, 2n+1}(V^A, V^A) \cong 0$  and*

$$\text{Ext}_{(V^A, V^A T)}^{1, 2n}(V^A, V^A) \cong A/H_n,$$

where  $H_n$  is the ideal in  $A$  generated by all elements of the form  $(a^n - 1)$  with  $a \in A^\times$ .

Equivalently, in terms of the moduli stack  $\mathcal{M}_{fmA}$  of one-dimensional formal  $A$ -modules over  $\text{Spec } A$ , and with notation as in Conventions 2.0.10: for each  $n \in \mathbb{N}$ , we have isomorphisms of  $A$ -modules  $\text{Ext}_{(V^A, V^A T)}^{1, 2n+1}(V^A, V^A) \cong 0$  and

$$H_{fl}^{1, 2n}(\mathcal{M}_{fmA}; \mathcal{O}) \cong A/H_n,$$

where  $H_n$  is the ideal in  $A$  generated by all elements of the form  $(a^n - 1)$  with  $a \in A^\times$ .

*Proof.* Let  $f$  be the residue degree of  $K/\mathbb{Q}_p$ , and let  $\pi$  be a uniformizer for  $A$ . If  $p^f - 1$  does not divide  $n$ , then there exists some nonzero element  $x$  in the residue field  $A/\pi$  such that  $x^n \neq 1$ , and hence there exists some element  $\bar{x} \in A^\times$  such that  $\bar{x}^n - 1$  has nonzero reduction modulo  $\pi$ , i.e.,  $\bar{x}^n - 1 \in A^\times$ . Hence  $A/H_n \cong 0$ , which is the value of  $\text{Ext}_{(V^A, V^A T)}^{1, 2n}(V^A, V^A)$  predicted by Theorem 3.0.16.

On the other hand, if  $p^f - 1$  divides  $n$ , then for any unit  $x \in A^\times$ , we have that  $x^{p^f - 1}$  is congruent to 1 modulo  $\pi$ , and conversely, if  $y \in A$  is congruent to 1 modulo  $\pi$ , then  $y$  has a  $(p^f - 1)$ st root  $x \in A^\times$ , by Hensel’s Lemma; consequently  $A/H_n$  coincides with  $A/I_{n/(p^f - 1)}$ , the value of  $\text{Ext}_{(V^A, V^A T)}^{1, 2n}(V^A, V^A)$  predicted by Theorem 3.0.16.  $\square$

#### 4. $n$ -CONGRUING IDEALS AND A PRECISE STATEMENT OF THE GLOBAL CONJECTURE.

**4.1. The Hasse principle for  $n$ -congruing ideals.** Ravenel’s original statement of the Global Conjecture, which is included verbatim in the present paper as Conjecture 1.0.2, is phrased loosely enough that one has some leeway in interpreting the conjecture so that some version of it can be proven: the conjecture seems to be that, up to some small factor,  $\text{Ext}_{(L^A, L^A B)}^{1, 2m}(L^A, L^A)$  is isomorphic to  $A/I$  where  $I$  is an ideal with the property that, if  $a \in A$ , then  $a^N(a^n - 1) \in I$  for some  $N \in \mathbb{N}$ . Ravenel’s statement does not make it clear exactly which ideal with that property  $I$  ought to be, however, or even if there is more than one such ideal. So at this point it is natural to make the following definition:



**Definition 4.1.1.** *Let  $A$  be a commutative ring and let  $n$  be a positive integer. We say that an ideal  $I$  in  $A$  is  $n$ -congruing if, for every  $a \in A$ , there exists some  $N \in \mathbb{N}$  such that  $a^N(a^n - 1) \in I$ .*

Our first task, in proving the Global Conjecture, is to make the conjecture precise. The most convincing way to make it precise would be to show that there is in fact a “universal”  $n$ -congruing ideal among the collection of all the  $n$ -congruing ideals in  $A$ . In that case, the “universal”  $n$ -congruing ideal would be the right one to appear in the statement of the Global Conjecture. Notice there always exists at least one (trivial) choice of  $n$ -congruing ideal of  $A$ , namely  $A$  itself. What is less clear is whether there are others, and in particular, whether there exists a *minimal*  $n$ -congruing ideal in  $A$ . (Note that the zero ideal is not  $n$ -congruing unless  $n = 0$ .)

In this section I prove that there is, in fact, a universal (specifically, *minimal*) ideal  $I$  in  $A$  with the property that if  $a \in A$ , then  $a^N(a^n - 1) \in I$  for some  $N \in \mathbb{N}$ ; this is a consequence of Theorem 4.1.7, which is also interesting in its own right, as it establishes a Hasse principle for  $n$ -congruing ideals. Consequently Conjecture 1.0.5 (and its stronger form, Conjecture 1.0.6) is the desired precise form of Ravenel’s Global Conjecture. In the next section (specifically Corollaries 5.0.9 and 5.0.10) the reader can find proofs that Conjecture 1.0.5 is actually true, and that Conjecture 1.0.6 is true when certain hypotheses are met.

**Lemma 4.1.2.** *Let  $A$  be a Dedekind domain, let  $t \in \mathbb{N}$ , and let  $\mathfrak{p}$  be a maximal ideal in  $A$ . Let  $H_t^{\mathfrak{p}}$  denote the ideal in  $\hat{A}_{\mathfrak{p}}$  generated by all elements of the form  $x^t - 1$  for  $x \in \hat{A}_{\mathfrak{p}}^{\times}$ . Let  $u_{\mathfrak{p},t}$  denote the  $\mathfrak{p}$ -adic valuation of the ideal  $H_t^{\mathfrak{p}}$ , i.e.,  $u_{\mathfrak{p},t}$  is the greatest integer  $u$  such that  $H_t^{\mathfrak{p}} \subseteq \mathfrak{p}^u \subseteq \hat{A}_{\mathfrak{p}}$ .*

*Then, for all  $k \in A$  not contained in  $\mathfrak{p}$ , we have the inequality*

$$(4.1.1) \quad v_{\mathfrak{p}}(k^t - 1) \geq u_{\mathfrak{p},t},$$

*and furthermore, there exists some  $k \in A$  not contained in  $\mathfrak{p}$  such that*

$$(4.1.2) \quad v_{\mathfrak{p}}(k^t - 1) = u_{\mathfrak{p},t}.$$

*Proof.* To produce the inequality 4.1.1, fix  $k \in A$  not contained in  $\mathfrak{p}$ , and observe that the image  $\bar{k}$  of  $k$  under the completion map  $A \rightarrow \hat{A}_{\mathfrak{p}}$  is a unit. Consequently

$$\bar{k}^t - 1 \in H_t^{\mathfrak{p}} \subseteq \mathfrak{p}^u,$$

so  $\bar{k}^t - 1$  has  $\mathfrak{p}$ -adic valuation at least  $u$ .

To produce the equality 4.1.2, we choose an element  $k \in A$  whose image  $\bar{k} \in \hat{A}_{\mathfrak{p}}$  is a unit and makes  $v_{\mathfrak{p}}(\bar{k}^t - 1)$  as small as possible, i.e.,  $\bar{k}^t - 1$  is an element of minimal  $\mathfrak{p}$ -adic valuation in  $H_t^{\mathfrak{p}}$ . (It is possible to make such a choice of  $k$  because we can simply start with an element  $x \in \hat{A}_{\mathfrak{p}}^{\times}$  such that  $x^t - 1$  is of minimal  $\mathfrak{p}$ -adic valuation in  $H_t^{\mathfrak{p}}$ , and then create an element  $\bar{k}$  of  $\hat{A}_{\mathfrak{p}}$  by truncating  $x$  by setting all the  $\mathfrak{p}^m$ -coefficients to zero for  $m \gg 0$ , in the  $\mathfrak{p}$ -adic expansion of  $x$ . Then  $\bar{k}$  is still a unit in  $\hat{A}_{\mathfrak{p}}$  since its mod  $\mathfrak{p}$  reduction is still nonzero, and  $\bar{k}^t - 1$  has the same  $\mathfrak{p}$ -adic valuation as  $x^t - 1$  if  $m$  was chosen to be large relative to  $t$ . Furthermore,  $\bar{k}$  has only finitely many nonzero coefficients in its  $\mathfrak{p}$ -adic expansion, so  $\bar{k}$  is the image in  $\hat{A}_{\mathfrak{p}}$  of some element  $k \in A$  with the desired properties.) Then  $v_{\mathfrak{p}}(k^t - 1) = v_{\mathfrak{p}}(\bar{k}^t - 1) = u_{\mathfrak{p},t}$ , as desired.  $\square$

**Lemma 4.1.3.** *Let  $\hat{A}_{\mathfrak{p}}$  be a complete (that is, complete in the  $\mathfrak{p}$ -adic topology) discrete valuation ring with maximal ideal  $\mathfrak{p}$ , and let  $n$  be a positive integer. Let  $H_n^{\mathfrak{p}}$  denote the*

ideal in  $\hat{A}_p$  generated by all elements of the form  $x^n - 1$  for  $x \in \hat{A}_p^\times$ . Then an ideal  $I$  in  $\hat{A}_p$  is  $n$ -congruing if and only if  $I$  contains  $H_n^p$ .

*Proof.* Suppose  $I$  is an ideal in  $\hat{A}_p$  containing  $H_n^p$ , and let  $a \in A$ . If  $a \notin \mathfrak{p}$ , then  $a$  is a unit in  $\hat{A}_p$ , so  $a^n - 1 \in H_n^p \subseteq I$ . If, on the other hand,  $a \in \mathfrak{p}$ , then some power of  $a$  is contained in  $I$ , since every ideal in a discrete valuation ring is a power of the maximal ideal, so  $I = \mathfrak{p}^m$  for some  $m$ , and then  $a^m \in \mathfrak{p}^m = I$ . Hence every ideal in  $\hat{A}_p$  containing  $H_n^p$  is  $n$ -congruing.

On the other hand, suppose that  $I$  is an  $n$ -congruing ideal in  $\hat{A}_p$ , and suppose that  $x \in \hat{A}_p^\times$ . Then there exists some integer  $N$  such that  $x^N(x^n - 1) \in I$ , since  $I$  is  $n$ -congruing. But  $x^N$  is a unit for all  $N$ , so  $x^N(x^n - 1) \in I$  if and only if  $x^n - 1 \in I$ . So  $x^n - 1 \in I$ . Hence  $H_n^p \subseteq I$ . Hence every  $n$ -congruing ideal in  $\hat{A}_p$  contains  $H_n^p$ .  $\square$

**Lemma 4.1.4.** *Let  $A$  be a Dedekind domain, and let  $n$  be a positive integer. Let  $k$  be a nonzero element of  $\hat{A}_p$ , and let  $k(j)$  denote the truncation of  $k$  in which we replace the  $\mathfrak{p}^m$ -coefficient in the  $\mathfrak{p}$ -adic expansion of  $k$  by zero, for all  $m > j$ . Put another way: the ring  $\hat{A}_p$  is defined as the limit  $\hat{A}_p = \lim_{i \rightarrow \infty} A/\mathfrak{p}^i$ , so to specify an element  $k$  in  $\hat{A}_p$ , we can specify a sequence of elements*

$$(k_0, k_1, k_2, \dots : \forall i \ k_i \in A/\mathfrak{p}^i, \text{ and } k_{i+1} \equiv k_i \pmod{\mathfrak{p}^i}).$$

By  $k(j)$  we mean the element of  $\hat{A}_p$  given by the sequence

$$(k_0, k_1, k_2, \dots, k_{j-2}, k_{j-1}, k_j, k_j, k_j, \dots).$$

Let  $I$  be an ideal in  $\hat{A}_p$ . Suppose that, for each positive integer  $j$ , there exists some  $N \in \mathbb{N}$  such that  $k(j)^N(k(j)^n - 1) \in I$ . Let  $N(j)$  be the smallest natural number  $N$  such that  $k(j)^N(k(j)^n - 1) \in I$ .

Then the sequence of natural numbers

$$(4.1.3) \quad N(1), N(2), N(3), \dots$$

is eventually constant.

*Proof.* Since  $\hat{A}_p$  is a discrete valuation ring, whether  $k(j)^N(k(j)^n - 1)$  is contained in  $I$  is determined entirely by the  $\mathfrak{p}$ -adic valuations of  $k(j)^N(k(j)^n - 1)$  and  $I$ , i.e.,  $N(j)$  is the least  $N$  such that

$$v_p(k(j)^N(k(j)^n - 1)) \geq v_p(I),$$

where we write  $v_p(I)$  for the least  $\mathfrak{p}$ -adic valuation of an element in  $I$ , i.e.,

$$v_p(I) = \min\{v_p(i) : i \in I\}.$$

Consequently,  $N(j)$  is the least  $N$  such that

$$Nv_p(k(j)) + v_p(k(j)^n - 1) \geq v_p(I),$$

i.e.,  $N(j)$  is the natural number ceiling

$$(4.1.4) \quad N(j) = \lceil \frac{v_p(I) - v_p(k(j)^n - 1)}{v_p(k(j))} \rceil.$$

(By the ‘‘natural number ceiling’’ of a real number  $x$  I mean the least natural number which is greater than or equal to  $x$ .) By basic properties of analysis in a complete discrete valuation ring,  $\lim_{j \rightarrow \infty} k(j) = k$ , so the sequence of natural numbers

$$(4.1.5) \quad (v_p(k(1)), v_p(k(2)), v_p(k(3)), \dots)$$

(which is indeed a sequence of natural numbers, never taking the value  $\infty$ , since we assumed that  $k \neq 0$ ), is eventually constant, converging to  $v_p(k)$ .

Continuity of addition and the  $n$ th power function on  $\hat{A}_p$  also gives us that

$$(4.1.6) \quad \lim_{j \rightarrow \infty} (k(j)^n - 1) = \left( \lim_{j \rightarrow \infty} k(j) \right)^n - 1 = k^n - 1.$$

The sequence

$$(4.1.7) \quad (v_p(k(1)^n - 1), v_p(k(2)^n - 1), v_p(k(3)^n - 1), \dots)$$

is a sequence of extended natural numbers, i.e., a sequence of elements of the set  $\mathbb{N} \cup \{\infty\}$ , and it is again (due to 4.1.6) eventually constant, converging to  $v_p(k^n - 1)$ .

Consequently one of two things happens:

- If  $v_p(k^n - 1) = \infty$  (i.e., if  $k^n - 1 = 0$ ), then the sequence 4.1.7 is an eventually constant sequence of extended natural numbers converging to  $\infty$ , and so  $N(j) = 0$  for all sufficiently large  $j$ . Consequently sequence 4.1.3 is eventually constant (and converges to zero).
- If  $v_p(k^n - 1) \neq \infty$  (i.e., if  $k^n - 1 \neq 0$ ), then the sequences 4.1.5 and 4.1.7 are both eventually constant, hence from equation 4.1.4 we see that the sequence 4.1.3 is eventually constant.

□

**Definition 4.1.5.** We will say that a Dedekind domain  $A$  is Minkowski if  $A$  satisfies the following two conditions:

- All the residue fields of  $A$  are finite, i.e.,  $A/\mathfrak{p}$  is finite for all maximal ideals  $\mathfrak{p}$ .
- For each natural number  $N$ , there exist only finitely many maximal ideals  $\mathfrak{p}$  of  $A$  such that  $N(\mathfrak{p}) < N$ . (Recall that the “absolute norm”  $N(\mathfrak{p})$  of a maximal ideal  $\mathfrak{p}$  is simply the cardinality of the residue field  $A/\mathfrak{p}$ .)

For example, the ring of integers in any number field is Minkowski, and any localization of a Minkowski Dedekind domain is also Minkowski.

**Lemma 4.1.6.** Let  $A$  be a Minkowski Dedekind domain, and let  $n$  be a positive integer. For each maximal ideal  $\mathfrak{p}$  in  $A$ , let  $H_n^{\mathfrak{p}}$  denote the ideal in  $\hat{A}_{\mathfrak{p}}$  generated by all elements of the form  $x^n - 1$  for  $x \in \hat{A}_{\mathfrak{p}}^{\times}$ . Let

$$pr_n : A \rightarrow \prod_{\text{maximal } \mathfrak{p} \subseteq A} \hat{A}_{\mathfrak{p}}/H_n^{\mathfrak{p}}$$

be the ring homomorphism whose  $\mathfrak{p}$ -component  $A \rightarrow \hat{A}_{\mathfrak{p}}/H_n^{\mathfrak{p}}$  is the composite of the completion map  $A \rightarrow \hat{A}_{\mathfrak{p}}$  with the modulo- $H_n^{\mathfrak{p}}$ -reduction map  $\hat{A}_{\mathfrak{p}} \rightarrow \hat{A}_{\mathfrak{p}}/H_n^{\mathfrak{p}}$ .

