RAVENEL'S ALGEBRAIC EXTENSIONS OF THE SPHERE SPECTRUM DO NOT EXIST

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ABSTRACT. In this paper we prove a topological nonrealizability theorem: certain classes of graded BP_*-modules are shown to never occur as the BP-homology of a spectrum. Many of these BP_*-modules admit the structure of BP_*BP-comodules, meaning that their topological nonrealizability does not follow from earlier results like Landweber's filtration theorem. As a consequence we solve Ravenel's 1983 problem on the existence of "algebraic extensions of the sphere spectrum": algebraic extensions of the sphere spectrum do not exist, except in trivial cases.

CONTENTS

1. INTRODUCTION.

This paper solves an open problem posed by D. Ravenel in 1983 on topological realizability of the classifying rings of formal groups with complex multiplication, i.e., "formal modules."[1] Given a commutative ring A, a *formal A-module* is a formal group law F equipped with a ring map $A \xrightarrow{\rho} \text{End}(F)$ such that $\rho(a)(X) \equiv aX \mod X^2$. There exists a classifying ring L^A of formal A-modules, and if A is a discrete valuation ring, there exists a classifying ring V^A of A-typical formal A-modules. See [24] and [23] for general properties and explicit computations of the rings L^A, and [8] and [19] for V^A. The ring L^A is naturally a commutative graded algebra over Lazard's classifying ring L of formal group laws, i.e., the coefficient ring MU_* of complex bordism, by the ring map $L \to L^A$ classifying the underlying formal group law of the universal formal A-module. For the same reasons, the ring V^A is naturally a commutative graded algebra over the classifying ring V of p-typical formal group laws, i.e., the coefficient ring BP_* of p-primary Brown-Peterson homology. See [18] for this relationship between L and MU_* and between V and BP_*.

Now here is a natural question:

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[1]In this paper, all formal groups and formal modules are assumed to be *one-dimensional*.

Question 1.1.   **Global realization:** Given the ring of integers A in a finite extension
$K/\mathbb{Q}$ of degree > 1, does there exist a spectrum S^A such that $MU_*(S^A) \cong L^A$?

    Local realization: Given the ring of integers A in a finite extension $K/\mathbb{Q}_p$ of degree
> 1, does there exist a spectrum S^A such that $MU_*(S^A) \cong L^A$?

    p-typical local realization: Given the ring of integers A in a finite extension $K/\mathbb{Q}_p$
of degree > 1, does there exist a spectrum S^A such that $BP_*(S^A) \cong V^A$?

Ravenel asked the "Local realization" part of Question 1.1 in [19]. The "Global real-
ization" and "p-typical local realization" parts of Question 1.1 are natural extensions of
Ravenel's original question. (The reason for the restriction to degree > 1 is that the degree
1 cases, $K = \mathbb{Q}$ and $K = \mathbb{Q}_p$, are known classically to have positive answers: the relevant
spectra are simply the sphere spectrum and its p-completion, respectively. See [18].)

Question 1.1 is a reasonable one because, on the one hand, L^A and V^A are very inter-
esting (number-theoretically quite meaningful) MU_*- and BP_*-algebras, so one wants to
know if they occur as the complex bordism or Brown-Peterson homology of some spec-
trum; and on the other hand, Ravenel remarks at the end of [19] that, if such a spectrum S^A
exists, then the cohomology $\mathrm{Ext}^{*,*}_{(L^A, L^A B)}(L^A, L^A)$ of the classifying Hopf algebroid of one-
dimensional formal A-modules (in other words, the flat cohomology of the moduli stack of
formal A-modules) is the E_2-term of the Adams-Novikov spectral sequence computing the
stable homotopy groups $\pi_*(S^A)$. It is Ravenel's suggestion that the spectra S^A ought to be
thought of as "algebraic extensions of the sphere spectrum."

The "p-typical local realization" part of Question 1.1 was taken up by A. Pearlman in
his 1986 Ph.D. thesis [17]. Pearlman was able to show that, for a totally ramified finite
extension $K/\mathbb{Q}_p$ with ring of integers A, there exists a BP-module spectrum Y and an iso-
morphism $\pi_*(Y) \cong V^A$ of graded BP_*-modules; however, Pearlman was not able to resolve
the p-typical local realization problem, i.e., he did not resolve the question of whether
Y splits as $BP \wedge S^A$ for some spectrum S^A. Pearlman also made some computations of
the Morava K-theories of such a spectrum S^A, assuming its existence; and in the 2007
paper [12], T. Lawson proved that, when $K/\mathbb{Q}_p$ is a finite field extension of degree > 1
and with ring of integers A, there does not exist an A_p-ring spectrum (i.e., associative-
up-to-pth-order-homotopies) X such that $MU_*(X) \cong L^A$ as graded rings. This is a negative
answer to a *multiplicative analogue* of the local case of Question 1.1. Lawson's multiplica-
tive argument uses a Kudo-Araki-Dyer-Lashof operation in an essential way (in the case of
an A_p-ring spectrum, by identifying it with a p-fold Massey product), and so it cannot be
reduced to an argument about spectra rather than ring spectra. Lawson also showed, by a
different argument, that there does not exist a spectrum (no ring structure assumed) X such
that $MU_*(X) \cong L^{\hat{\mathbb{Z}}_2[i]}$, where $\hat{\mathbb{Z}}_2[i]$ is the ring of integers in the 2-adic rationals with a prim-
itive fourth root of unity adjoined, but Lawson remarks that his argument does not seem to
generalize (to odd primes, for example). So Ravenel's original question, Question 1.1, has
remained open except in the case $A = \hat{\mathbb{Z}}_2[i]$, where it has a negative answer.

In the present paper we settle all cases, global[2] and local and p-typical local, of Ques-
tion 1.1. The answer turns out to be:

Answer 1.2. No.

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[2]In the global case—but not in the p-typical local case or Ravenel's original case, the local case—there is
a small caveat: we assume that the extension $K/\mathbb{Q}$ is either cyclic of prime power order, or Galois. This is so
that we can find a prime number p which neither ramifies nor splits completely in $\mathcal{O}_K$, and such that every prime
of $\mathcal{O}_K$ over p has the same degree. If one knows, for a given extension $K/\mathbb{Q}$, that such a prime exists, then it is
unnecessary to assume that $K/\mathbb{Q}$ is cyclic of prime power order or Galois in order to get a negative answer to the
global case of Question 1.1.

In the p-typical local case, the method of proof is an analysis of what kinds of BP_*-modules can actually occur as the BP-homology of a spectrum. The arguments use some topology, and are not purely algebraic, unlike earlier topological nonrealizability results on BP_*-modules. Landweber's filtration theorem [11], for example, gives a condition that a BP_*-module must satisfy in order to be a BP_*BP-comodule and hence possibly the BP-homology of a spectrum, but the BP_*-module V^A of Question 1.1 actually *is* a BP_*BP-comodule, so purely algebraic (non-topological) criteria like Landweber's filtration theorem do not shed any light on Question 1.1.

The local and global cases then follows from the p-typical local case, but not in a totally trivial way: the proof I give for the global case, which is the simplest proof I know, uses the fact that in every finite extension of $\mathbb{Q}$ there are infinitely many primes which do not split completely, which indicates that there is at least a little bit of nontrivial (although well-known) number theory involved.