Then  $pr_n$  is surjective, and the quotient ring  $A/\ker pr_n$  is finite.

*Proof.* I claim that, for all but finitely many maximal ideals  $\mathfrak{p}$  in  $A$ , the quotient  $\hat{A}_{\mathfrak{p}}/H_n^{\mathfrak{p}}$  is the zero ring. (Of course, which of the maximal ideals  $\mathfrak{p}$  have nonzero quotient  $\hat{A}_{\mathfrak{p}}/H_n^{\mathfrak{p}}$  depends on the choice of  $n$ ; but for any given choice of  $n$ , there are only finitely many such maximal ideals  $\mathfrak{p}$ .) Suppose  $\mathfrak{p}$  is a maximal ideal in  $A$  such that  $n$  is not divisible by  $N(\mathfrak{p}) - 1 = \#((A/\mathfrak{p})^{\times})$ . Then there exists an element  $x$  in  $\hat{A}_{\mathfrak{p}}^{\times}$  such that  $x^n$  is not congruent to 1 modulo  $\mathfrak{p}$ , hence  $x^n - 1$  is a unit in  $\hat{A}_{\mathfrak{p}}$ , hence the ideal  $H_n^{\mathfrak{p}} \subseteq \hat{A}_{\mathfrak{p}}$  contains a unit. Consequently  $\hat{A}_{\mathfrak{p}}/H_n^{\mathfrak{p}}$  is only nonzero when  $n$  is divisible by  $\#(A/\mathfrak{p}) - 1$ , which only happens for finitely many  $\mathfrak{p}$ , since  $A$  is Minkowski.

Clearly the ideal  $H_n^{\mathfrak{p}}$  is also never the zero ideal, hence  $A/H_n^{\mathfrak{p}}$  is always finite. Now the Chinese Remainder Theorem implies surjectivity of the map  $pr_n$ , and hence  $\prod_{\text{maximal } \mathfrak{p} \subseteq A} \hat{A}_{\mathfrak{p}}/H_n^{\mathfrak{p}} \cong A/\ker pr_n$  is finite.  $\square$

**Theorem 4.1.7. (Hasse principle for  $n$ -congruing ideals.)** *Let  $A$  be a Minkowski Dedekind domain, and let  $n$  be a positive integer. Let  $pr_n$  be as in Lemma 4.1.6.*

*Then the following conditions on an ideal  $I$  of  $A$  are equivalent:*

- (1)  $I$  is  $n$ -congruing.
- (2) For all maximal ideals  $\mathfrak{p}$  of  $A$ , the ideal  $\hat{I}_{\mathfrak{p}}$  of  $\hat{A}_{\mathfrak{p}}$  is  $n$ -congruing.
- (3)  $\ker pr_n \subseteq I$ .

*Consequently, there exists a unique minimal  $n$ -congruing ideal of  $A$ , namely,  $\ker pr_n$ .*

*Proof.* • That condition 2 is equivalent to condition 3 is Lemma 4.1.3.

- That condition 3 implies condition 1 is as follows: suppose that  $I$  is an ideal of  $A$ , and suppose that  $\ker pr_n \subseteq I$ . For each maximal ideal  $\mathfrak{p}$  of  $A$ , we write  $c_{\mathfrak{p}}$  for the completion map  $c_{\mathfrak{p}} : A \rightarrow \hat{A}_{\mathfrak{p}}$ . Then  $I$  contains every element  $a$  of  $A$  with the property that, for all maximal ideals  $\mathfrak{p}$  of  $A$ ,  $c_{\mathfrak{p}}(a)$  is divisible by  $x^n - 1$  for some unit  $x \in \hat{A}_{\mathfrak{p}}^{\times}$ .

Consequently, if  $a \in A^{\times}$ , then for each maximal ideal  $\mathfrak{p}$  of  $A$ ,  $c_{\mathfrak{p}}(a)^n \in \hat{A}_{\mathfrak{p}}^{\times}$  and hence  $c_{\mathfrak{p}}(a)^n - 1$  is divisible by  $x^n - 1$  for some unit  $x \in \hat{A}_{\mathfrak{p}}^{\times}$  (namely,  $x = c_{\mathfrak{p}}(a)$ ). This is the case for all maximal ideals  $\mathfrak{p}$  of  $A$ , so  $a^n - 1 \in I$ .

On the other hand, if  $a \notin A^{\times}$ , then either  $a^N(a^n - 1) \in I$  for some  $N \in \mathbb{N}$ , or the quotient ring  $A/I$  has an element  $a$  of infinite order. But this second case is impossible, since  $I \supseteq \ker pr_n$  and so  $A/I$  is a quotient of  $A/\ker pr_n$ , which is finite by Lemma 4.1.6. So, if  $a \notin A^{\times}$ , then  $a^N(a^n - 1) \in I$ .

Consequently, for all  $a \in A$ ,  $a^N(a^n - 1) \in I$  for some  $N \in \mathbb{N}$ . Hence  $I$  is  $n$ -congruing.

- That condition 1 implies condition 2 is as follows: suppose that  $I$  is an  $n$ -congruing ideal in  $A$ . Choose a maximal ideal  $\mathfrak{p}$  of  $A$ . We continue to write  $c_{\mathfrak{p}}$  for the completion map  $c_{\mathfrak{p}} : A \rightarrow \hat{A}_{\mathfrak{p}}$ . Let  $k \in \hat{A}_{\mathfrak{p}}$  be nonzero. Then we apply Lemma 4.1.4 to  $k$ : each  $k(j) \in \hat{A}_{\mathfrak{p}}$  for  $j \in \mathbb{N}$  is in the image of the completion map  $c_{\mathfrak{p}}$ , consequently for each  $j$  there exists some  $N \in \mathbb{N}$  such that  $k(j)^N(k(j)^n - 1) \in I$ , since  $I$  is  $n$ -congruing. So the assumptions made in the statement of Lemma 4.1.4 are satisfied in our case.

Now Lemma 4.1.4 tells us that, if we write  $N(j)$  for the least natural number  $N$  such that  $k(j)^N(k(j)^n - 1) \in I$ , then the sequence  $(N(1), N(2), N(3), \dots)$  is eventually constant. So we can choose some natural number  $j$  such that  $N(j) = N(j + c)$  for all  $c \geq 0$ . Then  $k(j + c)^{N(j)}(k(j + c)^n - 1) \in \hat{I}_{\mathfrak{p}}$  for all  $c \geq 0$ . Taking the limit over  $c$ , we have that

$$\begin{aligned} \lim_{c \rightarrow \infty} \left( k(j + c)^{N(j)}(k(j + c)^n - 1) \right) &= \left( \lim_{c \rightarrow \infty} k(j + c) \right)^{N(j)} \left( \lim_{c \rightarrow \infty} k(j + c)^n - 1 \right) \\ &= k^{N(j)}(k^n - 1) \in \hat{I}_{\mathfrak{p}}, \end{aligned}$$

with the containment in  $\hat{I}_{\mathfrak{p}}$  because  $\hat{I}_{\mathfrak{p}}$  is closed (again, a standard fact about nonzero ideals in a complete discrete valuation ring with finite residue field, or more generally, finite-index subgroups of a profinite group: they are both open and closed) and hence contains all limits of sequences contained in itself.

Consequently, for any  $k \in \hat{A}_p$ , there exists some integer  $N$  (namely, the  $N(j)$  produced above) such that  $k^{N(j)}(k^n - 1) \in \hat{I}_p \subseteq \hat{A}_p$ . So  $\hat{I}_p$  is an  $n$ -congruing ideal in  $\hat{A}_p$ .  $\square$

Consequently, “universal” (in the sense that the  $n$ -congruing ideal involved in the conjecture is the “universal” one, i.e., the minimal one) precise versions of Ravenel’s Global Conjecture, Conjecture 1.0.2, are Conjectures 1.0.5 and 1.0.6. I prove many cases of Conjecture 1.0.6 in Corollary 5.0.10, and I prove completely (handling all cases) the weaker Conjecture 1.0.5, in Corollary 5.0.9.

**4.2. Relation to Adams’s  $h(f, t)$ .** In [1], Adams makes the following definition:

**Definition 4.2.1.** *Let  $f : \mathbb{Z} \rightarrow \mathbb{N}$  be a function. Then we define  $h(f, t)$  to be the greatest common divisor of the set of integers*

$$\{k^{f(k)}(k^n - 1) : k \in \mathbb{Z}\}.$$

Adams then uses his numbers  $h(f, n)$  to prove essentially the  $K = \mathbb{Q}$  case of our Theorem 5.0.8, that is, the original computation which motivated Ravenel to make the Global Conjecture (and indeed, the only case of the Global Conjecture which was known to hold, before the results of the present paper were obtained).

In our setting, where  $K$  is a finite extension of  $\mathbb{Q}$ , it is somewhat problematic to try to use Adams’s numbers  $h(f, n)$ , because really what we are after is the ideal that they generate, and when the class number of the ring of integers of  $K$  is greater than one, the ideal one really wants to work with may not be principal. So it is reasonable to instead make the definition of the ideals  $H(f, t)$  in the following proposition:

**Proposition 4.2.2.** *Let  $K/\mathbb{Q}$  be a finite extension with ring of integers  $A$ . Let  $f : A \rightarrow \mathbb{N}$  be a function. If  $t \in \mathbb{N}$ , we let  $H(f, t)$  denote the ideal of  $A$  which is the sum, over all  $k \in A$ , of the principal ideals  $(k^{f(k)}(k^t - 1))$ , i.e.,*

$$H(f, t) = \sum_{k \in A} (k^{f(k)}(k^t - 1))$$

Let

$$pr_t : A \rightarrow \prod_{\text{maximal } \mathfrak{p} \subseteq A} \hat{A}_p/H_t^{\mathfrak{p}}$$

be the ring homomorphism whose  $\mathfrak{p}$ -component  $A \rightarrow \hat{A}_p/H_t^{\mathfrak{p}}$  is the composite of the completion map  $A \rightarrow \hat{A}_p$  with the modulo- $H_t^{\mathfrak{p}}$ -reduction map  $\hat{A}_p \rightarrow \hat{A}_p/H_t^{\mathfrak{p}}$ .

Then  $\ker pr_t \subseteq H(f, t)$ .

*Proof.* Choose a function  $f : A \rightarrow \mathbb{N}$ , and let  $\mathfrak{p}$  be a maximal ideal in  $A$ . Let  $w_{\mathfrak{p}}$  be the  $\mathfrak{p}$ -adic valuation of the ideal  $H(f, t)$ , i.e.,  $w_{\mathfrak{p}}$  is the greatest integer  $w$  such that  $H(f, t) \subseteq \mathfrak{p}^w$ . By Lemma 4.1.2, there exists some  $k$  not in  $\mathfrak{p}$  such that

$$v_{\mathfrak{p}}(k^{f(t)}(k^t - 1)) = v_{\mathfrak{p}}(k^t - 1) = u_{\mathfrak{p}, t},$$

and all other choices of  $k_0$  not in  $\mathfrak{p}$  yield

$$v_{\mathfrak{p}}(k_0^{f(t)}(k_0^t - 1)) = v_{\mathfrak{p}}(k_0^t - 1) \geq u_{\mathfrak{p}, t}.$$

The valuation behavior of elements  $k_0 \in \mathfrak{p}$  is irrelevant now: since there exists some  $k$  such that  $v_{\mathfrak{p}}(k^{f(t)}(k^t - 1)) = u_{\mathfrak{p}, t}$ , we have

$$(4.2.1) \quad v_{\mathfrak{p}}(H(f, t)) = w_{\mathfrak{p}, t} \leq u_{\mathfrak{p}, t} = v_{\mathfrak{p}}(H_t^{\mathfrak{p}}),$$

which is what we want.

Since we have inequality 4.2.1 for all maximal primes  $\mathfrak{p}$  in  $A$ , we then have that  $\ker pr_t \subseteq H(f, t)$ .  $\square$

The ideals  $H(f, n)$  still are not central to the ideas in the present paper, but I include Proposition 4.2.2 and the following Corollary 4.2.3 to make more clear what the relationship is between our  $n$ -congruence ideals and the numbers  $h(f, n)$  defined by Adams.

**Corollary 4.2.3.** *Let  $K/\mathbb{Q}$  be a finite extension with ring of integers  $A$ , and let  $n$  be a positive integer. For every function  $f : A \rightarrow \mathbb{N}$ , the ideal  $H(f, n)$  satisfies the equivalent conditions of Theorem 4.1.7. In particular,  $H(f, n)$  is  $n$ -congruence.*

*Proof.* By Proposition 4.2.2,  $H(f, n)$  satisfies condition 3 of Theorem 4.1.7.  $\square$

I do not know if the converse to Corollary 4.2.3 holds. That is, I do not know if every  $n$ -congruence ideal in  $A$  is of the form  $H(f, n)$  for some function  $f : A \rightarrow \mathbb{N}$ . It seems likely that this is so (Adams proves it for  $A = \mathbb{Z}$ , in [1]), but it is not necessary for the arguments in the present paper, so I do not pursue the question here.

## 5. PROOF OF RAVENEL'S GLOBAL CONJECTURE.

In order to prove the Global Conjecture, I need to invoke A. Pearlman's proof of Ravenel's Local-Global Conjecture (originally conjectured in [20], but note that the statement below is slightly different from Ravenel's statement in [20], which seems to have a typographical error in the coefficients), from Pearlman's unpublished thesis [16]:

**Theorem 5.0.4. (Ravenel's Local-Global conjecture.)** *Suppose  $K/\mathbb{Q}$  is a finite Galois extension, and let  $A$  be the ring of integers of  $K$ . Then, for any  $L^A B$ -comodule  $M$  and any choice of prime  $\mathfrak{p}$  in  $A$ , we have an isomorphism of graded  $A_{\mathfrak{p}}$ -modules:*

$$A_{\mathfrak{p}} \otimes_A \text{Ext}_{(L^A, L^A B)}^*(L^A, M) \cong \text{Ext}_{(V^{A_{\mathfrak{p}}}, V^{A_{\mathfrak{p}} T})}^*(V^{A_{\mathfrak{p}}}, V^{A_{\mathfrak{p}}} \otimes_{L^A} M).$$

*Proof.* See [16].  $\square$

Now we define a useful invariant of an extension  $K/\mathbb{Q}$ :

**Definition 5.0.5.** *Let  $K/\mathbb{Q}$  be a finite field extension with ring of integers  $A$ . Then by the prime-power-ramification discriminant  $\underline{\Delta}_{K/\mathbb{Q}}$  we mean the number  $\underline{\Delta}_{K/\mathbb{Q}} \in \mathbb{N}$  which is the product of all the prime numbers  $p$  with the property that there exists a prime ideal  $\mathfrak{p}$  of  $A$  over  $p$  with ramification degree  $e_{\mathfrak{p}}$  such that  $\log_p(\frac{e_{\mathfrak{p}}}{p-1})$  is an integer.*

**Example 5.0.6.** •  $\underline{\Delta}_{\mathbb{Q}/\mathbb{Q}} = 2$ , since  $\log_2(\frac{1}{2-1}) = 0$ , an integer, while  $\log_p(\frac{1}{p-1})$  is not an integer if  $p > 2$ .

- For quadratic  $K/\mathbb{Q}$ ,  $\underline{\Delta}_{K/\mathbb{Q}} = 6$  if 3 ramifies, and  $\underline{\Delta}_{K/\mathbb{Q}} = 2$  if 3 does not ramify.
- For cubic  $K/\mathbb{Q}$ , we have  $2 \mid \underline{\Delta}_{K/\mathbb{Q}}$  if and only if 2 does not ramify totally in  $K$ , and  $3 \mid \underline{\Delta}_{K/\mathbb{Q}}$  if and only if some prime over 3 has ramification degree 2 over 3. No other primes divide  $\underline{\Delta}_{K/\mathbb{Q}}$ .
- In particular, for cubic Galois  $K/\mathbb{Q}$ , we have  $\underline{\Delta}_{K/\mathbb{Q}} = 1$  if 2 ramifies in  $K$ , and  $\underline{\Delta}_{K/\mathbb{Q}} = 2$  otherwise.
- More generally, if  $\ell$  is an odd prime number and  $K/\mathbb{Q}$  is a Galois extension of degree  $\ell$ , then we have  $\underline{\Delta}_{K/\mathbb{Q}} = 1$  if 2 ramifies in  $K$ , and  $\underline{\Delta}_{K/\mathbb{Q}} = 2$  otherwise.
- Let  $K$  be the splitting field of  $x^5 + x + 2$  over  $\mathbb{Q}$ . Then  $K/\mathbb{Q}$  is Galois and  $[K : \mathbb{Q}] = 24$ . The only primes of  $\mathbb{Z}$  which ramify in the ring of integers of  $K$  are 2 and 349; the primes over 2 have ramification degree 3, and the primes over 349

have ramification degree 2, so the prime-power-ramification discriminant of  $K/\mathbb{Q}$  is 1. Furthermore, the Galois group  $G_{K/\mathbb{Q}}$  is an order 24 subgroup of the symmetric group on 5 letters, hence is nonabelian. So  $K$  is an example of a nonabelian Galois extension of  $\mathbb{Q}$  with trivial prime-power-ramification discriminant.

**Observation 5.0.7.** If  $K \neq \mathbb{Q}$ , then the prime-power-ramification discriminant divides two times the classical discriminant, that is,  $\underline{\Delta}_{K/\mathbb{Q}} \mid 2\Delta_{K/\mathbb{Q}}$ .

*Proof.* This follows from three easy observations: the primes that ramify in  $K$  divide the classical discriminant, no odd prime divides the prime-power-ramification discriminant unless it ramifies in  $K$ , and the prime-power-ramification discriminant is square-free.  $\square$

One can generalize Definition 5.0.5 by defining a “relative prime-power-ramification discriminant” for an extension of number fields  $L/K$ , which is an ideal in the ring of integers of  $K$ , but I do not know any examples of things you can do with the extra generality in the definition.

**Theorem 5.0.8.** Suppose  $K/\mathbb{Q}$  is a finite Galois extension, and let  $A$  be the ring of integers of  $K$ . Let  $\underline{\Delta}_{K/\mathbb{Q}}$  denote the prime-power-ramification discriminant of  $K/\mathbb{Q}$  (defined in Definition 5.0.5). Then, for each  $n \in \mathbb{N}$ ,

$$\mathrm{Ext}_{(L^A, L^A B)}^{1, 2n+1}(L^A, L^A) \cong 0,$$

and the  $A[\underline{\Delta}^{-1}]$ -module

$$\mathrm{Ext}_{(L^A, L^A B)}^{1, 2n}(L^A, L^A)[\underline{\Delta}^{-1}]$$

is isomorphic to the localized direct sum

$$\left( \bigoplus_{\max' l} l_{\mathfrak{p} \subseteq A} \hat{A}_{\mathfrak{p}} / H_n^{\mathfrak{p}} \right) [\underline{\Delta}^{-1}],$$

i.e., the localization (inverting  $\underline{\Delta}$ ) of the direct sum, over all maximal ideals  $\mathfrak{p}$  of  $A$ , of the modules  $\hat{A}_{\mathfrak{p}} / H_n^{\mathfrak{p}}$ , where  $H_n^{\mathfrak{p}}$  is defined as in Lemma 4.1.2.

*Proof.* First, a few easy observations:

- (1)  $\mathrm{Ext}_{(L^A, L^A B)}^{n, i}(L^A, L^A)$  vanishes for  $i$  odd, since  $L^A$  and  $L^A B$  are concentrated in even grading degrees. In particular,  $\mathrm{Ext}_{(L^A, L^A B)}^{1, i}(L^A, L^A) \cong 0$  for odd  $i$ , as claimed.
- (2) By Theorem 3.2 of [20],  $\mathrm{Ext}_{(L^A, L^A B)}^i(L^A, L^A)$  is a torsion  $A$ -module for all  $i > 0$ . In particular,  $\mathrm{Ext}_{(L^A, L^A B)}^1(L^A, L^A)$  is a torsion  $A$ -module, hence each of its summands  $\mathrm{Ext}_{(L^A, L^A B)}^{1, n}(L^A, L^A)$  is a torsion  $A$ -module.
- (3) For all  $i$ , the grading degree  $i$  summand  $(L^A B)^i$  of  $L^A B$  is a finitely generated  $A$ -module.
- (4) For all  $i$ ,  $\mathrm{Ext}_{(L^A, L^A B)}^{1, i}(L^A, L^A)$  is a subquotient of  $(L^A B)^i$ , since the cobar complex of  $(L^A, L^A B)$  computes  $\mathrm{Ext}_{(L^A, L^A B)}^{1, i}(L^A, L^A)$ .

Consequently  $\mathrm{Ext}_{(L^A, L^A B)}^{1, i}(L^A, L^A)$  is a finitely generated torsion  $A$ -module for all  $i$ . Since  $A$  is the ring of integers in a number field,  $A$  is a Dedekind domain, and now the classification of finitely generated modules over a Dedekind domain tells us that  $\mathrm{Ext}_{(L^A, L^A B)}^{1, i}(L^A, L^A)$  splits as a direct sum of finitely many cyclic submodules of the form  $A/\mathfrak{p}^n$ , for various prime ideals  $\mathfrak{p}$  of  $A$  and various positive integers  $n$ .

Now we use Pearlman's result. As a consequence of Theorem 5.0.4, for any prime ideal  $\mathfrak{p}$  of  $A$ , we have an isomorphism of graded  $\hat{A}_{\mathfrak{p}}$ -modules

$$\mathrm{Ext}_{(L^A, L^A B)}^*(L^A, M)_{\mathfrak{p}} \cong \mathrm{Ext}_{(V^{\hat{A}_{\mathfrak{p}}}, V^{\hat{A}_{\mathfrak{p}} T})}^*(V^{\hat{A}_{\mathfrak{p}}}, V^{\hat{A}_{\mathfrak{p}}} \otimes_{L^A} M).$$

(We do not distinguish between  $\hat{A}_p \otimes_A \text{Ext}_{(L^A, L^A B)}^*(L^A, M)$  and  $\text{Ext}_{(L^A, L^A B)}^*(L^A, M)_{\hat{p}}$  because  $A$  is Noetherian and  $\text{Ext}_{(L^A, L^A B)}^*(L^A, M)$  is a finitely generated  $A$ -module in each bidegree  $\text{Ext}^{i,j}$ , for the same reasons described above for  $\text{Ext}_{(V^A, V^A T)}^{1,i}(V^A, V^A)$ ; consequently  $p$ -adic completion on  $\text{Ext}_{(L^A, L^A B)}^*(L^A, M)$  coincides with taking the tensor product  $\hat{A}_p \otimes_A \text{Ext}_{(L^A, L^A B)}^*(L^A, M)$ . See Proposition 10.13 of [3] for the classical result on comparison of adic completion of modules to the base-change to the adic completion of the base ring.)

Now each cyclic summand  $A/\underline{q}^n$  of  $\text{Ext}_{(V^A, V^A T)}^{1,2i}(V^A, V^A)$  is annihilated by  $p$ -adic completion for all maximal ideals  $\mathfrak{p}$  of  $A$  except for one, namely, the prime  $\mathfrak{p} = \underline{q}$ . Furthermore  $A/\underline{q}^n$  is isomorphic to its own  $\underline{q}$ -adic completion. Consequently, for all  $i$ , the  $A$ -module  $\text{Ext}_{(L^A, L^A B)}^{1,2i}(L^A, L^A)$  splits as a direct sum of its own  $p$ -adic completions:

$$(5.0.2) \quad \text{Ext}_{(L^A, L^A B)}^{1,2i}(L^A, L^A) \cong \bigoplus_{\mathfrak{p}} \text{Ext}_{(L^A, L^A B)}^{1,2i}(L^A, L^A)_{\hat{\mathfrak{p}}}$$

$$(5.0.3) \quad \cong \bigoplus_{\mathfrak{p}} \text{Ext}_{(V^{\hat{A}_p}, V^{\hat{A}_p} T)}^{1,2i}(V^{\hat{A}_p}, V^{\hat{A}_p}),$$

with 5.0.3 the consequence of Theorem 5.0.4, as described above.

Consequently, since  $\text{Ext}_{(L^A, L^A B)}^{1,2i}(L^A, L^A)$  splits as a direct sum of summands, each of which is  $\text{Ext}_{(V^{\hat{A}_p}, V^{\hat{A}_p} T)}^{1,2i}(V^{\hat{A}_p}, V^{\hat{A}_p})$  and hence  $p$ -adically complete for some maximal ideal  $\mathfrak{p}$  in  $A$ , the effect of inverting the prime-power-ramification discriminant  $\underline{\Delta}_{K/\mathbb{Q}}$  is that the localized Ext  $A$ -module  $\text{Ext}_{(L^A, L^A B)}^{1,2i}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$  splits as a direct sum

$$\text{Ext}_{(L^A, L^A B)}^{1,2i}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}] \cong \bigoplus_{\underline{\Delta}_{K/\mathbb{Q}} \notin \mathfrak{p}} \text{Ext}_{(V^{\hat{A}_p}, V^{\hat{A}_p} T)}^{1,2i}(V^{\hat{A}_p}, V^{\hat{A}_p}),$$

with the direct sum taken over all maximal ideals  $\mathfrak{p}$  in  $A$  which do not contain the prime-power-ramification discriminant  $\underline{\Delta}_{K/\mathbb{Q}}$ .

Finally, by the cases of the Local Conjecture proved in Theorem 3.0.16 (and, in particular, the formulation in Corollary 3.0.17) for each  $i \in \mathbb{Z}$  and for each maximal ideal  $\mathfrak{p}$  in  $A$  which sits over a prime  $p$  in  $\mathbb{Z}$  not in  $\underline{\Delta}_{K/\mathbb{Q}}$ , we have the isomorphism of  $A$ -modules

$$\text{Ext}_{(V^{\hat{A}_p}, V^{\hat{A}_p} T)}^{1,2i}(V^{\hat{A}_p}, V^{\hat{A}_p}) \cong A/H_i^p$$

Consequently  $\text{Ext}_{(L^A, L^A B)}^{1,2i}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$  is isomorphic, as an  $A$ -module, to the direct product of the quotients  $\prod_{\mathfrak{p}} A/H_i^p$ , for all maximal ideals  $\mathfrak{p}$  in  $A$  which do not contain  $\underline{\Delta}_{K/\mathbb{Q}}$ . Inverting  $\underline{\Delta}_{K/\mathbb{Q}}$  then yields the claimed isomorphism.  $\square$

**Corollary 5.0.9.** *The weaker form of Ravel's Global Conjecture, Conjecture 1.0.5, is true.*

*Specifically: let  $K/\mathbb{Q}$  be a finite field extension with ring of integers  $A$ . Then there exists some number  $c \in \mathbb{N}$  such that, for all  $m \in \mathbb{N}$ ,*

$$\text{Ext}_{(L^A, L^A B)}^{1,2m}(L^A, L^A)[c^{-1}] \cong A/(J_m)[c^{-1}],$$

*where  $J_m$  is the minimal  $m$ -congruence ideal of  $A$ . The number  $c$  can be taken to be the prime-power-ramification discriminant,  $c = \underline{\Delta}_{K/\mathbb{Q}}$ .*

*Equivalently, in terms of the moduli stack  $\mathcal{M}_{fmA}$  of one-dimensional formal  $A$ -modules over  $\text{Spec } A$ , and with notation as in Conventions 2.0.10: there exists some number  $c \in \mathbb{N}$  such that, for all  $m \in \mathbb{N}$ ,*

$$H_{fl}^{1,2m}(\mathcal{M}_{fmA}; O)[c^{-1}] \cong A/(J_m)[c^{-1}],$$



where  $J_m$  is the minimal  $m$ -congruence ideal of  $A$ . The number  $c$  can be taken to be the prime-power-ramification discriminant,  $c = \Delta_{K/\mathbb{Q}}$ .

*Proof.* To get this result from Theorem 5.0.8, just apply Theorem 4.1.7 to the Minkowski Dedekind domain  $A[\Delta_{K/\mathbb{Q}}^{-1}]$  (the fact that we have to localize  $A$  is the reason we phrased Theorem 4.1.7 in terms of Minkowski Dedekind domains, e.g. localized number rings, rather than just number rings!), to get that  $\prod_{\mathfrak{p}} A/H_{\mathfrak{p}}^p$  is isomorphic to  $A/J_i$ ,  $A$  modulo the maximal  $i$ -congruence ideal in  $A$ .  $\square$

**Corollary 5.0.10.** *The strong form of Ravenel's Global Conjecture, Conjecture 1.0.6, is true for number fields  $K$  whose prime-power-ramification discriminant  $\Delta_{K/\mathbb{Q}}$  is equal to one. The "correcting factor" in the statement of Conjecture 1.0.6 can be taken to be one (the strongest possible, that is, no correcting factor is necessary!).*

*Specifically: let  $K/\mathbb{Q}$  be a finite field extension with ring of integers  $A$ . Suppose that  $\Delta_{K/\mathbb{Q}} = 1$ , i.e., suppose that, for every maximal ideal  $\mathfrak{p}$  in  $A$ , the number  $\log_p(\frac{e}{p-1})$  is not an integer, where  $p$  is the prime of  $\mathbb{Z}$  under  $\mathfrak{p}$ , and  $e$  is the ramification degree of  $\mathfrak{p}$ . Then, for all  $m \in \mathbb{N}$ ,*

$$\mathrm{Ext}_{(L^A, L^A B)}^{1,2m}(L^A, L^A) \cong A/J_m,$$

where  $J_m$  is the minimal  $m$ -congruence ideal of  $A$ .

The next two corollaries recall the classes of examples, from Example 5.0.6, in which the prime-power-ramification discriminant is one, and hence Corollary 5.0.10 applies.

**Corollary 5.0.11.** *Suppose  $K/\mathbb{Q}$  is a Galois extension of odd primary degree, and suppose that 2 ramifies in  $K$ . Let  $A$  be the ring of integers of  $K$ . Then, for all  $m \in \mathbb{N}$ ,*

$$\mathrm{Ext}_{(L^A, L^A B)}^{1,2m}(L^A, L^A) \cong A/J_m,$$

where  $J_m$  is the minimal  $m$ -congruence ideal of  $A$ .

*Equivalently, in terms of the moduli stack  $\mathcal{M}_{fmA}$  of one-dimensional formal  $A$ -modules over  $\mathrm{Spec} A$ , and with notation as in Conventions 2.0.10: for all  $m \in \mathbb{N}$ ,*

$$H_{fl}^{1,2m}(\mathcal{M}_{fmA}; \mathcal{O}) \cong A/J_m,$$

where  $J_m$  is the minimal  $m$ -congruence ideal of  $A$ .

**Corollary 5.0.12.** *A nonabelian example: let  $K$  be the splitting field of  $x^5 + x + 2$  over  $\mathbb{Q}$ . Let  $A$  be the ring of integers of  $K$ . Then, for all  $m \in \mathbb{N}$ ,*

$$\mathrm{Ext}_{(L^A, L^A B)}^{1,2m}(L^A, L^A) \cong A/J_m,$$

where  $J_m$  is the minimal  $m$ -congruence ideal of  $A$ .

*Equivalently, for all  $m \in \mathbb{N}$ ,*

$$H_{fl}^{1,2m}(\mathcal{M}_{fmA}; \mathcal{O}) \cong A/J_m,$$

where  $J_m$  is the minimal  $m$ -congruence ideal of  $A$ .