The proof I give in the p-typical local case is really an answer to a more general question, which is as follows: we have the ring map $BP_* \to V^A$ classifying the underlying p-typical formal group law of the universal A-typical formal A-module, and via this ring map, every V^A-module is also a BP_*-module. So one can ask what spectra X have the property that $BP_*(X)$ is a V^A-module, not just a BP_*-module, where again A is the ring of integers in a finite extension $K/\mathbb{Q}_p$ of degree > 1. The answer to this question is given in Theorem 3.9: if $K/\mathbb{Q}_p$ is not totally ramified, then the only spectra X such that $BP_*(X)$ is a V^A-module are dissonant spectra ("dissonant" in the sense of [20]; I give a brief recap of what this means in Remark 3.6). Such spectra do indeed exist, for example, $X = H\mathbb{F}_p$; see the comments at the beginning of section 3. Consequently the only V^A-modules which occur as the BP-homology of a spectrum are V^A-modules on which all of the Hazewinkel generators v_n of BP_* act nilpotently, i.e., for each element $m \in M$, there exists $i \in \mathbb{N}$ such that $v_n^i m = 0$. If $K/\mathbb{Q}_p$ is totally ramified, then the situation is a little more forgiving: in Theorem 4.8 we find that the only spectra X such that $BP_*(X)$ is a V^A-module are extensions of rational spectra by dissonant spectra.

In the case that X *is* a spectrum whose BP-homology admits the structure of a V^A-module, then, at least when $K/\mathbb{Q}_p$ is totally ramified, there exists a change-of-rings isomorphism describing the E_2-term of the Adams-Novikov spectral sequence computing $\pi_*(X)_{(p)}$ in terms of the classifying Hopf algebroid (V^A, V^AT) of formal A-modules:

$$(1.0.1) \qquad \operatorname{Ext}^{*,*}_{(BP_*, BP_*BP)}(BP_*, BP_*(X)) \cong \operatorname{Ext}^{*,*}_{(V^A, V^AT)}(V^A, BP_*(X)),$$

as long as $BP_*(X)$ is flat over BP_*. This is proven in [24]. In the case $BP_*(X) = V^A$, the existence of such a change-of-rings isomorphism seems to have been known to Ravenel (who says it follows from "standard arguments" at the end of [19]), and to Pearlman (who gives a sketch of a proof in [17]). However, in the present paper it is proven that there is no spectrum X such that $BP_*(X) \cong V^A$ unless $K = \mathbb{Q}_p$, rendering that special case moot. But actually writing out the "standard arguments," as is done in [24], leads to the more general isomorphism 1.0.1, which does indeed apply in some cases, e.g. $X = H\mathbb{Q}_p^{\vee d}$ where $d = [K : \mathbb{Q}_p]$, when $K/\mathbb{Q}_p$ is totally ramified.

The conclusion here (with the understanding that, in the global case, we assume that the extension $K/\mathbb{Q}$ is either cyclic of prime power order, or Galois) is Corollary 5.7:

Conclusion 1.3. Except in the trivial cases where $A = \mathbb{Z}$ or $A = \hat{\mathbb{Z}}_p$, Ravenel's algebraic extensions of the sphere spectrum do not exist.

This paper ends with a short appendix on the most basic ideas from local cohomology, for the reader who does not already know them. These ideas are easy, relevant, and useful, and I make use of them starting with the proof of Theorem 4.8.

Remark 1.4. I want to take a moment, however, to defend the use of formal modules in homotopy theory: despite the nonexistence of Ravenel's algebraic spheres, formal modules are very useful for a homotopy theorist! There exist families of "height-shifting" spectral sequences which, generally speaking, take as input a height d formal A-module computation and output a height n formal group law computation, where A is the ring of integers in a field extension $K/\mathbb{Q}_p$, and $[K : \mathbb{Q}_p] \cdot d = n$. There also exist families of spectral sequences which take as input a height d formal group computation and output a height d formal A-module computation. See [25] for this material, where both methods are used in order to compute the cohomology of the height 4 Morava stabilizer group at large primes, using the cohomology of the height 2 Morava stabilizer group as input. In fact, it is the most *useful* properties of formal modules—that the underlying formal group of an A-height n formal A-module has p-height $n \cdot [K : \mathbb{Q}_p]$, i.e., the map of moduli stacks $\mathcal{M}_{fmA} \to \mathcal{M}_{fg}$ sends the height n stratum to the height $n \cdot [K : \mathbb{Q}_p]$ stratum, allowing "height-shifting" techniques to be used in the computation of the flat cohomology of $\mathcal{M}_{fmA}$ and its various substacks— which are responsible for the map $MU_* \cong L \to L^A$ having the algebraic properties which result in the nonexistence results proven in the present paper. So while we do not have the certain kind of topological usefulness of formal modules (as in Question 1.1) which Ravenel wanted, we instead have other kinds of topological usefulness of formal modules.

I am grateful to E. Friedman for suggesting to me that it is possible to remove the assumption that $K/\mathbb{Q}$ is Galois from my original proof that "global" algebraic extensions of the sphere do not exist. I am especially grateful to D. Ravenel for teaching me a great deal about homotopy theory and formal groups over the years.

Conventions 1.5.   (1) Whenever I refer to v_n or BP or $E(n)$, I am implicitly choosing a prime number p, and referring to the p-primary v_n or BP or $E(n)$. Unless otherwise specified, $v_1, v_2, \dots$ will be Hazewinkel's generators for BP_*, rather than Araki's.
   (2) When (A, Γ) is a graded Hopf algebroid, M, N are graded Γ-comodules, and I write $\mathrm{Ext}^{s,t}_{(A,\Gamma)}(M, N)$, I am referring to the sth relative right-derived functor of $\hom_{\mathrm{gr\,Comod}(\Gamma)}(\Sigma^t M, -)$ applied to N; here $\mathrm{gr\,Comod}(\Gamma)$ is the category of graded Γ-comodules, and the allowable class (in the sense of relative homological algebra, as in [6] or chapter IX of [13]) in question is the class of extensions of graded Γ-comodules whose underlying extensions of graded A-modules are split. Appendix 1 of [21] is the standard reference for Ext in categories of comodules relative to this allowable class.
   (3) When R is a commutative ring, $r \in R$, and M is an R-module, I will write $r^{-1}M$ for the localization of M inverting the principal ideal (r), i.e., $r^{-1}M \cong M \otimes_R R[\frac{1}{r}]$.

2. The map $BP_* \to V^A$.

Recall (from [8] or from [19]) that M. Hazewinkel proved that, if A is a discrete valuation ring with finite residue field, then the classifying ring V^A of A-typical formal A-modules is isomorphic to $A[v_1^A, v_2^A, \dots]$, a polynomial algebra over A on countably infinitely many generators. Now suppose we have an extension L/K of p-adic number fields, and suppose that A, B are the rings of integers in K, L, respectively. Then there are induced

maps of classifying rings

$$V^A \xrightarrow{\gamma} V^B \quad \text{and}$$

$$V^A \otimes_A B \xrightarrow{\gamma^\sharp} V^B$$

given by classifying the A-typical formal A-module law underlying the universal B-typical formal B-module law on V^B. One naturally wants to know whether $\gamma^\sharp$ is injective, surjective, or has other recognizable properties (this is necessary in order to prove the topological nonrealizability theorems in section 3 and section 4, for example).

In this section, the two main results are:

- Corollary 2.3: if L/K is unramified, then $\gamma^\sharp$ is surjective, and we compute V^B as a quotient of $V^A \otimes_A B$. (That particular computation is not really new; the case $K = \mathbb{Q}_p$ appeared as Corollary 2.7 in [19], and the proof for general K is essentially the same.)
- Theorem 2.9: if L/K is totally ramified, then $\gamma^\sharp$ is a rational isomorphism. That is, $\gamma^\sharp$ is injective, and $\gamma^\sharp$ is an isomorphism after inverting the prime p. (This result, on the other hand, is new.)

Remark 2.1. In this section we make heavy use of Hazewinkel's generators v_n^A for V^A. They are defined as follows: let A be a discrete valuation ring of characteristic zero and with finite residue field, let q denote the cardinality of the residue field of A, let π denote a uniformizer for A, and let

$$\log_{F^A}(X) = \sum_{n \geq 1} \ell_n^A X^{q^n}$$

be the logarithm of the universal A-typical formal A-module F^A. Then the logarithm coefficients $\ell_n^A \in \mathbb{Q} \otimes_{\mathbb{Z}} A$ satisfy the equations

$$\ell_0^A = 1,$$

(2.0.1) $$\pi \ell_n^A = \sum_{i=0}^{n-1} \ell_i^A (v_{n-i}^A)^{q^i}$$

for some set of parameters $v_0^A = \pi, v_1^A, v_2^A, \ldots$, and these are the Hazewinkel generators for V^A. This material appears in 21.5.4 of [7]. It follows from Hazewinkel's construction of the universal A-typical formal A-module, using Hazewinkel's functional equation lemma, that the map $V^A \xrightarrow{\gamma} V^B$ is "natural in the logarithm," that is, γ sends the coefficient of the degree n term in $\log_{F_A}$ to the coefficient of the degree n term of $\log_{F_B}$, for all n. (It is really $\mathbb{Q} \otimes_{\mathbb{Z}} \gamma : \mathbb{Q} \otimes_{\mathbb{Z}} V^A \to \mathbb{Q} \otimes_{\mathbb{Z}} V^B$ that can be evaluated on the coefficients in the power series $\log_{F_A}$, not γ itself, since the logarithm coefficients involve denominators; when convenient I will write $\gamma(\ell_n^A)$ as shorthand for $(\mathbb{Q} \otimes_{\mathbb{Z}} \gamma)(\ell_n^A)$.) Basically all of the arguments in the rest of this section consist in then solving the resulting equations to see what γ does to the Hazewinkel generators $v_0^A, v_1^A, v_2^A, \ldots$.

The case $K = \mathbb{Q}_p$ of Proposition 2.2 also appears as Corollary 2.7 in [19].

Proposition 2.2. *Let K, L be p-adic number fields, and let L/K be unramified of degree f. Let A, B denote the rings of integers of K and L, respectively. Using the Hazewinkel generators for V^A and V^B, the map $V^A \xrightarrow{\gamma} V^B$ is determined by*

$$\gamma(v_i^A) = \begin{cases} v_{i/f}^B & \text{if } f \mid i \\ 0 & \text{if } f \nmid i \end{cases}.$$

Proof. Let π be a uniformizer for A. Since L/K is unramified, π is also a uniformizer for B. Let q be the cardinality of the residue field of A. We immediately have that $v_0^A = \pi = v_0^B$, so $\gamma(v_0^A) = v_0^B$.

Now, since the residue field of B is $\mathbb{F}_{q^f}$, the logarithm $\log_{F^B}(X) = \sum_{n \geq 1} \ell_n^B X^{q^{fn}}$ has no nonzero terms X^{q^i} for $f \nmid i$. Hence $\gamma(\ell_i^A) = 0$ if $f \nmid i$. So we have that $\gamma(v_i^A) = 0$ for $0 < i < f$, and

$$
\begin{aligned}
\frac{1}{\pi} v_1^B &= \ell_1^B \\
&= \gamma(\ell_f^A) \\
&= \gamma\left(\frac{1}{\pi} \sum_{i=0}^{f-1} \ell_i^A (v_{f-i}^A)^{q^i} \right) \\
&= \frac{1}{\pi} \gamma(v_f^A),
\end{aligned}
$$

so $\gamma(v_f^A) = v_1^B$.

We proceed by induction. Suppose that $\gamma(v_i^A) = 0$ if $f \nmid i$ and $0 < i < hf$, and that $\gamma(v_i^A) = v_{i/f}^B$ if $f \mid i$ and $i < hf$. Then

$$
\begin{aligned}
v_h^B + \sum_{i=1}^{h-1} \ell_i^B (v_{h-i}^B)^{q^{if}} &= \ell_h^B \\
&= \gamma(\ell_{hf}^A) \\
&= \gamma\left(\sum_{i=0}^{hf-1} \ell_i^A (v_{hf-i}^A)^{q^i} \right) \\
&= \sum_{i=0}^{h-1} \ell_i^B \gamma(v_{(h-i)f}^A)^{q^{if}} \\
&= \gamma(v_{hf}^A) + \sum_{i=1}^{h-1} \ell_i^B (v_{h-i}^B)^{q^{if}},
\end{aligned}
$$

so $\gamma(v_{hf}^A) = v_h^B$.

Now suppose that $0 < j < f$ and $\gamma(v_i^A) = 0$ if $f \nmid i$ and $i < hf + j$, and suppose that $\gamma(v_i^A) = v_{i/f}^B$ if $f \mid i$ and $i < hf$. We want to show that $\gamma(v_{hf+j}^A) = 0$. We see that

$$
\begin{aligned}
0 &= \gamma(\ell_{hf+j}^A) \\
&= \gamma\left(\sum_{i=0}^{hf+j-1} \ell_i^A (v_{hf+j-i}^A)^{q^i} \right) \\
&= \sum_{i=0}^{h} \ell_i^B \gamma(v_{(h-i)f+j}^A)^{q^{if}} \\
&= \gamma(v_{hf+j}^A).
\end{aligned}
$$

This concludes the induction. $\square$

Corollary 2.3. *Let K, L be p-adic number fields with rings of integers A, B, respectively, and let L/K be unramified of degree f. Then*

$$V^B \cong (V^A \otimes_A B)/\left(\{v_i^A : f \nmid i\}\right).$$

When L/K is ramified, the map γ is much more complicated. One can always use the above formulas to solve the equation $\gamma(\ell_i^A) = \ell_i^B$ to get a formula for $\gamma(v_i^A)$, as long as one knows $\gamma(v_j^A)$ for all $j < i$; but it seems that there is no simple closed form for $\gamma(v_i^A)$ for general i. Here are formulas for low i:

Proposition 2.4. *Let K, L be p-adic number fields with rings of integers A and B, and let L/K be totally ramified. Let A (and consequently also B) have residue field $\mathbb{F}_q$. Let π_A, π_B be uniformizers for A, B, respectively. Let γ be the ring homomorphism $V^A \xrightarrow{\gamma} V^B$ classifying the underlying formal A-module law of the universal formal B-module law. Then*

$$\gamma(v_1^A) = \frac{\pi_A}{\pi_B} v_1^B,$$

$$\gamma(v_2^A) = \frac{\pi_A}{\pi_B} v_2^B + \left(\frac{\pi_A}{\pi_B^2} - \frac{\pi_A^q}{\pi_B^{q+1}}\right)(v_1^B)^{q+1}.$$

Proof. These formulas follow immediately from setting $\gamma(\ell_i^A) = \ell_i^B$ and solving, using formula 2.0.1. $\square$

Proposition 2.5. *Let K, L be p-adic number fields with rings of integers A, B, respectively, and let L/K be totally ramified. Write $\gamma^\sharp$ for the ring map $V^A \otimes_A B \to V^B$ classifying the underlying A-typical formal A-module of the universal B-typical formal B-module. Then $\gamma^\sharp$ is rationally surjective, that is, the map*

$$V^A \otimes_A B \otimes_\mathbb{Z} \mathbb{Q} \xrightarrow{\gamma^\sharp \otimes_\mathbb{Z} \mathbb{Q}} V^B \otimes_\mathbb{Z} \mathbb{Q}$$

is surjective.