## 6. ZETA-FUNCTIONS.

**6.1. Motivation.** In [20], Ravenel writes: "The numbers  $j_m$  of 3.8 are also related to Bernoulli numbers and the values of the Riemann zeta function at negative integers, but these properties do not appear to generalize to other number fields. For example if the field is not totally real its Dedekind zeta function vanishes at all negative integers." Ravenel's observation, that a direct connection between the order of  $\mathrm{Ext}_{(L^A, L^A B)}^{1,2m}(L^A, L^A)$  and  $\zeta_K(1-m)$

is impossible unless  $K$  is totally real since otherwise  $\zeta_K$  vanishes at negative integers, is quite correct.

However, a good relationship between numerical invariants of  $\zeta_K$  and the order of  $\text{Ext}_{(L^A, L^A)}^{1,2m}(L^A, L^A)$  is quite possible, if one is willing to work with invariants of  $\zeta$ -functions other than special values. The idea is as follows: in an Euler product admitting an analytic continuation, the  $p$ -local Euler factor typically controls prime-to- $p$  denominators of special values at negative integers. One wants to apply some kind of transform to such an Euler product which “straightens out” the relationship between the  $p$ -local Euler factors and the  $p$ -primary factors in the special values of the function, causing the  $p$ -local Euler factor to contribute to  $p$ -primary factors, rather than prime-to- $p$  factors, in the values of the transformed function.

In the remaining sections of this paper, I give two distinct ways of doing this, which I call “unramified straightening” and “Galois-Dedekind straightening.”

- The “unramified straightening transform”  $\mathbb{S}(L(s))(n)$  of an  $L$ -function  $L(s)$  has the appeal of having an extremely simple and natural definition. On the other hand, it has the disadvantage that, when  $A$  is the ring of integers of a finite extension  $K/\mathbb{Q}$  with Dedekind  $\zeta$ -function  $\zeta_K(s)$ , the unramified straightening transform  $\mathbb{S}(\zeta_K(s))(n)$  only describes the order of  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)$  after inverting two times the classical discriminant  $\Delta_{K/\mathbb{Q}}$ .
- On the other hand, the “Galois-Dedekind straightening transform”  $\mathbb{S}_{GD}(L(s))(n)$  is only defined for certain (sufficiently “nice”)  $L$ -functions  $L(s)$ , and its definition is more complicated and less natural-seeming than the unramified straightening transform. On the other hand, the Galois-Dedekind straightening transform has the advantage that, when  $A$  is the ring of integers of a finite *Galois* extension  $K/\mathbb{Q}$  with Dedekind  $\zeta$ -function  $\zeta_K(s)$ , the Galois-Dedekind straightening transform  $\mathbb{S}_{GD}(\zeta_K(s))(n)$  is defined, and it describes the order of  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)$  after inverting the prime-power-ramification discriminant  $\underline{\Delta}_{K/\mathbb{Q}}$ , which is a less destructive operation than inverting two times the classical discriminant.

As I mentioned in the introduction to this paper, these straightening transforms also have some applications elsewhere, in the theory of  $L$ -functions associated to Bousfield-localized stable homotopy types of finite CW-complexes, which I will say much more about in a later paper, currently in preparation. The definitions of these straightening transforms are chosen in order to have convenient properties which relate them to  $\text{Ext}_{(L^A, L^A)}^1(L^A, L^A)$  and to stable homotopy types of finite CW-complexes; I do not know if these straightening transforms are of any interest or use for number theorists.

## 6.2. The unramified straightening transform.

**Definition 6.2.1.** *If  $p$  is a prime number, then by an elementary  $p$ -local Euler factor I mean an expression of the form*

$$\frac{1}{1 - p^{-as}}$$

*for some positive integer  $a$ . By an elementary Euler factor I mean elementary  $p$ -local Euler factor for some prime number  $p$ .*

*Suppose  $L(s)$  is an  $L$ -function which is equal (for  $s$  with sufficiently large real part) to an Euler product which is a product of elementary Euler factors. By the unramified straightening transform of  $L(s)$  I mean the function  $\mathbb{S}(L(s)) : \mathbb{N} \rightarrow \mathbb{N}$  given as follows:*

$$\mathbb{S}\left(\frac{1}{1 - p^{-as}}\right)(n) = \begin{cases} 1 & \text{if } p^a - 1 \nmid n \\ p^{(1+v_p(n))a} & \text{if } p^a - 1 \mid n, \end{cases}$$

and  $\mathbb{S}$  of a product of elementary Euler factors is the product of  $\mathbb{S}$  applied to each Euler factor.

Obviously this definition is limited to only a very restricted class of  $L$ -functions, namely, those with an Euler product which is a product of elementary Euler factors. In a later paper (currently in preparation) I extend the definition of this straightening transform to a larger class of Euler products, and I demonstrate that this slightly more general unramified  $\mathbb{S}$ -transform has some agreeable properties (e.g. recovering the orders of  $KU$ -local stable homotopy groups) when evaluated on certain  $L$ -functions associated to finite CW-complexes.

The following proposition, Proposition 6.2.2, is equivalent to a computation of the denominators of the numbers  $\zeta(1 - n)$  for positive integers  $n$ . This computation is very well-known: in the terms most familiar to a topologist, it is the fact that the orders of the homotopy groups  $\pi_*(j)$  in the image of the  $J$ -homomorphism in degrees congruent to  $-1 \pmod{4}$  (which Adams identified, in [2], with the denominators of the special values at negative integers of the Riemann  $\zeta$ -function) have the correct  $p$ -adic valuations to coincide, after  $p$ -adic completion for  $p > 2$ , with the familiar pattern one sees in the homotopy groups of the  $K(1)$ -local (or  $E(1)$ -local) sphere, computed by Adams and Baird (see [21]). But of course Proposition 6.2.2 was known to number theorists much, much earlier. I do not know where to find a proof in the literature, so I supply one, below.

**Proposition 6.2.2.** *Let  $\zeta$  be the Riemann  $\zeta$ -function. Then, for all positive integers  $n$ , the denominator of  $\zeta(1 - 2n)$  is equal to  $\mathbb{S}(\zeta(s))(2n)$ .*

*Suppose further that  $m$  is an odd positive integer. Then  $\zeta(1 - m) = 0$ , so it is a matter of convention what the denominator of  $\zeta(1 - m)$  is. However,  $\mathbb{S}(\zeta(s))(m) = 2$  for all odd positive integers  $m$ , so if we adopt the convention that the denominator of  $\zeta(1 - m) = 0$  is 2, then the denominator of  $\zeta(1 - n)$  is equal to  $\mathbb{S}(\zeta(s))(n)$  for all positive integers  $n$ .*

*Proof.* First, recall that the Euler product for  $\zeta$  is:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

when  $\Re s > 1$ . Consequently, for each prime  $p$ , the  $p$ -adic valuation of  $\mathbb{S}(\zeta(s))(n)$  is given by

$$(6.2.1) \quad v_p(\mathbb{S}(\zeta(s))(n)) = \begin{cases} 0 & \text{if } (p-1) \nmid n \\ 1 + v_p(n) & \text{if } (p-1) \mid n. \end{cases}$$

Now we will show that the  $p$ -adic valuation of  $\mathbb{S}(\zeta(s))(2n)$  agrees with the  $p$ -adic valuation of the denominator of  $\zeta(1 - 2n)$ . It is classical that

$$(6.2.2) \quad \zeta(1 - n) = \frac{-B_n}{n}$$

for positive integers  $n$ , and that  $B_n = 0$  if  $n$  is odd and  $n > 1$ . By the von Staudt-Clausen theorem, the denominator of the Bernoulli number  $B_{2n}$  is equal to the product of all primes  $p$  such that  $p - 1$  divides  $2n$ . Suppose that  $p$  is a prime and  $n$  a positive integer such that  $(p - 1) \nmid 2n$ . Let  $N_{2n}$  denote the numerator and  $D_{2n}$  the denominator of  $B_{2n}$ . Then it is known that

$$v_p(2n) \leq v_p(N_{2n}),$$

that is, the  $p$ -adic valuation of  $N_{2n}$  is at least as large as the  $p$ -adic valuation of  $2n$ ; [8] provides, in the introduction, a brief history of this result, which goes back to von Staudt but has been independently rediscovered many times. Consequently, if  $p - 1$  does not

divide  $2n$ , then  $D_{2n}$  is not divisible by  $p$ , and  $N_{2n}$  is divisible by at least as large a power of  $p$  as  $2n$  is. Hence the denominator of  $\frac{B_{2n}}{2n}$  is not divisible by  $p$ .

On the other hand, suppose that  $p$  is a prime and  $n$  a positive integer such that  $(p-1) \mid 2n$ . Then  $D_{2n}$  is divisible by  $p$ , and hence  $N_{2n}$  isn't divisible by  $p$ ; consequently we have the equality of  $p$ -adic valuations

$$\begin{aligned} v_p\left(\frac{B_{2n}}{2n}\right) &= -v_p(D_{2n}) - v_p(2n) \\ &= -v_p(2n) - 1. \end{aligned}$$

Consequently, using the equation 6.2.2, if we let  $D'_{2n}$  denote the denominator of  $\zeta(1 - 2n)$ , then for all primes  $p$  and positive integers  $n$  we get the formula:

$$v_p(D_{2n}) = \begin{cases} 0 & \text{if } (p-1) \nmid 2n \\ v_p(2n) + 1 & \text{if } (p-1) \mid 2n. \end{cases}$$

This is equal to  $\mathbb{S}(\zeta(s))(2n)$  by equation 6.2.1.

If  $m$  is odd, we already observed that  $\zeta(1 - m) = 0$  since the odd Bernolli numbers  $B_m$  are zero. On the other hand, for odd  $m$ , there exists a (unique) prime number  $p$  such that  $(p-1) \mid m$ , namely,  $p = 2$ . Furthermore,  $1 + v_2(m) = 1$  for odd  $m$ , so by equation 6.2.1, we have that  $\mathbb{S}(\zeta(s))(m) = 2$  for all positive odd integers  $m$ , as claimed.  $\square$

Now we need to review a few facts about Hecke  $L$ -functions. We will only need the Hecke  $L$ -functions of trivial Größencharakteren (with varying moduli, however), so the treatment I give here is much more limited than what one can find in a good number theory textbook, such as [15].

**Definition-Proposition 6.2.3. (Classical.)** *Let  $K/\mathbb{Q}$  be a number field with ring of integers  $A$ , and let  $m$  be a proper ideal of  $A$ . We write  $\chi$  for the trivial Größencharakter of  $K$  with modulus  $m$ . By the Hecke  $L$ -function of  $\chi$  we mean the complex-valued function*

$$L(s, \chi) = \sum_I \frac{1}{N(I)^{-s}},$$

for all complex numbers  $s$  with real part  $\Re s > 1$ , where  $I$  ranges across all nonzero ideals of  $A$  which are coprime to  $m$ , and  $N(I)$  is the absolute norm of  $I$ , i.e.,  $N(I)$  is the cardinality of the residue ring  $A/I$ .

Then  $L(s, \chi)$  admits analytic continuation to a meromorphic function on the complex plane, and  $L(s, \chi)$  admits an Euler product

$$L(s, \chi) = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}}$$

for all complex numbers  $s$  with real part  $\Re s > 1$ , where the product is taken over all maximal ideals  $\mathfrak{p}$  of  $A$  which are coprime to  $m$ .

Hence

$$L(s, \chi) = \left( \prod_{\mathfrak{p} \supseteq m} (1 - N(\mathfrak{p})^{-s}) \right) \zeta_K(s),$$

where  $\zeta_K(s)$  is the Dedekind  $\zeta$ -function of  $K$ , and the product is taken over all maximal ideals  $\mathfrak{p}$  of  $A$  which contain  $m$ . In particular, if  $m = 1$ , then  $L(s, \chi) = \zeta_K(s)$ .

So Hecke  $L$ -functions of trivial Größencharakteren are a convenient “language” for working with Dedekind  $\zeta$ -functions in which certain Euler factors—those associated to the primes containing the modulus ideal—have been omitted.

From Proposition 6.2.2, one knows that the unramified  $\mathbb{S}$ -transform of the Riemann  $\zeta$ -function recovers the denominators of the special values of  $\zeta(s)$  at negative integers, and consequently (by the classical  $K = \mathbb{Q}$  case of Theorem 5.0.8 which motivated Ravenel's Global Conjecture), it recovers the order of  $\text{Ext}_{(L, LB)}^{1,*}(L, L)$  up to multiplication by a power of 2. A more remarkable fact is that, for the Dedekind  $\zeta$ -function  $\zeta_K(s)$  of a number field  $K/\mathbb{Q}$ , the unramified  $\mathbb{S}$ -transform  $\mathbb{S}(\zeta_K(s))$  *does not* recover the denominators of the special values of  $\zeta(s)$  at negative integers, but instead, if the discriminant *and* the prime-power-ramification discriminant of  $K/\mathbb{Q}$  are both trivial, then  $\mathbb{S}(\zeta_K(s))$  recovers the order of  $\text{Ext}_{(L^A, L^A)}^{1,*}(L^A, L^A)$ , where  $A$  is the ring of integers in  $K$ . In fact a stronger statement is true, and here is a precise statement (and proof):

**Theorem 6.2.4.** *Let  $K/\mathbb{Q}$  be a finite field extension with ring of integers  $A$ . Let  $\chi_{2\Delta_{K/\mathbb{Q}}}$  be the trivial Größencharakter of  $K$  of modulus equal to  $2\Delta_{K/\mathbb{Q}}$ , two times the classical discriminant of  $K/\mathbb{Q}$ , and let  $L(s, \chi_{2\Delta_{K/\mathbb{Q}}})$  be its associated Hecke  $L$ -function.*

*Then, for all positive  $n \in \mathbb{N}$ , the following numbers are all equal:*

- *the order of the abelian group  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$ ,*
- *the order of the abelian group  $H_{f_1}^{1,2n}(\mathcal{M}_{f_1 A}; \mathcal{O})[(2\Delta_{K/\mathbb{Q}})^{-1}]$ ,*
- *the order of the abelian group  $A[(2\Delta_{K/\mathbb{Q}})^{-1}]/J_n$ , where  $J_n$  is the minimal  $n$ -congruence ideal in  $A[(2\Delta_{K/\mathbb{Q}})^{-1}]$ , and*
- *the number  $\mathbb{S}(L(s, \chi_{2\Delta_{K/\mathbb{Q}}})) (n)$ .*

*Proof.* First, since  $(\underline{\Delta}_{K/\mathbb{Q}}) \supseteq (2\Delta_{K/\mathbb{Q}})$  by Observation 5.0.7, inverting  $2\Delta_{K/\mathbb{Q}}$  implies also inverting  $\underline{\Delta}_{K/\mathbb{Q}}$ , so Corollary 5.0.9 implies that we have an isomorphism of  $A$ -modules

$$\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}] \cong A[(2\Delta_{K/\mathbb{Q}})^{-1}]/J_n,$$

consequently these two abelian groups have equal order. So for the rest of the proof we will concern ourselves with the question of why the order of  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  is equal to  $\mathbb{S}(L(s, \chi_{2\Delta_{K/\mathbb{Q}}})) (n)$ .

Now we need to see why  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  has finite order! We know that  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)$  is a finitely generated torsion  $A$ -module (see the beginning of the proof of Theorem 5.0.8), hence has finite order. Furthermore, by the fundamental theorem of finitely generated modules over a Dedekind domain,  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)$  is a direct sum of finitely many  $A$ -modules of the form  $A/\mathfrak{p}^i$ , for various prime ideals  $\mathfrak{p}$  of  $A$  and various positive integers  $i$ . Consequently, the effect of inverting  $2\Delta_{K/\mathbb{Q}}$  on  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)$  is the same as quotienting out those summands of  $A$  of the form  $A/\mathfrak{p}^i$  where  $2\Delta_{K/\mathbb{Q}} \in \mathfrak{p}$ . So  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  is a quotient  $A$ -module of the finite  $A$ -module  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)$ . So  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  is itself finite.

Now clearly  $\mathbb{S}(L(s, \chi_{2\Delta_{K/\mathbb{Q}}})) (n)$  and the order of  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  are both positive integers, so if we show that the  $p$ -adic valuation of  $\mathbb{S}(L(s, \chi_{2\Delta_{K/\mathbb{Q}}})) (n)$  agrees with the  $p$ -adic valuation of the order of  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  for all prime numbers  $p$ , then we are done.

Suppose  $p$  is a prime number such that some prime  $\mathfrak{p}$  of  $A$  over  $p$  is ramified. Then  $\mathfrak{p} \supseteq (\Delta_{K/\mathbb{Q}}) \supseteq (2\Delta_{K/\mathbb{Q}})$ , and consequently the Euler product for  $L(s, \chi_{2\Delta_{K/\mathbb{Q}}})$  contains no  $p$ -local Euler factors. Hence  $v_p(\mathbb{S}(L(s, \chi_{2\Delta_{K/\mathbb{Q}}})) (n)) = 0$ . On the other hand, for each prime  $\ell$  of  $A$  over a given prime number  $\ell$ , the completion  $(\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A))_{\hat{\ell}}$  is a finite  $\hat{A}_{\ell}$ -module, hence has order equal to some power of  $\ell$ , and  $\text{Ext}_{(L^A, L^A)}^{1,2n}(L^A, L^A)$  splits as a direct

sum of its completions at the various maximal ideals  $\mathfrak{p}$  of  $A$  (this is isomorphism 5.0.2 from the proof of Theorem 5.0.8). Consequently  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  has no summands of order a power of  $p$ . Consequently the  $p$ -adic valuation of the order of  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  is zero.

The more important situation is when  $p$  is a prime number such that no primes of  $A$  over  $p$  are ramified. Suppose  $p$  is such a prime. Then the  $p$ -adic valuation of the order of  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  is equal to the  $p$ -adic valuation of the order of the completion  $(\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A))_{\mathfrak{p}}$ , i.e., the product of the  $p$ -adic valuations of the orders of the completions  $(\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A))_{\mathfrak{p}}$  for all primes  $\mathfrak{p}$  of  $A$  over  $p$ . By Theorem 5.0.4, the completion  $(\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A))_{\mathfrak{p}}$  is isomorphic as a  $\hat{A}_{\mathfrak{p}}$ -module to  $\text{Ext}_{(V^{\hat{A}_{\mathfrak{p}}}, V^{\hat{A}_{\mathfrak{p}}})}^{1,2n}(V^{\hat{A}_{\mathfrak{p}}}, V^{\hat{A}_{\mathfrak{p}}})$ , which is in turn isomorphic, by Corollary 3.0.17, to  $\hat{A}_{\mathfrak{p}}/H_n$ , where  $H_n$  is the ideal of  $A_{\mathfrak{p}}$  generated by all elements of the form  $x^n - 1$ , where  $x \in \hat{A}_{\mathfrak{p}}^{\times}$ . As we showed in Theorem 3.0.16,  $\hat{A}_{\mathfrak{p}}/H_n \cong 0$  if  $p^f - 1 \nmid n$  where  $p^f$  is the cardinality of the residue field of  $\hat{A}_{\mathfrak{p}}$ , and if  $p^f - 1 \mid n$ , then  $H_n = I_{n/(p^f-1)}$ , where  $I_{n/(p^f-1)}$  is the ideal in  $\hat{A}_{\mathfrak{p}}$  generated by all elements of the form  $a^{n/(p^f-1)} - 1$  for elements  $a \in \hat{A}_{\mathfrak{p}}$  congruent to 1 modulo  $\mathfrak{p}$  (this is by the Hensel's Lemma argument at the end of the proof of Corollary 3.0.17).

Finally, Lemma 3.0.13 gives us that the order of  $\hat{A}_{\mathfrak{p}}/I_{n/(p^f-1)}$  is equal to  $p^{fj}$ , where

$$j = \begin{cases} e \left( v_p(n/(p^f - 1)) - \lceil \log_p \frac{e}{p-1} \rceil \right) + p^{\lceil \log_p \frac{e}{p-1} \rceil} & \text{if } \lceil \log_p \frac{e}{p-1} \rceil \leq v_p(n/(p^f - 1)) \\ p^{v_p(n/(p^f - 1))} & \text{if } \lceil \log_p \frac{e}{p-1} \rceil > v_p(n/(p^f - 1)) \end{cases}$$

$$= v_p(n) + 1,$$

since by assumption  $\mathfrak{p}$  is not ramified and hence  $e = 1$ . Consequently our formula for the  $p$ -adic valuation of the order of  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}]$  is:

$$v_p \left( \# \left( \text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}] \right) \right) = \sum_{\mathfrak{p}} \left( (v_p(n) + 1) f_{\mathfrak{p}} \right),$$

where the sum is taken over all primes  $\mathfrak{p}$  of  $A$  over  $p$  such that the cardinality  $p^{f_{\mathfrak{p}}}$  of the residue field  $A/\mathfrak{p}$  satisfies  $(p^{f_{\mathfrak{p}}} - 1) \mid n$ .

Now we turn to the  $p$ -adic valuation of  $\mathbb{S}(L(s, \chi_{2\Delta_{K/\mathbb{Q}}})(n)$ . This follows easily from the definition of  $\mathbb{S}$  and the Euler product for  $L(s, \chi_{2\Delta_{K/\mathbb{Q}}})$ :

$$v_p \left( \mathbb{S}(L(s, \chi_{2\Delta_{K/\mathbb{Q}}})(n) \right) = \sum_{\mathfrak{p}} \left( (v_p(n) + 1) f_{\mathfrak{p}} \right),$$

where the sum is taken over all primes  $\mathfrak{p}$  of  $A$  over  $p$  such that the cardinality  $p^{f_{\mathfrak{p}}}$  of the residue field  $A/\mathfrak{p}$  satisfies  $(p^{f_{\mathfrak{p}}} - 1) \mid n$ , i.e.,

$$v_p \left( \mathbb{S}(L(s, \chi_{2\Delta_{K/\mathbb{Q}}})(n) \right) = v_p \left( \# \left( \text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[(2\Delta_{K/\mathbb{Q}})^{-1}] \right) \right),$$

as desired.  $\square$

**Definition 6.2.5.** Recall that two number fields  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are said to be arithmetically equivalent if their Dedekind  $\zeta$ -functions  $\zeta_{K_1}(s)$  and  $\zeta_{K_2}(s)$  are equal.

See [?] and [4] for some nice examples of arithmetically equivalent number fields; in particular, Perlis shows that the smallest  $n$  such that there exist distinct arithmetically equivalent number fields  $K_1, K_2$  of degree  $n$  over  $\mathbb{Q}$  is  $n = 7$ , and Bosma and de Smit compute various number-theoretic invariants of degree 7 arithmetically equivalent extensions of  $\mathbb{Q}$ .

**Theorem 6.2.6. (Perlis.)** *Suppose  $K_1, K_2$  are arithmetically equivalent number fields. Then  $[K_1 : \mathbb{Q}] = [K_2 : \mathbb{Q}]$ . If  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are both Galois, then  $K_1$  is isomorphic to  $K_2$ .*

*Proof.* These results are found in [?]. □

If  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are number fields, a weaker notion than arithmetic equivalence is to ask whether the Dedekind  $\zeta$ -functions of  $K_1$  and of  $K_2$  agree when certain Euler factors are ignored. In other words:

**Definition 6.2.7.** *Let  $K_1, K_2$  be number fields, and let  $m$  be a positive integer. We will say that  $K_1$  and  $K_2$  are arithmetically equivalent modulo  $m$  if the Hecke  $L$ -function of the trivial Größencharakter of  $K_1$  with modulus  $m$  is equal to the Hecke  $L$ -function of the trivial Größencharakter of  $K_2$  with modulus  $m$ .*

Now, since  $\text{Ext}_{(L, L^A B)}^{1, n}(L^A, L^A)$  vanishes for odd  $n$ , we have the following corollary of Theorem 6.2.4:

**Corollary 6.2.8.** *Let  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  be finite field extensions with ring of integers  $A_1$  and  $A_2$ , respectively. Let  $m$  be any integer which is divisible by 2,  $\Delta_{K_1/\mathbb{Q}}$ , and  $\Delta_{K_2/\mathbb{Q}}$ . If  $K_1$  and  $K_2$  are arithmetically equivalent modulo  $m$ , then for all positive  $n \in \mathbb{N}$ , the order of the abelian group*

$$\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1, n}(L^{A_1}, L^{A_1})[m^{-1}]$$

*is equal to the order of the abelian group*

$$\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1, n}(L^{A_2}, L^{A_2})[m^{-1}].$$

*Equivalently, in terms of the moduli stack  $\mathcal{M}_{f mA}$  of one-dimensional formal  $A$ -modules over  $\text{Spec } A$ , and with notation as in Conventions 2.0.10: the order of the abelian group*

$$H_{fl}^{1, n}(\mathcal{M}_{f mA_1}; \mathcal{O})[m^{-1}]$$

*is equal to the order of the abelian group*

$$H_{fl}^{1, n}(\mathcal{M}_{f mA_2}; \mathcal{O})[m^{-1}].$$

**Remark 6.2.9.** I do not know if the  $\mathbb{S}$ -transform is of any independent interest for number theorists (it is a very naive construction, so perhaps not!), but I do know of another topological situation in which the  $\mathbb{S}$ -transform is very useful, namely, in the problem of associating an  $L$ -function  $L(s, X)$  to a finite CW-complex  $X$  in such a way that the special values  $L(-n, X)$  recover the orders of the homotopy groups of various Bousfield localizations (say, the  $KU$ -localization) of  $X$ : this problem is in general not solvable (except for very particular choices of  $X$ ) if one insists on working with special values, but if one instead asks that the values  $\mathbb{S}(L(s, X))(n)$  of the  $\mathbb{S}$ -transform coincide with the orders of the homotopy groups of the  $KU$ -localization of  $X$ , then this problem is very tractable, and I will give my solution to this problem in a paper which is currently in preparation.

**6.3. The Galois-Dedekind straightening transform.** Theorem 6.2.4, identifying the values of the unramified straightening transform of a Hecke  $L$ -function of  $K/\mathbb{Q}$  in terms of orders of  $\text{Ext}_{(L^A, L^A B)}^1$  groups after inverting the modulus of the Größencharakter, is an interesting result but there is something not totally satisfying about it: the modulus of the Größencharakter is required to be divisible by twice the (classical) discriminant of  $K/\mathbb{Q}$ , meaning that the result always requires that we invert at least one prime (2) as well as any primes that ramify in  $K$ . It would be nice to have a generalization of Theorem 6.2.4 which

holds without having to invert so much, in particular, one which holds for at least some number fields  $K$  without having to invert *any* primes at all.

In this section I state and prove Theorem 6.3.5, which is just such a generalization of Theorem 6.2.4. Theorem 6.3.5 comes at a price, however: it is phrased in terms of the Galois-Dedekind straightening transform (also defined in this section), which is significantly less simple and natural-looking than the unramified straightening transform.

Recall that elementary Euler factors were defined in Definition 6.2.1.

**Definition 6.3.1.** *Suppose  $L(s)$  is an  $L$ -function which is equal (for  $s$  with sufficiently large real part) to an Euler product which is a product of elementary Euler factors. I will say that the Euler product is of Galois-Dedekind  $d$ -type if the following two conditions hold:*

- (Galois axiom:) *For each prime number  $p$ , if*

$$\frac{1}{1 - p^{-as}} \quad \text{and} \quad \frac{1}{1 - p^{-bs}}$$

*are factors appearing in the Euler product for  $L(s)$ , then  $a = b$ .*

- (Degree axiom:) *For all but finitely many prime numbers  $p$ , there exists at least one  $p$ -local elementary Euler factor in the Euler product for  $L(s)$ , and, writing  $\frac{1}{1 - p^{ns}}$  for any such  $p$ -local elementary Euler factor, we have the equation  $an = d$ , where  $n$  is the number of factors  $\frac{1}{1 - p^{ns}}$  appearing in the Euler product for  $L(s)$ .*

*I will say that an Euler product is of Galois-Dedekind type if it is of Galois-Dedekind  $d$ -type for some positive integer  $d$ .*

**Example 6.3.2.** If  $K/\mathbb{Q}$  is a finite extension, then the Dedekind  $\zeta$ -function  $\zeta_K(s)$  is not necessarily of Galois-Dedekind type. However, if  $K/\mathbb{Q}$  is also Galois, then the Dedekind  $\zeta$ -function *is* of Galois-Dedekind type, specifically of Galois-Dedekind  $[K : \mathbb{Q}]$ -type.

In fact, a more general statement is true: suppose  $K/\mathbb{Q}$  is a finite field extension which is Galois. Let  $A$  be the ring of integers of  $K$ , and let  $m \in \mathbb{Z}$ . Then the Hecke  $L$ -function  $L(s, \chi_m)$  of the trivial Größencharakter of  $A$  with modulus  $m$  is of Galois-Dedekind  $[K : \mathbb{Q}]$ -type. The reason for this is as follows: clearly all the factors in the Euler product for  $L(s, \chi_m)$  are elementary. If  $p$  is a prime number and  $p \mid m$ , then  $L(s, \chi_m)$  has no  $p$ -local Euler factors, so the Galois axiom is automatically satisfied for such  $p$ .

If  $p \nmid m$ , then  $L(s, \chi_m)$  has  $p$ -local Euler factors, one for each prime  $\mathfrak{p}$  of  $A$  over  $p$ . Since  $K/\mathbb{Q}$  is Galois, the Galois group  $G_{K/\mathbb{Q}}$  acts transitively on the set of primes of  $A$  over  $p$ , each of which has the same residue degree and ramification degree. Consequently, for fixed  $p$ , the elementary  $p$ -local Euler factors of  $L(s, \chi_m)$  are all equal. So the Galois axiom is satisfied for such  $p$ . Furthermore there will be exactly one elementary  $p$ -local Euler factor  $\frac{1}{1 - p^{-f_p s}}$  for each prime  $\mathfrak{p}$  of  $A$  over  $p$ , and consequently the number of such factors will be equal to  $\frac{[K:\mathbb{Q}]}{e_p f_p}$ , where  $e_p$  is the ramification degree and  $f_p$  the residue degree of any of the primes  $\mathfrak{p}$  over  $p$ .

However, only finitely many primes  $p$  will divide the classical discriminant  $\Delta_{K/\mathbb{Q}}$ , hence only finitely many primes  $p$  will have  $e_p > 1$ . Hence, for all but finitely many prime numbers  $p$  (those finitely many prime numbers  $p$  are the ones which either ramify in  $A$  or which divide the modulus  $m$ ), the number of elementary  $p$ -local Euler factors  $\frac{1}{1 - p^{-f_p s}}$  in  $L(s, \chi_m)$  is equal to  $\frac{[K:\mathbb{Q}]}{f_p}$ . Hence the degree axiom is satisfied.

In the case  $m = 1$ , this shows that the Dedekind  $\zeta$ -function  $\zeta_K(s)$  of a finite Galois extension  $K/\mathbb{Q}$  is of Galois-Dedekind type  $[K : \mathbb{Q}]$ .



**Definition 6.3.3.** Let  $d$  be a positive integer and let  $L(s)$  be an  $L$ -function of Galois-Dedekind  $d$ -type. By the Galois-Dedekind straightening transform of  $L(s)$  I mean the function  $\mathbb{S}_{GD}(L(s)) : \mathbb{N} \rightarrow \mathbb{N}$  given as follows:

$$\mathbb{S}_{GD}\left(\frac{1}{1-p^{-as}}\right)(n) = \begin{cases} 1 & \text{if } p^a - 1 \nmid n \\ p^{\frac{d}{g_p}(v_p(n) - \lceil \log_p \frac{d}{ag_p(p-1)} \rceil) + ap^{\lceil \log_p \frac{d}{ag_p(p-1)} \rceil}} & \text{if } p^a - 1 \mid n \text{ and } \lceil \log_p \frac{d}{ag_p(p-1)} \rceil \leq v_p(n), \\ p^{ap^{v_p(n)}} & \text{if } p^a - 1 \mid n \text{ and } \lceil \log_p \frac{d}{ag_p(p-1)} \rceil > v_p(n), \end{cases}$$

where  $g_p$  is the number of elementary  $p$ -local Euler factors in  $L(s)$ , and  $\mathbb{S}$  of a product of Euler factors is the product of  $\mathbb{S}$  applied to each Euler factor.

**Example 6.3.4.** Let  $\zeta(s)$  be the Riemann  $\zeta$ -function. Certainly its Galois-Dedekind straightening transform  $\mathbb{S}_{GD}(\zeta(s))$  is defined, since  $\zeta(s)$  is a special case of a Dedekind  $\zeta$ -function, so it is of Galois-Dedekind 1-type by Example 6.3.2.