Proof. Let π_A be a uniformizer for A, and let π_B be a uniformizer for B. Given any v_i^B, we want to produce some element of $V^A \otimes_A B \otimes_\mathbb{Z} \mathbb{Q} \cong V^A \otimes_A L$ which maps to it. We begin with $i = 1$. By Proposition 2.4, we have $(\gamma^\sharp \otimes_\mathbb{Z} \mathbb{Q})(\frac{\pi_B}{\pi_A} v_1^A) = v_1^B$.

Now we proceed by induction. Suppose that we have shown that there is an element in $(v_1^A, \ldots, v_{j-1}^A) \subseteq V^A \otimes_\mathbb{A} \mathbb{L}$ which maps via $(\gamma^\sharp \otimes_\mathbb{Z} \mathbb{Q})$ to v_{j-1}^B. We have $(\gamma^\sharp \otimes_\mathbb{Z} \mathbb{Q})(\ell_j^A) = \ell_j^B$, i.e.,

$$(2.0.2) \qquad (\gamma^\sharp \otimes_\mathbb{Z} \mathbb{Q})\left(\pi_A^{-1} \sum_{i=0}^{j-1} \ell_i^A (v_{j-i}^A)^{q^i}\right) = \pi_B^{-1} \sum_{i=0}^{j-1} \ell_i^B (v_{j-i}^B)^{q^i}$$

and hence

$$(\gamma^\sharp \otimes_\mathbb{Z} \mathbb{Q})(\pi_A^{-1} v_j^A) = \pi_B^{-1} \sum_{i=0}^{j-1} \ell_i^B (v_{j-i}^B)^{q^i} - (\gamma^\sharp \otimes_\mathbb{Z} \mathbb{Q})(\pi_A^{-1} \sum_{i=1}^{j-1} \ell_i^A (v_{j-i}^A)^{q^i})$$

$$(2.0.3) \qquad \equiv \pi_B^{-1} v_j^B \mod (v_1^B, v_2^B, \ldots, v_{j-1}^B) \subseteq V^B \otimes_\mathbb{Z} \mathbb{Q}.$$

So $(\gamma^\sharp \otimes_\mathbb{Z} \mathbb{Q})(\frac{\pi_B}{\pi_A} v_j^A) \equiv v_j^B$ modulo terms hit by elements in the ideal generated by Hazewinkel generators of lower degree. This completes the induction. $\square$

For the proof of the next proposition we will use a monomial ordering on V^A.

Definition 2.6. *We put the following ordering on the Hazewinkel generators of V^A:*

$$v_i^A \leq v_j^A \quad \text{iff} \quad i \leq j$$

and we put the lexicographic order on the monomials of V^A. This ordering on the monomials in V^A is a total ordering (see e.g. [4] for these kinds of methods).

We extend this ordering to an ordering on all of V^A by letting $x \leq y$ if and only if the maximal (in the given total ordering on monomials) monomial in x is less than or equal to the maximal monomial in y.

So, for example, v_3^A is "greater than" every monomial in the generators v_1^A and v_2^A; the monomial $(v_2^A)^2$ is "greater than" $(v_1^A)^n v_2^A$ for all n; and so on.

Lemma 2.7. *Let K, L be finite extensions of $\mathbb{Q}_p$ with rings of integers A, B, respectively, and let L/K be a totally ramified extension. Let x, y be monomials in V^A. Then $x \leq y$ implies that $\gamma^\sharp(x) \leq \gamma^\sharp(y)$.*

Proof. Let q be the cardinality of the residue field of A (hence also B), and let π_A, π_B be uniformizers of A, B, respectively. Equation 2.0.2 implies that

$$(2.0.4) \qquad\qquad \gamma(v_n^A) = \frac{\pi_A}{\pi_B} v_n^B + \sum_{i=1}^{n-1} \ell_i^B \left(\frac{\pi_A}{\pi_B} (v_{n-i}^B)^{q^i} - \gamma(v_{n-i}^A)^{q^i} \right).$$

Suppose that $x \leq y$. Let n_x be the largest integer n such that v_n^A divides x, and similar for n_y. If $n_x < n_y$, then $\gamma^\sharp(y)$ has monomials divisible by $v_{n_y}^B$ but there is no monomial of $\gamma^\sharp(x)$ divisible by v_n^B for $n \geq n_y$. Consequently $\gamma^\sharp(x) \leq \gamma^\sharp(y)$. If $n_x = n_y$, then $x/v_{n_x}^A \leq y/v_{n_y}^A$, and we replace x, y with $x/v_{n_x}^A \leq y/v_{n_y}^A$ in the above argument; the obvious downward induction now implies that $\gamma^\sharp(x) \leq \gamma^\sharp(y)$. $\qquad\qquad\square$

Proposition 2.8. *Let K, L be p-adic number fields with rings of integers A, B, and let L/K be a totally ramified, finite extension. Then $V^A \otimes_A B \xrightarrow{\gamma^\sharp} V^B$ is injective.*

Proof. Let π_A, π_B be uniformizers for A, B respectively. By equation 2.0.3, for each n, $\gamma^\sharp(v_n^A) \equiv \frac{\pi_A}{\pi_B} v_n^B$ modulo $(v_1^B, v_2^B, \dots, v_{n-1}^B)$. Suppose that $f \in V^A$ is nonzero. Order the monomials in f using the lexicographic order from Definition 2.6. Since this is a total ordering, there exists a monomial g of f which is maximal. By equation 2.0.4, γ is nonvanishing on each monomial in V^A, so $\gamma^\sharp(g) \neq 0$. Choose a maximal monomial $\overline{g}$ of $\gamma^\sharp(g)$. Adding lesser (in the partial ordering) monomials to $\overline{g}$ cannot yield zero, so Lemma 2.7 implies that $\gamma^\sharp(f) \neq 0$. So $\gamma^\sharp$ is injective. $\qquad\qquad\square$

Theorem 2.9. *Let $K/\mathbb{Q}_p$ be a totally ramified finite extension, and let A be the ring of integers of K. Then the resulting map $BP_* \otimes_{\mathbb{Z}_{(p)}} A \longrightarrow V^A$ is a rational isomorphism, i.e., the graded ring homomorphism*

$$p^{-1}(BP_* \otimes_{\mathbb{Z}_{(p)}} A) \longrightarrow p^{-1} V^A$$

is an isomorphism.

Proof. The map $BP_* \otimes_{\mathbb{Z}_{(p)}} \hat{\mathbb{Z}}_p \to V^{\hat{\mathbb{Z}}_p}$ classifying the underlying p-typical formal group law of the universal $\hat{\mathbb{Z}}_p$-typical formal $\hat{\mathbb{Z}}_p$-module is an isomorphism, by Hazewinkel's computation of V^R for an arbitrary discrete valuation ring R with finite residue field; see section 21.5 of [8]. The map $\gamma^\sharp : V^{\hat{\mathbb{Z}}_p} \otimes_{\hat{\mathbb{Z}}_p} A \to V^A$ is injective by Proposition 2.8, and it is rationally surjective by Proposition 2.5. $\qquad\qquad\square$

3. Nonrealizability in the not-totally-ramified case.

When A is a discrete valuation ring with finite residue field, I write V^A for the classifying ring of A-typical formal A-modules, which as Hazewinkel proved, is isomorphic to a polynomial A-algebra on countably infinitely many generators: $V^A \cong A[v_1^A, v_2^A, \dots]$ (see chapter 21 of [8] for this fact).