I claim that  $\mathbb{S}(\zeta(s)) = \mathbb{S}_{GD}(\zeta(s))$ . For each prime number  $p$ , we will have  $g_p = 1$ , i.e.,  $\zeta(s)$  has a unique  $p$ -local elementary Euler factor, and that elementary Euler factor is

$$\frac{1}{1-p^{-s}},$$

i.e., the number  $a$  is 1 in Definition 6.3.3. The number  $d$  is also 1 since  $\zeta(s)$  is of Galois-Dedekind 1-type. Consequently

$$\lceil \log_p \frac{d}{ag_p(p-1)} \rceil = \lceil \log_p \frac{1}{p-1} \rceil = 0,$$

so the condition  $\lceil \log_p \frac{d}{ag_p(p-1)} \rceil \leq v_p(n)$  is always satisfied, and we have that

$$\mathbb{S}_{GD}\left(\frac{1}{1-p^{-s}}\right)(n) = p^{av_p(n)+a}$$

if  $(p-1) \mid n$ ; finally, this agrees with Definition 6.2.1.

Consequently, by Proposition 6.2.2, the denominator of  $\zeta(1-2n)$  is equal to  $\mathbb{S}_{GD}(\zeta(s))(2n)$  as well as  $\mathbb{S}_{GD}(\zeta(s))(2n)$ .

**Theorem 6.3.5.** Let  $K/\mathbb{Q}$  be a finite field extension with ring of integers  $A$ . Suppose  $K/\mathbb{Q}$  is Galois. Let  $\chi_{\Delta_{K/\mathbb{Q}}}$  be the trivial Größencharakter of  $K$  of modulus equal to the prime-power-ramification discriminant  $\Delta_{K/\mathbb{Q}}$ , and let  $L(s, \chi_{\Delta_{K/\mathbb{Q}}})$  be its associated Hecke  $L$ -function.

Then, for all positive  $n \in \mathbb{N}$ , the following numbers are all equal:

- the order of the abelian group  $\text{Ext}_{(L^A, L^A B)}^{1, 2n}(L^A, L^A)[\Delta_{K/\mathbb{Q}}^{-1}]$ ,
- the order of the abelian group  $H_{f_l}^{1, 2n}(\mathcal{M}_{f_l A}; \mathcal{O})[\Delta_{K/\mathbb{Q}}^{-1}]$ ,
- the order of the abelian group  $A[\Delta_{K/\mathbb{Q}}^{-1}]/J_n$ , where  $J_n$  is the minimal  $n$ -congruence ideal in  $A[\Delta_{K/\mathbb{Q}}^{-1}]$ , and
- the number  $\mathbb{S}_{GD}(L(s, \chi_{\Delta_{K/\mathbb{Q}}}})(n)$ .

*Proof.* This proof is much the same as that of Theorem 6.2.4. Corollary 5.0.9 implies that we have an isomorphism of  $A$ -modules

$$\text{Ext}_{(L^A, L^A B)}^{1, 2n}(L^A, L^A)[\Delta_{K/\mathbb{Q}}^{-1}] \cong A[\Delta_{K/\mathbb{Q}}^{-1}]/J_n,$$

consequently these two abelian groups have equal order. So for the rest of the proof we will concern ourselves with the question of why the order of  $\text{Ext}_{(L^A, L^A B)}^{1, 2n}(L^A, L^A)[\Delta_{K/\mathbb{Q}}^{-1}]$  is equal to  $\mathbb{S}_{GD}(L(s, \chi_{\Delta_{K/\mathbb{Q}}}})(n)$ .

That  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$  has finite order is proven by the same argument as in the proof of Theorem 6.2.4. Now clearly  $\mathbb{S}_{GD}(L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})) (n)$  and the order of  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$  are both positive integers, so if we show that the  $p$ -adic valuation of  $\mathbb{S}_{GD}(L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})) (n)$  agrees with the  $p$ -adic valuation of the order of  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$  for all prime numbers  $p$ , then we are done.

Suppose  $p$  is a prime number such that some prime  $\mathfrak{p}$  of  $A$  over  $p$  has ramification degree  $e$  with the property that  $\log_p \frac{e}{p-1}$  is an integer. Then  $\mathfrak{p} \supseteq (p) \supseteq (\underline{\Delta}_{K/\mathbb{Q}})$ , and consequently the Euler product for  $L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})$  contains no  $p$ -local Euler factors. Hence  $v_p(\mathbb{S}_{GD}(L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})) (n)) = 0$ . On the other hand, for each prime  $\underline{\ell}$  of  $A$  over a given prime number  $\ell$ , the completion  $(\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A))_{\underline{\ell}}^{\wedge}$  is a finite  $\hat{A}_{\underline{\ell}}$ -module, hence has order equal to some power of  $\ell$ , and  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)$  splits as a direct sum of its completions at the various maximal ideals  $\mathfrak{p}$  of  $A$  (this is isomorphism 5.0.2 from the proof of Theorem 5.0.8). Consequently  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$  has no summands of order a power of  $p$ . Consequently the  $p$ -adic valuation of the order of  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$  is zero.

The more important situation is when  $p$  is a prime number such that no primes of  $A$  over  $p$  have ramification degree  $e$  such that  $\log_p \frac{e}{p-1}$  is an integer. Suppose  $p$  is such a prime. Then the  $p$ -adic valuation of the order of  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$  is equal to the  $p$ -adic valuation of the order of the completion  $(\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A))_p^{\wedge}$ , i.e., the product of the  $p$ -adic valuations of the orders of the completions  $(\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A))_{\mathfrak{p}}^{\wedge}$  for all primes  $\mathfrak{p}$  of  $A$  over  $p$ . By Theorem 5.0.4, the completion  $(\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A))_{\mathfrak{p}}^{\wedge}$  is isomorphic as a  $\hat{A}_{\mathfrak{p}}$ -module to  $\text{Ext}_{(V^{\hat{A}_{\mathfrak{p}}}, V^{\hat{A}_{\mathfrak{p}}})}^{1,2n}(V^{\hat{A}_{\mathfrak{p}}}, V^{\hat{A}_{\mathfrak{p}}})$ , which is in turn isomorphic, by Corollary 3.0.17, to  $\hat{A}_{\mathfrak{p}}/H_n$ , where  $H_n$  is the ideal of  $\hat{A}_{\mathfrak{p}}$  generated by all elements of the form  $x^n - 1$ , where  $x \in \hat{A}_{\mathfrak{p}}^{\times}$ . As we showed in Theorem 3.0.16,  $\hat{A}_{\mathfrak{p}}/H_n \cong 0$  if  $p^f - 1 \nmid n$  where  $p^f$  is the cardinality of the residue field of  $\hat{A}_{\mathfrak{p}}$ , and if  $p^f - 1 \mid n$ , then  $H_n = I_{n/(p^f-1)}$ , where  $I_{n/(p^f-1)}$  is the ideal in  $\hat{A}_{\mathfrak{p}}$  generated by all elements of the form  $a^{n/(p^f-1)} - 1$  for elements  $a \in \hat{A}_{\mathfrak{p}}$  congruent to 1 modulo  $\mathfrak{p}$  (this is by the Hensel's Lemma argument at the end of the proof of Corollary 3.0.17).

Finally, Lemma 3.0.13 gives us that the order of  $\hat{A}_{\mathfrak{p}}/I_{n/(p^f-1)}$  is equal to  $p^{fj}$ , where

$$j = \begin{cases} e \left( v_p(n/(p^f - 1)) - \lceil \log_p \frac{e}{p-1} \rceil \right) + p^{\lceil \log_p \frac{e}{p-1} \rceil} & \text{if } \lceil \log_p \frac{e}{p-1} \rceil \leq v_p(n/(p^f - 1)) \\ p^{v_p(n/(p^f - 1))} & \text{if } \lceil \log_p \frac{e}{p-1} \rceil > v_p(n/(p^f - 1)). \end{cases}$$

Consequently our formula for the  $p$ -adic valuation of the order of  $\text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}]$  is:

$$v_p \left( \# \left( \text{Ext}_{(L^A, L^A B)}^{1,2n}(L^A, L^A)[\underline{\Delta}_{K/\mathbb{Q}}^{-1}] \right) \right) = \sum_{\mathfrak{p}} (f_{\mathfrak{p}} j_{\mathfrak{p}}(n)),$$

where the sum is taken over all primes  $\mathfrak{p}$  of  $A$  over  $p$  such that the cardinality  $p^{f_{\mathfrak{p}}}$  of the residue field  $A/\mathfrak{p}$  satisfies  $(p^{f_{\mathfrak{p}}} - 1) \mid n$ , and where  $j_{\mathfrak{p}}(n)$  is defined by the formula

$$(6.3.1) \quad j_{\mathfrak{p}}(n) = \begin{cases} e_{\mathfrak{p}} \left( v_p(n) - \lceil \log_p \frac{e_{\mathfrak{p}}}{p-1} \rceil \right) + p^{\lceil \log_p \frac{e_{\mathfrak{p}}}{p-1} \rceil} & \text{if } \lceil \log_p \frac{e_{\mathfrak{p}}}{p-1} \rceil \leq v_p(n) \\ p^{v_p(n)} & \text{if } \lceil \log_p \frac{e_{\mathfrak{p}}}{p-1} \rceil > v_p(n). \end{cases}$$

But  $K/\mathbb{Q}$  is assumed Galois, so the Galois group  $G_{K/\mathbb{Q}}$  acts transitively on the set of primes  $\mathfrak{p}$  of  $A$  over  $p$ , and any two such primes  $\mathfrak{p}$  have equal ramification degrees and equal residue degrees, and finally, the number of distinct primes  $\mathfrak{p}$  of  $A$  over  $p$  is equal to  $\frac{[K:\mathbb{Q}]}{e_p f_p}$ , where  $e_p$

and  $f_p$  are the ramification degree and residue degree, respectively, of any such prime  $p$ . Consequently,

$$v_p \left( \# \left( \text{Ext}_{(L^A, L^A B)}^{1, 2n} (L^A, L^A) [\underline{\Delta}_{K/\mathbb{Q}}^{-1}] \right) \right) = \frac{[K : \mathbb{Q}]}{e_p} j_p(n),$$

where  $j_p(n)$  is equal to the number  $j_p(n)$  defined in 6.3.1 for any prime  $p$  of  $A$  over  $p$ .

Now we turn to the  $p$ -adic valuation of  $\mathbb{S}_{GD}(L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})) (n)$ . We are still assuming that  $p$  does not divide  $\underline{\Delta}_{K/\mathbb{Q}}$ . Again, since  $K/\mathbb{Q}$  is Galois, every prime  $p$  of  $A$  over  $p$  has the same residue degree, so the  $p$ -local Euler factors in  $L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})$ , taken all together, are of the form

$$\left( \frac{1}{1 - p^{-f_p s}} \right)^{[K:\mathbb{Q}]/(e_p f_p)},$$

where  $e_p, f_p$  are (still) the ramification degree and the residue degree, respectively, of any prime  $p$  of  $A$  over  $p$ . Consequently:

$$v_p \left( \mathbb{S}_{GD}(L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})) (n) \right) = \frac{[K : \mathbb{Q}]}{e_p f_p} h_p(n),$$

where  $h_p(n)$  is defined by the formula

$$h_p(n) = \begin{cases} 0 & \text{if } p^{f_p} - 1 \nmid n \\ \frac{[K:\mathbb{Q}]}{g_p} \left( v_p(n) - \lceil \log_p \frac{[K:\mathbb{Q}]}{f_p g_p (p-1)} \rceil \right) + f_p p^{\lceil \log_p \frac{[K:\mathbb{Q}]}{f_p g_p (p-1)} \rceil} & \text{if } p^{f_p} - 1 \mid n \text{ and } \lceil \log_p \frac{[K:\mathbb{Q}]}{f_p g_p (p-1)} \rceil \leq v_p(n), \\ f_p p^{v_p(n)} & \text{if } p^{f_p} - 1 \mid n \text{ and } \lceil \log_p \frac{[K:\mathbb{Q}]}{f_p g_p (p-1)} \rceil > v_p(n) \end{cases} \quad (6.3.2)$$

$$= \begin{cases} 0 & \text{if } p^{f_p} - 1 \nmid n \\ e_p f_p \left( v_p(n) - \lceil \log_p \frac{e_p}{p-1} \rceil \right) + f_p p^{\lceil \log_p \frac{e_p}{p-1} \rceil} & \text{if } p^{f_p} - 1 \mid n \text{ and } \lceil \log_p \frac{e_p}{p-1} \rceil \leq v_p(n), \\ f_p p^{v_p(n)} & \text{if } p^{f_p} - 1 \mid n \text{ and } \lceil \log_p \frac{e_p}{p-1} \rceil > v_p(n), \end{cases} \quad (6.3.3)$$

where  $g_p$  is the number of distinct primes of  $A$  over  $p$ , i.e.,  $g_p = \frac{[K:\mathbb{Q}]}{e_p f_p}$ .

Now it takes only a moment's worth of arithmetic to verify, using equations 6.3.1 and 6.3.2, that

$$\frac{[K : \mathbb{Q}]}{e_p} j_p(n) = \frac{[K : \mathbb{Q}]}{e_p f_p} h_p(n),$$

and hence that the  $p$ -adic valuation of  $\# \left( \text{Ext}_{(L^A, L^A B)}^{1, 2n} (L^A, L^A) [\underline{\Delta}_{K/\mathbb{Q}}^{-1}] \right)$  agrees with the  $p$ -adic valuation of  $\mathbb{S}_{GD}(L(s, \chi_{\underline{\Delta}_{K/\mathbb{Q}}})) (n)$ .  $\square$

Theorem 6.3.5 is, to me, much more satisfying than Theorem 6.2.4, because Theorem 6.3.5 implies really “global” statements that do not require any localization whatsoever, while Theorem 6.2.4 always requires that, at the very least, 2 be inverted. So Theorem 6.3.5 allows us (when the prime-power-ramification discriminant is trivial) to work with the Dedekind  $\zeta$ -function and not just a Hecke  $L$ -function for some Größencharakter with restrictions on its modulus. For example:

**Corollary 6.3.6.** *Let  $K/\mathbb{Q}$  be a finite field extension with ring of integers  $A$ . Suppose  $K/\mathbb{Q}$  is Galois and suppose that the prime-power-ramification discriminant  $\underline{\Delta}_{K/\mathbb{Q}}$  is equal to one.*

*Then, for all positive  $n \in \mathbb{N}$ , the following numbers are all equal:*

- the order of the abelian group  $\text{Ext}_{(L^A, L^A B)}^{1, 2n} (L^A, L^A)$ ,

- the order of the abelian group  $H_{f_1}^{1,2n}(\mathcal{M}_{f_m A}; \mathcal{O})$ ,
- the order of the abelian group  $A/J_n$ , where  $J_n$  is the minimal  $n$ -congruence ideal in  $A$ , and
- the number  $\mathbb{S}_{GD}(\zeta_K(s))(n)$ .

Arithmetic equivalence and arithmetic equivalence modulo  $m$  were defined in Definitions 6.2.5 and 6.2.7, respectively.