Here is an easy observation: the spectrum $H\mathbb{F}_p$ has the property that its BP-homology $BP_*(H\mathbb{F}_p)$, as a graded BP_*-module, splits as a direct sum of suspensions of the residue field $\mathbb{F}_p$ of BP_*. Consequently, if A is the ring of integers in a degree d unramified field extension of $\mathbb{Q}_p$, then $BP_*(H\mathbb{F}_p^{\vee d})$ admits the structure of a V^A-module, since the residue field of V^A is $\mathbb{F}_{p^d}$, which is isomorphic to $\mathbb{F}_p^{\oplus d}$ as a BP_*-module, and we can simply let all the generators $\pi, v_1^A, v_2^A, \dots$ of V^A act trivially on $BP_*(H\mathbb{F}_p^{\vee d})$.

This is a pretty silly example, though: $H\mathbb{F}_p$ is a *dissonant* spectrum (see Remark 3.6 for a brief review of dissonant spectra and their pathological properties), so one cannot use (and does not need) chromatic techniques to learn anything at all about $H\mathbb{F}_p$. One would like to know if there exist any non-dissonant spectra X such that $BP_*(X)$ is the underlying BP_*-module of a V^A-module.

In this section we see that the answer is *no*: if X is a spectrum and $BP_*(X)$ is the underlying BP_*-module of a V^A-module, where A is the ring of integers of some field extension $K/\mathbb{Q}_p$, then either $K/\mathbb{Q}_p$ is totally ramified or X is dissonant (this is Theorem 3.9). This tells us that $BP_*(X)$ cannot be a V^A-module, for $K/\mathbb{Q}_p$ not totally ramified, unless X has some rather pathological properties; for example, $BP_*(X)$ cannot be finitely-presented as a BP_*-module (this is Corollary 3.11), and by Corollary 3.12, the BP-homology of a *space* which is not p-locally contractible is never a V^A-module unless $K/\mathbb{Q}_p$ is totally ramified, due to the Hopkins-Ravenel result [9] that suspension spectra are harmonic. Finally, Corollary 3.10 gives us that V^A itself is not the BP-homology of any spectrum.

Before we begin with new results, here are three known results we will use, Lemma 3.1 and Theorems 3.2 and 3.3:

Lemma 3.1. *The Bousfield class of $L_{E(n)}BP$ is equal to the Bousfield class of $v_n^{-1}BP$, the telescope of v_n on BP.*

Proof. This is Lemma 8.1.4 in Ravenel's book [22]. □

Theorem 3.2. (Ravenel's localization conjecture; proven by Hopkins and Ravenel.) *Let Y be a spectrum and n a positive integer. Then $BP \wedge L_{E(n)}Y \cong Y \wedge L_{E(n)}BP$. Consequently, if $v_{n-1}^{-1}BP_*(Y) \cong 0$, then $BP_*(L_{E(n)}Y) \cong v_n^{-1}BP_*(Y)$.*

Proof. See section 8.1 of [22]. □

Theorem 3.3. (Ravenel's smashing conjecture; proven by Hopkins and Ravenel.) *Let n be a nonnegative integer. Then every p-local spectrum is $E(n)$-prenilpotent. Consequently:*

- *$E(n)$-localization is smashing, i.e., $L_{E(n)}X \simeq L_{E(n)}S \wedge X$ for all p-local spectra X, and*
- *$E(n)$-nilpotent completion coincides with $E(n)$-localization, i.e., $\hat{X}_{E(n)} \simeq L_{E(n)}X$ for all p-local spectra X.*

Proof. See section 8.2 of [22]. □

Definition 3.4. *Let R be a graded ring, and let M be a graded left R-module. Let $r \in R$ be a homogeneous element.*

- *We say that M is r-power-torsion if, for each homogeneous $x \in M$, there exists some nonnegative integer m such that $r^m x = 0$.*
- *We say that r acts trivially on M if $rx = 0$ for all homogeneous $x \in M$.*
- *We say that r acts without torsion on M if for all homogeneous nonzero $x \in M$, $rx \neq 0$.*
- *We say that r acts invertibly on M if, for each homogeneous $x \in M$, there exists some $y \in M$ such that $ry = x$.*
- *Suppose $s \in R$ is a homogeneous element. We say that r acts with eventual s-division on M if, for each homogeneous $x \in M$, there exists some nonnegative integer n and some $y \in M$ such that $sy = r^n x$.*
- *Suppose $s \in R$ is a homogeneous element, and $I \subseteq R$ is a homogeneous ideal. We say that r acts with eventual s-division on M modulo I if, for each homogeneous $x \in M$, there exists some nonnegative integer n and some $y \in M$ such that $sy \equiv r^n x$ modulo IM.*

Lemmas 3.5 and 3.8 and Proposition 3.7 are all easy and surely well-known, but I do not know where to find any of them in the literature:

Lemma 3.5. *Let X be a spectrum, and let n be a nonnegative integer. Then the Bousfield localization $L_{E(n)}X$ is contractible if and only if $BP_*(X)$ is v_n-power-torsion.*

Proof. Suppose that $BP_*(X)$ is v_n-power-torsion. Since $E(n)_*$ is a Landweber exact homology theory (see [11]), we have isomorphisms of BP_*-modules

$$E(n)_*(X) \cong E(n)_* \otimes_{BP_*} BP_*(X)$$
$$\cong v_n^{-1}\left(BP_*(X)/(v_{n+1}, v_{n+2}, \dots)BP_*(X)\right)$$

and $BP_*(X)/(v_{n+1}, v_{n+2}, \dots)BP_*(X)$ is v_n-power-torsion by Lemma 4.1. So $E(n)_*(X) \cong 0$, so X is $E(n)$-acyclic, so $L_{E(n)}X$ is contractible.

On the other hand, if $L_{E(n)}X \cong 0$, then $BP \wedge L_{E(n)}X$ is contractible, hence $L_{E(n)}BP \wedge X$ is contractible by Theorem 3.2. Hence $(v_n^{-1}BP) \wedge X$ is contractible by Lemma 3.1. Now, since $v_n^{-1}BP_*$ is a Landweber-exact homology theory, we have

$$0 \cong \pi_*\left((v_n^{-1}BP) \wedge X\right)$$
$$\cong \left(v_n^{-1}BP_*\right) \otimes_{BP_*} BP_*(X).$$

So inverting v_n kills $BP_*(X)$, so $BP_*(X)$ is v_n-power-torsion. $\square$

Remark 3.6. Recall (from e.g. [20]) that a spectrum X is said to be *dissonant at p* (or just *dissonant*, if the prime p is understood from the context) if it is acyclic with respect to the wedge $\bigvee_{n\in\mathbb{N}} E(n)$ of all the p-local Johnson-Wilson theories, equivalently, if it is acyclic with respect to the wedge $\bigvee_{n\in\mathbb{N}} K(n)$ of all the p-local Morava K-theories. Dissonant spectra are "invisible" to the typical methods of chromatic localization. In some sense dissonant spectra are uncommon; for example, it was proven in [9] that all p-local suspension spectra are *harmonic,* that is, local with respect to the wedge of the p-local Johnson-Wilson theories, hence as far from dissonant as possible. On the other hand, there are some familiar examples of dissonant spectra, for example, the Eilenberg-Mac Lane spectrum HA of any torsion abelian group A (this example appears in [20]).

Proposition 3.7. *Let p be a prime number and let X be a spectrum. Then the following conditions are equivalent:*

- *X is dissonant at p.*

- $BP_*(X)$ is v_n-power-torsion for infinitely many n.
- $BP_*(X)$ is v_n-power-torsion for all n.