Now, since  $\text{Ext}_{(L, L^A B)}^{1,n}(L^A, L^A)$  vanishes for odd  $n$ , we have the corollary:

**Corollary 6.3.7.** *Let  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  be finite field extensions with ring of integers  $A_1$  and  $A_2$ , respectively. Suppose that  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are both Galois. Let  $m$  be any integer which is divisible by both  $\underline{\Delta}_{K_1/\mathbb{Q}}$  and  $\underline{\Delta}_{K_2/\mathbb{Q}}$ . If  $K_1$  and  $K_2$  are arithmetically equivalent modulo  $m$ , then for all positive  $n \in \mathbb{N}$ , the order of the abelian group*

$$\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,n}(L^{A_1}, L^{A_1})[m^{-1}]$$

is equal to the order of the abelian group

$$\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,n}(L^{A_2}, L^{A_2})[m^{-1}].$$

**Corollary 6.3.8.** *Let  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  be finite field extensions with ring of integers  $A_1$  and  $A_2$ , respectively. Suppose that  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are both Galois, and suppose that  $\underline{\Delta}_{K_1/\mathbb{Q}}$  and  $\underline{\Delta}_{K_2/\mathbb{Q}}$  are both equal to 1. If  $K_1$  and  $K_2$  are arithmetically equivalent, then for all positive  $n \in \mathbb{N}$ , the order of the abelian group*

$$\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,n}(L^{A_1}, L^{A_1})$$

is equal to the order of the abelian group

$$\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,n}(L^{A_2}, L^{A_2}).$$

Equivalently, in terms of the moduli stack  $\mathcal{M}_{f_m A}$  of one-dimensional formal  $A$ -modules over  $\text{Spec } A$ , and with notation as in Conventions 2.0.10: the order of the abelian group

$$H_{f_1}^{1,n}(\mathcal{M}_{f_m A_1}; \mathcal{O})$$

is equal to the order of the abelian group

$$H_{f_1}^{1,n}(\mathcal{M}_{f_m A_2}; \mathcal{O}).$$

The converse of Corollary 6.3.7 is proven in Theorem 6.4.7, and the converse of Corollary 6.3.8 is in Corollary 6.4.8. Theorem 6.4.7 and Corollary 6.4.8 also imply strengthened versions of Corollaries 6.3.7 and 6.3.8: specifically, in the conclusions of Corollaries 6.3.7 and 6.3.8 we get more than just that the two abelian groups are of equal order: we even get that they are *isomorphic*.

**6.4. The inverse Galois-Dedekind straightening transform.** In this section I construct an inverse to the Galois-Dedekind straightening transform, and I use it to prove the main results of this section, Theorem 6.4.7 and Corollary 6.4.8.

First I have to say what kind of functions the inverse Galois-Dedekind straightening transform can be applied to. This requires a preliminary definition:

**Definition 6.4.1.** *Let  $h : \mathbb{N} \rightarrow \mathbb{N}$  be a function, and let  $p$  be a prime number. I will write  $\Xi_p(h)$  for the least positive integer  $n$  such that  $v_p(h(n)) > 0$ , or  $\Xi_p(h) = \infty$  if  $v_p(h(n)) = 0$  for all  $n$ .*

**Definition 6.4.2.** *Let  $h : \mathbb{N} \rightarrow \mathbb{N}$  be a function. I will say that  $h$  is of Dedekind type if  $h$  satisfies all of the following conditions:*

- For all  $n \in \mathbb{N}$ , the integer  $h(n)$  is positive.
- For each prime number  $p$ , either  $\Xi_p(h) = \infty$  or the number  $\log_p(1 + \Xi_p(h))$  is an integer which divides  $\nu_p(h(\Xi_p(h)))$ .

**Example 6.4.3.** Let  $K/\mathbb{Q}$  be a finite Galois extension with ring of integers  $A$ , let  $m$  be a positive integer, and let  $L(s, \chi_m)$  be the Hecke  $L$ -function of the trivial Größencharakter of  $K$  of modulus  $m$ . Then the Galois-Dedekind straightening transform  $\mathbb{S}_{GD}(L(s, \chi_m))$  of  $L(s, \chi_m)$  is of Dedekind type: clearly  $\mathbb{S}_{GD}(L(s, \chi_m))(n) > 0$  for all  $n$ , and for any prime number  $p$ , the smallest integer  $n$  such that  $\mathbb{S}_{GD}(L(s, \chi_m))(n)$  is divisible by  $p$  is  $p^{f_p} - 1$ , where  $f_p$  is the residue degree of any prime of  $A$  over  $p$ . Hence  $\Xi_p(\mathbb{S}_{GD}(L(s, \chi_m))) + 1$  is a power of  $p$ , hence  $\log_p(\Xi_p(\mathbb{S}_{GD}(L(s, \chi_m))) + 1)$  is an integer, namely,  $f_p$ . One can easily use Definition 6.3.3 to check that

$$(6.4.1) \quad \begin{aligned} \nu_p(\mathbb{S}_{GD}(L(s, \chi_m))(\Xi_p(\mathbb{S}_{GD}(L(s, \chi_m)))))) &= \nu_p(\mathbb{S}_{GD}(L(s, \chi_m))(p^{f_p} - 1)) \\ &= f_p g_p, \end{aligned}$$

where  $g_p$  is the number of primes of  $A$  over  $p$ . Hence  $\log_p(\Xi_p(\mathbb{S}_{GD}(L(s, \chi_m))) + 1)$  is indeed an integer which divides  $\nu_p(\mathbb{S}_{GD}(L(s, \chi_m))(\Xi_p(\mathbb{S}_{GD}(L(s, \chi_m)))))$ .

As the special case when  $m = 1$ , the Galois-Dedekind straightening transform  $\mathbb{S}_{GD}(\zeta_K(s))$  of any finite Galois extension  $K/\mathbb{Q}$  is of Dedekind type.

**Definition 6.4.4.** Let  $h : \mathbb{N} \rightarrow \mathbb{N}$  be a function of Dedekind type. By the inverse Galois-Dedekind straightening transform of  $h$  I mean the formal product

$$\mathbb{S}_{GD}^{-1}(h)(s) = \prod_p \frac{1}{(1 - p^{-f_p s})^{g_p}},$$

where  $f_p$  and  $g_p$  are defined by the formulas

$$\begin{aligned} f_p &= \log_p(1 + \Xi_p(h)), \\ g_p &= \frac{\nu_p(h(\Xi_p(h)))}{f_p}. \end{aligned}$$

It is not at all clear that the inverse Galois-Dedekind straightening transform of a function of Dedekind type will converge anywhere in the complex plane at all. However, inverse Galois-Dedekind straightening transform of a function which is itself the Galois-Dedekind straightening transform of a Hecke  $L$ -function of a trivial Größencharakter of some modulus recovers that Hecke  $L$ -function, hence converges:

**Proposition 6.4.5.** Let  $K/\mathbb{Q}$  be a finite Galois extension, let  $m$  be a positive integer, and let  $L(s, \chi_m)$  be the Hecke  $L$ -function of the trivial Größencharakter of  $K$  of modulus  $m$ . Then the formal product  $\mathbb{S}_{GD}^{-1}(\mathbb{S}_{GD}L(s, \chi_m))(t)$  converges for all complex numbers  $t$  with real part  $\Re t > 1$ , and furthermore,

$$\mathbb{S}_{GD}^{-1}(\mathbb{S}_{GD}L(s, \chi_m))(t) = L(t, \chi_m).$$

*Proof.* Both  $\mathbb{S}_{GD}^{-1}(\mathbb{S}_{GD}L(s, \chi_m))(t)$  and  $L(t, \chi_m)$  are products of elementary Euler factors, so if we can show that the two have the same elementary  $p$ -local Euler factors for all prime numbers  $p$ , then we will be done.

Let  $p$  be a prime number dividing  $m$ . Then  $L(s, \chi_m)$  has no  $p$ -local elementary Euler factors, hence  $\mathbb{S}_{GD}(L(s, \chi_m))(n)$  is prime to  $p$  for all  $n$ , hence  $\Xi_p(\mathbb{S}_{GD}L(s, \chi_m)) = \infty$ . Hence  $\mathbb{S}_{GD}^{-1}(\mathbb{S}_{GD}L(s, \chi_m))$  also has no  $p$ -local elementary Euler factors.

Now let  $p$  be a prime number not dividing  $m$ , and write  $A$  for the ring of integers of  $K$ . Then there is one  $p$ -local elementary Euler factors of  $L(s, \chi_m)$  for each distinct prime  $\mathfrak{p}$  of  $A$  over  $p$ , and that  $p$ -local elementary Euler factor is equal to  $\frac{1}{1-p^{-f_{\mathfrak{p}}s}}$ , where  $f_{\mathfrak{p}}$  is the residue degree of  $\mathfrak{p}$ . Since  $K/\mathbb{Q}$  was assumed Galois, any two primes of  $A$  over  $p$  have equal residue degrees, so the  $p$ -local elementary Euler factors of  $L(s, \chi_m)$ , taken all together, are equal to

$$(6.4.2) \quad \frac{1}{(1-p^{-f_p s})^{g_p}},$$

where  $g_p$  is the number of primes of  $A$  over  $p$ , and  $f_p$  is equal to  $f_{\mathfrak{p}}$  for any such prime  $\mathfrak{p}$ .

Hence, from the definition of  $\mathbb{S}_{GD}$  in Definition 6.3.3, the smallest positive integer  $n$  such that  $\mathbb{S}_{GD}(L(s, \chi_m))(n)$  is divisible by  $p$  is  $n = p^{f_p} - 1$ , and

$$\mathbb{S}_{GD}(L(s, \chi_m))(p^{f_p} - 1) = p^{f_p g_p},$$

as one can easily check from Definition 6.3.3. Hence  $\log_p(1 + \Xi_p(\mathbb{S}_{GD}(L(s, \chi_m)))) = f_p$ , and

$$v_p(\mathbb{S}_{GD}(L(s, \chi_m))(\Xi_p(\mathbb{S}_{GD}(L(s, \chi_m)))) = f_p g_p,$$

as already observed in equation 6.4.1. Consequently the  $f_p$  and  $g_p$  appearing in the formula 6.4.2 are equal to the  $f_p$  and  $g_p$  appearing in Definition 6.4.4, i.e., the elementary  $p$ -local Euler factors in

$$\mathbb{S}_{GD}^{-1}(\mathbb{S}_{GD}(L(s, \chi_m)))(t),$$

taken all together, are equal to 6.4.2.  $\square$

**Lemma 6.4.6.** *Let  $K/\mathbb{Q}_p$  be a finite field extension with ring of integers  $A$ , ramification degree  $e$ , and residue degree  $f$ . Let  $\pi$  be a uniformizer for  $A$ , and let  $n$  be a positive integer, and write  $n$  as  $n = qe + r$ , with  $q$  a nonnegative integer and  $r$  an integer satisfying  $0 \leq r < e$ . Then we have an isomorphism of abelian groups:*

$$(6.4.3) \quad A/\pi^n \cong \left( (\mathbb{Z}/p^{q+1}\mathbb{Z})^{\oplus f} \right)^{\oplus r} \oplus \left( (\mathbb{Z}/p^q\mathbb{Z})^{\oplus f} \right)^{\oplus (e-r)}.$$

*Proof.* Since  $A$  is the maximal  $\hat{\mathbb{Z}}_p$ -order in  $K$  and  $[K : \mathbb{Q}_p] = ef$ , we have that  $A$  is isomorphic to  $\hat{\mathbb{Z}}_p^{\oplus ef}$  as an abelian group. Hence  $A/(p^m)$  is isomorphic to  $(\mathbb{Z}/p^m\mathbb{Z})^{\oplus ef}$  as an abelian group. Since  $K/\mathbb{Q}_p$  has ramification degree  $e$ , we have that  $(\pi^e) = (\pi)^e = (p)$ , and consequently

$$A/\pi^{qe} \cong (\mathbb{Z}/p^q\mathbb{Z})^{\oplus ef}$$

as abelian groups, i.e., formula 6.4.3 holds for  $r = 0$ .

If  $0 < r < e$ , then  $A/\pi^n$  sits in a sequence of surjections of  $A$ -modules

$$A/\pi^{(q+1)e} \twoheadrightarrow A/\pi^n \twoheadrightarrow A/\pi^{qe},$$

with  $A/\pi^{(q+1)e} \cong (\mathbb{Z}/p^{q+1}\mathbb{Z})^{\oplus ef}$  and  $A/\pi^{qe} \cong (\mathbb{Z}/p^q\mathbb{Z})^{\oplus ef}$  as abelian groups. This observation is the initial step in an induction: suppose  $s$  is an integer,  $0 < s < e$ , and isomorphism 6.4.3 is known to hold for all  $r < s$ .

Now we make three observations about  $A/\pi^{qe+s}$ :

- $A/\pi^{qe+s}$  sits in a short exact sequence of  $A$ -modules

$$0 \rightarrow \pi^{qe+s-1}/\pi^{qe+s} \rightarrow A/\pi^{qe+s} \rightarrow A/\pi^{qe+s-1} \rightarrow 0,$$

and  $\pi^{qe+s-1}/\pi^{qe+s} \cong \mathbb{F}_{p^f} \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus f}$  as abelian groups.

- By the inductive hypothesis, we have the isomorphism of abelian groups

$$A/\pi^{qe+s-1} \cong \left( (\mathbb{Z}/p^{q+1}\mathbb{Z})^{\oplus f} \right)^{\oplus (s-1)} \oplus \left( (\mathbb{Z}/p^q\mathbb{Z})^{\oplus f} \right)^{\oplus (e-s+1)}.$$

- $A/\pi^{(q+1)e} \cong ((\mathbb{Z}/p^{q+1}\mathbb{Z})^{\oplus f})^{\oplus e}$  surjects on to  $A/\pi^{qe+s}$ .

Up to isomorphism, there is only one abelian group  $A/\pi^{qe+s}$  which satisfies all three of these properties, namely,

$$A/\pi^{qe+s} \cong ((\mathbb{Z}/p^{q+1}\mathbb{Z})^{\oplus f})^{\oplus(s)} \oplus ((\mathbb{Z}/p^q\mathbb{Z})^{\oplus f})^{\oplus(e-s)}.$$

This completes the induction and the proof.  $\square$

Now Proposition 6.4.5 implies the converse of Corollaries 6.3.7 and 6.3.8:

**Theorem 6.4.7.** *Let  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  be finite field extensions with ring of integers  $A_1$  and  $A_2$ , respectively. Suppose that  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are both Galois. Let  $m$  be any integer which is divisible by both  $\Delta_{K_1/\mathbb{Q}}$  and  $\Delta_{K_2/\mathbb{Q}}$ . Then the following statements are all equivalent:*

- (1)  $K_1$  and  $K_2$  are arithmetically equivalent modulo  $m$ .
- (2) For all positive  $n \in \mathbb{N}$ , the order of the abelian group

$$\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,n}(L^{A_1}, L^{A_1})[m^{-1}]$$

is equal to the order of the abelian group

$$\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,n}(L^{A_2}, L^{A_2})[m^{-1}].$$

- (3) For all positive  $n \in \mathbb{N}$ , the order of the abelian group

$$H_{fl}^{1,n}(\mathcal{M}_{fmA_1}; O)[m^{-1}]$$

is equal to the order of the abelian group

$$H_{fl}^{1,n}(\mathcal{M}_{fmA_2}; O)[m^{-1}].$$

- (4) The graded abelian group

$$\text{Ext}_{(L^{A_1}, L^{A_1} B)}^1(L^{A_1}, L^{A_1})[m^{-1}]$$

is isomorphic to the graded abelian group

$$\text{Ext}_{(L^{A_2}, L^{A_2} B)}^1(L^{A_2}, L^{A_2})[m^{-1}].$$

- (5) The graded abelian group

$$H_{fl}^1(\mathcal{M}_{fmA_1}; O)[m^{-1}]$$

is isomorphic to the graded abelian group

$$H_{fl}^1(\mathcal{M}_{fmA_2}; O)[m^{-1}].$$

- (6) For all positive  $n \in \mathbb{N}$ , the order of the abelian group  $A_1/(J_{n,1})[m^{-1}]$  is equal to the order of the abelian group  $A_2/(J_{n,2})[m^{-1}]$  where  $J_{n,1}$  is the minimal  $n$ -congruence ideal of  $A_1[m^{-1}]$ . and  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_2[m^{-1}]$ .
- (7) For all positive  $n \in \mathbb{N}$ , the abelian group  $A_1/(J_{n,1})[m^{-1}]$  is isomorphic to the abelian group  $A_2/(J_{n,2})[m^{-1}]$  where  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_1[m^{-1}]$ . and  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_2[m^{-1}]$ .

*Proof.* For the duration of this proof, I will write  $L(s, \chi_{m,i})$  for the Hecke  $L$ -function of the trivial Größencharakter of modulus  $m$  on  $K_i$ , where  $i \in \{1, 2\}$ . Since  $\text{Ext}_{(L^A, L^A B)}^{1,n}(L^A, L^A) \cong H_{fl}^{1,n}(\mathcal{M}_{fmA}; O)$  vanishes for odd  $n$ , I will work only with  $\text{Ext}^{1,2n}$  and  $H^{1,2n}$  in this proof.