Proof.
- Suppose that $BP_*(X)$ is v_n-power-torsion for infinitely many n. For each nonnegative integer m, there exists some integer n such that $n \geq m$ and such that $BP_*(X)$ is v_n-power-torsion. Hence, by Lemma 3.5, $L_{E(n)}X$ is contractible. Hence

$$L_{E(m)}X \simeq L_{E(m)}L_{E(n)}X \simeq L_{E(m)}0 \simeq 0.$$

Hence X is $E(m)$-acyclic for all m. Hence

$$\left(\bigvee_{n \in \mathbb{N}} E(n)\right) \wedge X \simeq \bigvee_{n \in \mathbb{N}} (E(n) \wedge X) \simeq \bigvee_{n \in \mathbb{N}} 0 \simeq 0.$$

So X is dissonant.
- Now suppose that X is dissonant. Then it is in particular $E(0)$-acyclic, so $BP \wedge L_{E(0)}X$ must be contractible, so $p^{-1}BP_*(X)$ must be trivial. So $BP_*(X)$ must be p-power-torsion. This is the initial step in an induction: suppose we have already shown that $BP_*(X)$ is v_i-power-torsion for all $i < n$. Then Ravenel's localization conjecture/theorem (see Theorem 7.5.2 of [22], or Theorem 3.2 in the present paper for the statement of the result) gives us that $BP_*(L_{E(n)}X) \cong v_n^{-1}BP_*(X)$, and since X is dissonant, $L_{E(n)}X$ must be contractible, so $v_n^{-1}BP_*(X)$ must be trivial. Hence $BP_*(X)$ must be v_n-power-torsion. Now by induction, $BP_*(X)$ is v_n-power-torsion for all n.
- Clearly if $BP_*(X)$ is v_n-power-torsion for all n, then it is v_n-power-torsion for infinitely many n.

$\square$

Lemma 3.8. *Let A be a local commutative ring, let $g : \mathbb{N} \to \mathbb{N}$ be a monotone strictly increasing function, and let R be the commutative graded ring*

$$R = A[x_0, x_1, x_2, \dots]$$

with x_i in grading degree $g(i)$. Suppose M is a nonzero finitely-presented graded R-module. Then there exists some n such that M is not x_n-power-torsion.

Proof. Choose a presentation

$$(3.0.1) \qquad \coprod_{s \in S_1} \Sigma^{d_1(s)} R\{e_s\} \xrightarrow{f} \coprod_{s \in S_0} \Sigma^{d_0(s)} R\{e_s\} \to M \to 0$$

for M, where S_1, S_0 are finite sets and $d_0 : S_0 \to \mathbb{Z}$ and $d_1 : S_1 \to \mathbb{Z}$ are simply to keep track of the grading degrees of the summands $R\{e_s\}$. Now for each $s_1 \in S_1$ and each $s_0 \in S_0$, the component of $f(e_{s_1})$ in the summand $R\{e_{s_1}\}$ of the codomain of f is some grading-homogeneous polynomial in R of grading degree $d_1(s_1) - d_0(s_0)$. Since g was assumed monotone strictly increasing, there are only finitely many x_i such that $g(x_i) \leq d_1(s_1) - d_0(s_0)$, so there exists some maximal integer i such that x_i appears in one of the monomial terms of the component of $f(e_{s_1})$ in the summand $R\{e_{s_1}\}$ of the codomain of f. Let i_{s_0,s_1} denote that integer i.

Now there are only finitely many elements s_0 of S_0 and only finitely many elements s_1 of S_1, so the maximum $m = \max\{i_{s_0,s_1} : s_0 \in S_0, s_1 \in S_1\}$ is some integer. Let $\tilde{R}$ denote the commutative graded ring

$$\tilde{R} = A[x_0, x_1, x_2, \dots, x_m],$$

with x_i again in grading degree $g(x_i)$, and regard R as a graded $\tilde{R}$-algebra by the evident ring map $\tilde{R} \to R$. then the map f from presentation 3.0.1 is isomorphic to

$$(3.0.2) \qquad \coprod_{s \in S_1} \Sigma^{d_1(s)} \tilde{R}\{e_s\} \xrightarrow{\tilde{f}} \coprod_{s \in S_0} \Sigma^{d_0(s)} \tilde{R}\{e_s\}$$

tensored over $\tilde{R}$ with R. Here $\tilde{f}(e_s)$ is defined to be have the same components as $f(e_s)$. Since R is flat over $\tilde{R}$, we now have that $M \cong R \otimes_{\tilde{R}} \tilde{M}$, where $\tilde{M}$ is the cokernel of 3.0.2. Hence x_i acts without torsion on M for all $i > m$. $\qquad\square$

Theorem 3.9. *Let $K/\mathbb{Q}_p$ be a finite extension of degree > 1, and which is not totally ramified. Let A denote the ring of integers of K, and let X be a spectrum such that $BP_*(X)$ is the underlying BP_*-module of a V^A-module. Then X is dissonant, and $BP_*(X)$ is v_n-power-torsion for all $n \geq 0$.*

Proof. Let f denote the residue degree of the extension $K/\mathbb{Q}_p$. Then the natural ring homomorphism $BP_* \to V^A$ factors as the composite of the natural maps

$$BP_* \to V^{A_{nr}} \to V^A,$$

where A_{nr} is the ring of integers of the maximal unramified subextension $K_{nr}/\mathbb{Q}$ of $K/\mathbb{Q}$. By Proposition 2.2, the natural map $BP_* \to V^{A_{nr}}$ sends v_n to zero if f does not divide n. Hence the natural ring homomorphism $BP_* \to V^A$ sends v_n to zero if f does not divide n. Hence, if $BP_*(X)$ is the underlying BP_*-module of some V^A-module, then $BP_*(X)$ is v_n-power-torsion for all n not divisible by f. Lemma 3.7 then implies that X is dissonant. Now Proposition 3.7 implies that $BP_*(X)$ is v_n-power-torsion for all $n \geq 0$. $\qquad\square$

Corollary 3.10. *Let $K/\mathbb{Q}_p$ be a finite extension of degree > 1, and which is not totally ramified. Let A denote the ring of integers of K. Then there does not exist a spectrum X such that $BP_*(X) \cong V^A$.*

Proof. It is not the case that V^A is p-power-torsion. $\qquad\square$

Corollary 3.11. *Let $K/\mathbb{Q}_p$ be a finite extension extension of degree > 1 which is not totally ramified. Let A denote the ring of integers of K. Suppose M is a graded V^A-module, and suppose that X is a spectrum such that $BP_*(X) \cong M$. Suppose that either*

- *M is finitely presented as a BP_*-module, or*
- *M is finitely presented as a V^A-module, and $K/\mathbb{Q}_p$ is unramified.*

Then $M \cong 0$.

Proof. By Theorem 3.9, M must be v_n-power-torsion for all $n \geq 0$. Lemma 3.8 then implies that M cannot be finitely presented as a BP_*-module unless $M \cong 0$.