- Condition 2 is equivalent to condition 3 and condition 4 is equivalent to condition 5: This is the usual cohomology-preserving equivalence of comodules over a Hopf algebra with quasicohereant modules over the associated algebraic stack. See Conventions 2.0.10.

- *Condition 1 implies condition 2:* If  $K_1$  and  $K_2$  are arithmetically equivalent modulo  $m$ , then

$$\# \left( \text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}] \right) = \# \left( \text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,2n}(L^{A_2}, L^{A_2})[m^{-1}] \right)$$

by Corollary 6.3.7.

- *Condition 2 implies condition 1:* By Theorem 6.3.5, for  $i \in \{1, 2\}$ , the order of  $\# \left( \text{Ext}_{(L^{A_i}, L^{A_i} B)}^{1,2n}(L^{A_i}, L^{A_i})[m^{-1}] \right)$  is equal to  $\mathbb{S}_{GD}(L(s, \chi_{m,i}))(n)$ . By Proposition 6.4.5, we can recover  $L(s, \chi_{m,i})$  from  $\mathbb{S}_{GD}(L(s, \chi_{m,i}))$  by applying the inverse Galois-Dedekind straightening transform. Consequently, if

$$\# \left( \text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}] \right) = \# \left( \text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,2n}(L^{A_2}, L^{A_2})[m^{-1}] \right),$$

then  $L(s, \chi_{m,1}) = L(s, \chi_{m,2})$ , so  $K_1$  and  $K_2$  are arithmetically equivalent modulo  $m$ .

- *Condition 2 is equivalent to condition 4:* If  $\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}]$  is isomorphic as an abelian group to  $\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,2n}(L^{A_2}, L^{A_2})[m^{-1}]$ , then clearly the two groups have the same order for all  $n > 0$ .

For the converse: suppose  $\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}]$  has the same order as  $\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,2n}(L^{A_2}, L^{A_2})[m^{-1}]$  for all  $n > 0$ . I want to show that this implies that  $\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,i}(L^{A_1}, L^{A_1})[m^{-1}]$  is isomorphic, as an abelian group, to  $\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,i}(L^{A_2}, L^{A_2})[m^{-1}]$ , for all integers  $i$ . I do this in several steps:

**When  $i$  is odd or negative:** For any commutative ring  $A$ , the graded Hopf algebra  $(L^A, L^A B)$  is concentrated in nonnegative grading degrees, so  $\text{Ext}_{(L^A, L^A B)}^1(L^A, L^A)$  is concentrated in nonnegative grading degrees as well (since there are no 1-cocycles in the cobar complex in negative grading degrees). Hence  $\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,i}(L^{A_1}, L^{A_1})[m^{-1}]$  and  $\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,i}(L^{A_2}, L^{A_2})[m^{-1}]$  both vanish, for all  $i < 0$ . By the same argument,  $\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,i}(L^{A_1}, L^{A_1})[m^{-1}]$  and  $\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,i}(L^{A_2}, L^{A_2})[m^{-1}]$  both vanish when  $i$  is odd.

**When  $i = 0$ :** We are assuming that  $\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}]$  has the same order as  $\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,2n}(L^{A_2}, L^{A_2})[m^{-1}]$  for all  $n > 0$ , and earlier in this proof, I have already shown that this implies that  $K_1$  and  $K_2$  are arithmetically equivalent modulo  $m$ . Hence (and since  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are assumed Galois), for infinitely many prime numbers  $p$  (namely, those prime numbers which do not ramify in  $A_1$  or  $A_2$  and which do not divide  $m$ ), the residue degree  $f_{p,1}$  of the primes over  $p$  in  $A_1$  coincides with the residue degree  $f_{p,2}$  of the primes over  $p$  in  $A_2$ , and the number of primes  $g_{p,1}$  in  $A_1$  over  $p$  coincides with the number of primes  $g_{p,2}$  over  $p$  in  $A_2$ , and furthermore,  $[K_1 : \mathbb{Q}] = f_{p,1}g_{p,1} = f_{p,2}g_{p,2} = [K_2 : \mathbb{Q}]$  for those primes.

Since  $K_1$  and  $K_2$  have the same degree over  $\mathbb{Q}$ , we now have that  $A_1$  and  $A_2$  are isomorphic as abelian groups (since each is isomorphic to a  $[K_1 : \mathbb{Q}] = [K_2 : \mathbb{Q}]$ -fold direct sum of copies of  $\mathbb{Z}$ ). From Corollary 5.0.9, we already know that  $\text{Ext}_{(L^A, L^A B)}^{1,0}(L^A, L^A)[m^{-1}] \cong A[m^{-1}]$ , since the minimal 0-congruencing ideal of  $A$  is, of course,  $(0)$ . Consequently  $\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,0}(L^{A_1}, L^{A_1})[m^{-1}]$  is isomorphic, as an abelian group, to  $\text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,0}(L^{A_2}, L^{A_2})[m^{-1}]$ .

**When  $i = 0$ :** Choose a prime number  $p$  not dividing  $m$ , and write  $\mathfrak{p}_{1,1}, \dots, \mathfrak{p}_{1,g}$  for the set of primes of  $A_1$  over  $p$ , and  $\mathfrak{p}_{2,1}, \dots, \mathfrak{p}_{2,g}$  for the set of primes of  $A_2$



over  $p$ . (These two sets do indeed have the same cardinality, namely, the set of  $p$ -local elementary Euler factors in  $L(s, \chi_{m,1})$ , equivalently,  $L(s, \chi_{m,1})$ .) By Corollary 3.0.17 and Theorem 5.0.4, the order of the abelian group

$$\left( \text{Ext}_{(L^{A_i}, L^{A_i} B)}^{1,2n}(L^{A_i}, L^{A_i})[m^{-1}] \right)_{\mathfrak{p}_{i,j}}^{\wedge} \cong \left( \text{Ext}_{(L^{A_i}, L^{A_i} B)}^{1,2n}(L^{A_i}, L^{A_i}) \right)_{\mathfrak{p}_{i,j}}^{\wedge},$$

the  $\mathfrak{p}_{i,j}$ -adic completion of  $\text{Ext}_{(L^{A_i}, L^{A_i} B)}^{1,2n}(L^{A_i}, L^{A_i})[m^{-1}]$ , is given by the formula in Lemma 3.0.13, and this formula depends only on the ramification degree and the residue degree of  $\mathfrak{p}_{i,j}$  in the field extension  $K_i/\mathbb{Q}$ . That ramification degree and that residue degree are both independent of  $j$ , since  $K_i/\mathbb{Q}$  is Galois, and also independent of  $i$ , since  $K_1, K_2$  are assumed arithmetically equivalent modulo  $m$ .

Now the  $A_i[m^{-1}]$ -module  $\text{Ext}_{(L^{A_i}, L^{A_i} B)}^{1,2n}(L^{A_i}, L^{A_i})[m^{-1}]$  is cyclic, by Corollary 3.0.17,

hence its  $\mathfrak{p}$ -adic completion  $\left( \text{Ext}_{(L^{A_i}, L^{A_i} B)}^{1,2n}(L^{A_i}, L^{A_i})[m^{-1}] \right)_{\mathfrak{p}}^{\wedge}$  is a cyclic torsion  $(A_i)_{\mathfrak{p}}^{\wedge}$ -module for all maximal ideals  $\mathfrak{p}$  of  $A_i[m^{-1}]$ . Since  $(A_i)_{\mathfrak{p}}^{\wedge}$  is a discrete valuation ring, every cyclic torsion module over it is isomorphic to  $(A_i)_{\mathfrak{p}}^{\wedge}/\pi^n$  for some  $n$ , where  $\pi$  is a uniformizer for  $(A_i)_{\mathfrak{p}}^{\wedge}$ . Now by Lemma 6.4.6, the structure of  $(A_i)_{\mathfrak{p}}^{\wedge}/\pi^n$  as an abelian group is determined entirely by  $n$  and the ramification degree and residue degree of the quotient field of  $(A_i)_{\mathfrak{p}}^{\wedge}$  as an extension of  $\mathbb{Q}_p$ . Since  $K_1$  and  $K_2$  are arithmetically equivalent modulo  $m$  and are each Galois over  $\mathbb{Q}$ , the  $\mathfrak{p}_1$ -adic completion of  $K_1$  has the same residue degree and ramification degree over  $\mathbb{Q}_p$  as the residue degree and ramification degree of the  $\mathfrak{p}_2$ -adic completion of  $K_2$  over  $\mathbb{Q}_p$ , for any choice of prime  $\mathfrak{p}_1$  of  $A_1$  and prime  $\mathfrak{p}_2$  of  $A_2$ , both over  $p$ . Consequently, knowing that

$$\left( \text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}] \right)_p^{\wedge} \cong \bigoplus_{\mathfrak{p}_1/p} \left( \text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}] \right)_{\mathfrak{p}_1}^{\wedge}$$

and

$$\left( \text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,2n}(L^{A_2}, L^{A_2})[m^{-1}] \right)_p^{\wedge} \cong \bigoplus_{\mathfrak{p}_2/p} \left( \text{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,2n}(L^{A_2}, L^{A_2})[m^{-1}] \right)_{\mathfrak{p}_2}^{\wedge}$$

have the same order implies that they are isomorphic as abelian groups.

So we have isomorphisms of abelian groups

$$(6.4.4) \quad \left( \text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}] \right)_p^{\wedge} \cong \left( \text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}] \right)_p^{\wedge} \\ \cong T^{\oplus g},$$

for some abelian group  $T$  of finite order (that order depends on  $n$ , and it is described by the formula in Lemma 3.0.13, but it is not important for this part of the proof). The groups  $\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}]$  and  $\text{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,2n}(L^{A_1}, L^{A_1})[m^{-1}]$  are  $\mathbb{Z}[m^{-1}]$ -modules of finite order, hence are isomorphic as abelian groups if their  $p$ -adic completions are isomorphic for all  $p$  not dividing  $m$ . This indeed happens, since we have the isomorphism 6.4.4 for all prime numbers  $p$  not dividing  $m$ .

- *Condition 2 is equivalent to condition 6, and condition 4 is equivalent to condition 7:* This is the content of Corollary 5.0.9.

□

**Corollary 6.4.8.** *Let  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  be finite field extensions with ring of integers  $A_1$  and  $A_2$ , respectively. Suppose that  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are both Galois, and suppose that the prime-power-ramification discriminants  $\underline{\Delta}_{K_1/\mathbb{Q}}$  and  $\underline{\Delta}_{K_2/\mathbb{Q}}$  are both equal to 1.*

- (1)  $K_1$  and  $K_2$  are arithmetically equivalent.
- (2) For all positive  $n \in \mathbb{N}$ , the order of the abelian group

$$\mathrm{Ext}_{(L^{A_1}, L^{A_1} B)}^{1,n}(L^{A_1}, L^{A_1})$$

is equal to the order of the abelian group

$$\mathrm{Ext}_{(L^{A_2}, L^{A_2} B)}^{1,n}(L^{A_2}, L^{A_2}).$$

- (3) For all positive  $n \in \mathbb{N}$ , the order of the abelian group

$$H_{f_1}^{1,n}(\mathcal{M}_{f_{m_{A_1}}}; \mathcal{O})$$

is equal to the order of the abelian group

$$H_{f_1}^{1,n}(\mathcal{M}_{f_{m_{A_2}}}; \mathcal{O}).$$

- (4) The graded abelian group

$$\mathrm{Ext}_{(L^{A_1}, L^{A_1} B)}^1(L^{A_1}, L^{A_1})$$

is isomorphic to the graded abelian group

$$\mathrm{Ext}_{(L^{A_2}, L^{A_2} B)}^1(L^{A_2}, L^{A_2}).$$

- (5) The graded abelian group

$$H_{f_1}^1(\mathcal{M}_{f_{m_{A_1}}}; \mathcal{O})$$

is isomorphic to the graded abelian group

$$H_{f_1}^1(\mathcal{M}_{f_{m_{A_2}}}; \mathcal{O}).$$

- (6) For all positive  $n \in \mathbb{N}$ , the order of the abelian group  $A_1/(J_{n,1})$  is equal to the order of the abelian group  $A_2/(J_{n,2})$  where  $J_{n,1}$  is the minimal  $n$ -congruence ideal of  $A_1$ . and  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_2$ .
- (7) For all positive  $n \in \mathbb{N}$ , the abelian group  $A_1/(J_{n,1})$  is isomorphic to the abelian group  $A_2/(J_{n,2})$  where  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_1$ . and  $J_{n,2}$  is the minimal  $n$ -congruence ideal of  $A_2$ .

**Observation 6.4.9.** Suppose  $K_1, K_2$  are finite extensions of  $\mathbb{Q}$ , with ring of integers  $A_1, A_2$ , respectively. Then the graded abelian group  $\mathrm{Ext}_{(L^{A_1}, L^{A_1} B)}^0(L^{A_1}, L^{A_1})$  is isomorphic to the graded abelian group  $\mathrm{Ext}_{(L^{A_2}, L^{A_2} B)}^0(L^{A_2}, L^{A_2})$  if and only if  $[K_1 : \mathbb{Q}] = [K_2 : \mathbb{Q}]$ . This is easy to prove: it is because  $\mathrm{Ext}_{(L^{A_i}, L^{A_i} B)}^{0,*}(L^{A_i}, L^{A_i}) \cong A_i$  concentrated in grading degree zero, and, as an abelian group,  $A_i$  is isomorphic to a  $[K_i : \mathbb{Q}]$ -fold direct sum of copies of  $\mathbb{Z}$ . As a slogan: “degree 0 cohomology of the moduli stack of formal  $A$ -modules detects the degree of  $K/\mathbb{Q}$ .”

Corollary 6.4.8 then becomes, as a slogan, “degree 1 cohomology of the moduli stack of formal  $A$ -modules detects the arithmetic equivalence class of  $K/\mathbb{Q}$ ,” at least when  $K/\mathbb{Q}$  is Galois and the prime-power-ramification discriminant  $\bar{\Delta}_{K/\mathbb{Q}}$  is equal to 1.

I know of no reason why the restriction on the prime-power-ramification discriminant in Observation 6.4.9 cannot be lifted, aside from that the necessary computations (generalizing those in the proof of Theorem 3.0.16) are simply much harder. I suspect that that restriction can indeed be lifted, i.e., that the following is true:

**Conjecture 6.4.10. (Arithmetic equivalence conjecture.)** *Let  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  be finite field extensions with ring of integers  $A_1$  and  $A_2$ , respectively. Suppose that  $K_1/\mathbb{Q}$  and  $K_2/\mathbb{Q}$  are both Galois. Then  $K_1$  and  $K_2$  are arithmetically equivalent if and only if the graded abelian group*

$$\mathrm{Ext}_{(L^{A_1}, L^{A_1} B)}^1(L^{A_1}, L^{A_1})$$

*is isomorphic to the graded abelian group*

$$\mathrm{Ext}_{(L^{A_2}, L^{A_2} B)}^1(L^{A_2}, L^{A_2}).$$

*Equivalently, in terms of the moduli stack  $\mathcal{M}_{fmA}$  of one-dimensional formal  $A$ -modules over  $\mathrm{Spec} A$ , and with notation as in Conventions 2.0.10:  $K_1$  and  $K_2$  are arithmetically equivalent if and only if the graded abelian group*

$$H_{f1}^1(\mathcal{M}_{fmA_1}; \mathcal{O})$$

*is isomorphic to the abelian group*

$$H_{f1}^1(\mathcal{M}_{fmA_2}; \mathcal{O}).$$

By Theorem 6.2.6, two Galois extensions of  $\mathbb{Q}$  are arithmetically equivalent if and only if they are isomorphic, so Conjecture 6.2.6 is equivalent to the statement that, if  $A$  is the ring of integers in a finite Galois extension  $K/\mathbb{Q}$ , then the graded abelian group  $\mathrm{Ext}_{(L^A, L^A B)}^1(L^A, L^A)$  uniquely determines  $K$ , up to isomorphism.

The assumptions in Conjecture 6.4.10 matter; in particular, if  $K_1, K_2$  are not both Galois, then I do not know whether to expect anything like Conjecture 6.4.10 to hold.

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