If $K/\mathbb{Q}_p$ is unramified, then $V^A \cong A[v_d, v_{2d}, v_{3d}, \dots]$ as a BP_*-module, by Proposition 2.2. Hence, if M is a graded V^A-module which is v_n-power-torsion for all integers $n \geq 0$, then by Lemma 3.8, M cannot be finitely presented as a V^A-module unless $M \cong 0$. $\qquad\square$

Corollary 3.12. *Let $K/\mathbb{Q}_p$ be a finite extension of degree > 1, and which is not totally ramified. Let A denote the ring of integers of K. Then the BP-homology of a* space *is never the underlying BP_*-module of a V^A-module, unless that space is stably contractible.*

Proof. By [9], the suspension spectrum of any space is harmonic. By Theorem 3.9, if $BP_*(\Sigma^\infty X)$ is a V^A-module, then $\Sigma^\infty X$ is dissonant. The only spectra that are both dissonant and harmonic are the contractible spectra. $\qquad\square$

Corollary 3.13. *Let $K/\mathbb{Q}_p$ be a finite extension of degree > 1, and which is not totally ramified. Let A denote the ring of integers of K. If Y is a p-local spectrum whose BP-homology is the underlying BP_*-module of a V^A-module, then every map from Y to a suspension spectrum is nulhomotopic. In particular, the stable cohomotopy groups $\pi^*(Y)$ of Y are all zero, and the Spanier-Whitehead dual DY of Y is contractible.*

Proof. By [9], the suspension spectrum of any space is harmonic. By Theorem 3.9, if $BP_*(Y)$ is a V^A-module, then Y is dissonant. Hence Y is acyclic with respect to the wedge of the p-primary Johnson-Wilson theories, while any suspension spectrum $\Sigma^\infty X$ is local with respect to the same wedge. Hence every map from Y to $\Sigma^\infty X$ is nulhomotopic.   □

4. Nonrealizability in the totally ramified case.

Here is another observation, parallel to the one at the beginning of the previous section: the spectrum $H\mathbb{F}_p$ has the property that its BP_*-homology $BP_*(H\mathbb{F}_p)$, as a graded BP_*-module, splits as a direct sum of suspensions of the residue field $\mathbb{F}_p$ of BP_*. Consequently, if A is the ring of integers in a totally ramified field extension of $\mathbb{Q}_p$, then $BP_*(H\mathbb{F}_p)$ admits the structure of a V^A-module, since the residue field of V^A is $\mathbb{F}_p$, and we can simply let all the generators $\pi, v_1^A, v_2^A, \ldots$ of V^A act trivially on $BP_*(H\mathbb{F}_p)$.

However, unlike the unramified situation, in the totally ramified setting there is another obvious source of spectra X such that $BP_*(X)$ is a V^A-module: if d is the degree of the field extension $K/\mathbb{Q}_p$ of which A is the ring of integers, then the Eilenberg-Mac Lane spectrum $H\mathbb{Q}_p^{\vee d}$ admits the structure of a V^A-module, since

$$BP_*(H\mathbb{Q}_p^{\vee d}) \cong (\mathbb{Q}_p \otimes_\mathbb{Z} BP_*)^{\oplus d} \cong p^{-1}V^A$$

as BP_*-modules, since the map $BP_* \otimes_{\mathbb{Z}_{(p)}} A \xrightarrow{\cong} V^{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} A \to V^A$ is a rational isomorphism whenever $K/\mathbb{Q}_p$ is totally ramified, by Theorem 2.9.

So, by taking dissonant spectra and rational spectra, we can produce easy examples of spectra X such that $BP_*(X)$ is the underlying $BP_*(X)$-module of a V^A-module. These examples are not very interesting: again, one does not need chromatic localization methods to study rational spectra, and one cannot use chromatic localization methods to study dissonant spectra. So one would like to know if there exist any spectra X which are not "built from" rational and dissonant spectra, and such that $BP_*(X)$ is the underlying BP_*-module of a V^A-module.

In this section we see that the answer is *no*: if X is a spectrum and $BP_*(X)$ is the underlying BP_*-module of a V^A-module, where A is the ring of integers of some totally ramified field extension $K/\mathbb{Q}_p$, then X is an extension of a rational spectrum by a dissonant spectrum (this is Theorem 4.8). This tells us that $BP_*(X)$ cannot be a V^A-module, for $K/\mathbb{Q}_p$ totally ramified, unless X has some rather pathological properties; for example, if $BP_*(X)$ is torsion-free as an abelian group, then it cannot be finitely-presented as a BP_*-module or as a V^A-module (this is Theorem 4.13), and by Corollary 4.9, the BP-homology of a *space* is not a V^A-module unless that space is stably rational, i.e., its stable homotopy groups are $\mathbb{Q}$-vector spaces. Finally, Corollary 4.14 gives us that V^A itself is not the BP-homology of any spectrum.

First, we need a little bit of easy algebra, Lemmas 4.1, 4.2, 4.3, 4.4, and 4.5:

Lemma 4.1. *Let R be a commutative graded ring, $r \in R$ a homogeneous element, and let M be a graded R-module which is r-power-torsion.*

- *If $M \xrightarrow{f} N$ be a surjective homomorphism of graded R-modules, then N is r-power-torsion.*

- *If $N \xrightarrow{f} M$ is an injective homomorphism of graded R-modules, then N is r-power-torsion.*

Proof. Very easy.   □

Lemma 4.2. *Let R be a commutative graded ring, and let $r \in R$ be a homogeneous element. Then any direct sum of r-power-torsion graded R-modules is also r-power-torsion.*

Proof. Let S be a set and, for each $s \in S$, let M_s be an r-power-torsion graded R-module. Any homogeneous element m of the direct sum $\coprod_{s \in S} M_s$ has only finitely many nonzero components; for each such component m_s, choose a nonnegative integer i_s such that $r^{i_s} m_s = 0$. Then the maximum $i = \max\{i_s : s \in S\}$ satisfies $r^i m_s = 0$ for all $s \in S$, hence $r^i m = 0$. Hence $\coprod_{s \in S} M_s$ is r-power-torsion.   □

Lemma 4.3. *Let R be a commutative graded ring, $r, s \in R$ homogeneous elements, and let M be a graded R-module which is r-power-torsion. Then $s^{-1}M$ is r-power-torsion.*

Proof. I will write $\coprod_{n \in \mathbb{N}} M\{e_n\}$ to mean the countably infinite coproduct $\coprod_{n \in \mathbb{N}} M$ but equipped with a choice of labels $e_0, e_1, e_2, \ldots$ for the summands of $\coprod_{n \in \mathbb{N}} M$.

The module $s^{-1}M$ is isomorphic to the cokernel of the map

$$\coprod_{n \in \mathbb{N}} M\{e_n\} \xrightarrow{\mathrm{id} - T} \coprod_{n \in \mathbb{N}} M\{e_n\}$$

where T is the map of graded R-modules given, for each $m \in \mathbb{N}$, on the component $M\{e_m\}$ by the composite map

$$M\{e_m\} \xrightarrow{s} M\{e_m\} \xrightarrow{\cong} M\{e_{m+1}\} \xrightarrow{i} \coprod_{n \in \mathbb{N}} M\{e_n\},$$

where i is the direct summand inclusion map.

Now $\coprod_{n \in \mathbb{N}} M\{e_n\}$ is r-power-torsion by Lemma 4.2, so $s^{-1}M$ is a quotient of an r-power-torsion module. So $s^{-1}M$ is r-power-torsion by Lemma 4.1.   □

Lemma 4.4. *Let R be a graded ring, and let $r \in R$ be a homogeneous element. Let*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be a short exact sequence of graded R-modules, and suppose that M' and M'' are both r-power-torsion. Then M is also r-power-torsion.

Proof. Let $m \in M$. Then $g(r^n m) = r^n g(m) = 0$ for some nonnegative integer n since M'' is r-power-torsion, so there exists some $\tilde{m} \in M'$ such that $f(\tilde{m}) = r^n m$. Then $r^\ell \tilde{m} = 0$ for some nonnegative integer ℓ, since M' is r-power-torsion. So $0 = f(r^\ell \tilde{m}) = r^\ell r^n m = r^{\ell+n} m$. So M is r-power-torsion.   □

Lemma 4.5. *Let R, S be a commutative graded ring, let I be a homogeneous ideal of R, let S be a commutative graded R-algebra, and let r, r' be homogeneous elements of R such that r acts with eventual r'-division on S modulo I. Then, for every graded S-module M, r acts with eventual r'-division on S modulo I.*

Proof. Choose some nonnegative integer n and homogeneous element $y \in S$ such that $r'y \equiv r^n 1$ modulo IS. Then, for each homogeneous element x of M, the element $yx \in M$ satisfies $r'(yx) \equiv r^n x$ modulo IM.   □

To my mind, Lemma 4.6 is a curious fact: it implies that, for example, the graded BP_*-module $BP_*/(p, v_2 - v_1^{p+1})$ is not the BP-homology of any spectrum, and more generally, $BP_*/(p^a, v_2^b - v_1^{(p+1)b})$ is not the BP-homology of any spectrum for any positive integers a, b. While there is a tremendous wealth of graded BP_*-modules, Lemma 4.6 implies that vastly many of them are not realizable as the BP-homology of any spectrum.

Lemma 4.6. *Suppose that p is a prime and n is a positive integer, and suppose that X is a p-local spectrum such that $BP_*(X)$ is v_i-power-torsion for $i = 0, 1, \ldots, n-1$. Suppose that v_{n+1} acts with eventual v_n-division on $BP_*(X)$ modulo I_n. Then $BP_*(X)$ is v_n-power-torsion, and the Bousfield localization $L_{E(n)}X$ is contractible.*

Proof. First, since I_n acts trivially on $v_n^{-1}BP_*(X)/I_nBP_*(X)$, the Miller-Ravenel change-of-rings isomorphism (see section 6.1 of [21], or [14] for the original reference), gives us an isomorphism of bigraded $\mathbb{Z}_{(p)}$-modules

$$\text{Ext}_{(BP_*, BP_*BP)}^{*,*}(BP_*, v_n^{-1}BP_*(X)/I_nBP_*(X))$$
$$\cong \text{Ext}_{(E(n)_*, E(n)_*E(n))}^{*,*}(E(n)_*, (BP_*(X)/I_nBP_*(X)) \otimes_{BP_*} E(n)_*).$$

Now

$$E(n)_* \cong v_n^{-1}BP_*/(v_{n+1}, v_{n+2}, \ldots),$$

hence

$$(BP_*(X)/I_nBP_*(X)) \otimes_{BP_*} E(n)_* \cong v_n^{-1}BP_*(X)/(p, v_1, \ldots, v_{n-1}, v_{n+1}, v_{n+2}, \ldots)BP_*(X).$$

Now if v_n is assumed to act with eventual v_{n+1}-division on BP_* modulo I_n, then for any given element of $BP_*(X)/(p, v_1, \ldots, v_{n-1}, v_{n+1}, v_{n+2}, \ldots)BP_*(X)$ there is some power of v_n that acts trivially on that element, and hence

$$v_n^{-1}BP_*(X)/(p, v_1, \ldots, v_{n-1}, v_{n+1}, v_{n+2}, \ldots)BP_*(X) \cong 0, \quad \text{so}$$
$$(4.0.1) \qquad \text{Ext}_{(BP_*, BP_*BP)}^{*,*}(BP_*, v_n^{-1}BP_*(X)/I_nBP_*(X)) \cong 0.$$

I claim that this implies that $\text{Ext}_{(BP_*, BP_*BP)}^{*,*}(BP_*, v_n^{-1}BP_*(X))$ also vanishes; here is a proof. For each integer j such that $0 < j \leq n$, filter $v_n^{-1}BP_*(X)/I_j$ by kernels of multiplication by powers of v_j: let $F^iBP_*(X)$ be the kernel of the map

$$(4.0.2) \qquad v_j^i : v_n^{-1}BP_*(X)/I_j \rightarrow v_n^{-1}BP_*(X)/I_j,$$

so that we have the increasing filtration

$$(4.0.3) \qquad 0 = F^0 v_n^{-1}BP_*(X)/I_j \subseteq F^1 v_n^{-1}BP_*(X)/I_j \subseteq F^2 v_n^{-1}BP_*(X)/I_j \subseteq \ldots.$$

The filtration 4.0.3 is separated/Hausdorff, since $\cap_i F^iBP_*(X) = 0$, and complete as well, since $\lim_{i \to -\infty}(BP_*(X)/I_nBP_*(X))/F^i(BP_*(X)/I_nBP_*(X)) \cong BP_*(X)/I_nBP_*(X)$. Furthermore, the map 4.0.2 is a BP_*BP-comodule homomorphism, so filtration 4.0.3 is a filtration of $v_n^{-1}BP_*(X)/I_j$ by sub-BP_*BP-comodules. We assumed that $BP_*(X)$ is v_i-power-torsion for $i = 0, 1, \ldots, n-1$, so Lemma 4.1 and Lemma 4.3 together imply that $v_n^{-1}BP_*(X)/I_j$ is v_i-power-torsion for $i = 0, 1, \ldots, n-1$ and for all j. Hence filtration 4.0.3 is exhaustive.

Now the usual functoriality properties of the cobar complex $C_{(A,\Gamma)}^\bullet(M)$ of a Γ-comodule M then tell us that

$$C_{(BP_*, BP_*BP)}^\bullet(F^0BP_*(X)) \subseteq C_{(BP_*, BP_*BP)}^\bullet(F^1BP_*(X)) \subseteq C_{(BP_*, BP_*BP)}^\bullet(F^2BP_*(X)) \subseteq \ldots$$

is a complete, exhaustive, and separated/Hausdorff filtration on the cobar complex $C_{(BP_*, BP_*BP)}^\bullet(v_n^{-1}BP_*(X)/I_j)$ computing $\text{Ext}_{(BP_*, BP_*BP)}^{*,*}(BP_*, v_n^{-1}BP_*(X)/I_j)$.

Hence, from e.g. Theorem 9.2 of [2], , we get a conditionally convergent spectral sequence

(4.0.4) $$E_1^{s,t,u} \cong \mathrm{Ext}_{(BP_*,BP_*BP)}^{s,u}(BP_*, (F^t v_n^{-1} BP_*(X)/I_j)/(F^{t-1} v_n^{-1} BP_*(X)/I_j))$$

(4.0.5) $$\Rightarrow \mathrm{Ext}_{(BP_*,BP_*BP)}^{s,u}(BP_*, v_n^{-1} BP_*(X)/I_j).$$

We have an isomorphism of graded BP_*BP-comodules

(4.0.6) $$(F^t v_n^{-1} BP_*(X)/I_j)/(F^{t-1} v_n^{-1} BP_*(X)/I_j) \cong v_n^{-1} BP_*(X)/I_{j+1}$$

for all t, and consequently the E_1-term of the spectral sequence 4.0.4 is isomorphic (up to appropriate regrading) to a direct sum of countably infinitely many copies of $\mathrm{Ext}_{(BP_*,BP_*BP)}^{*,*}(BP_*, v_n^{-1} BP_*(X)/I_{j+1})$.

Now we just need an easy downward induction: isomorphisms 4.0.1 and 4.0.6 tell us that the E_1-term of spectral sequence 4.0.4 vanishes when $j = n - 1$. That was the initial step. For the inductive step, suppose we already know that

$$\mathrm{Ext}_{(BP_*,BP_*BP)}^{*,*}(BP_*, v_n^{-1} BP_*(X)/I_j) \cong 0$$

for some $j \geq 0$; then spectral sequence 4.0.4 and isomorphism 4.0.6 tell us that

$$\mathrm{Ext}_{(BP_*,BP_*BP)}^{*,*}(BP_*, v_n^{-1} BP_*(X)/I_{j-1}) \cong 0$$

as well. By downward induction,

$$\mathrm{Ext}_{(BP_*,BP_*BP)}^{*,*}(BP_*, v_n^{-1} BP_*(X)/I_0) \cong 0,$$

i.e. (since $I_0 = (0)$),

(4.0.7) $$\mathrm{Ext}_{(BP_*,BP_*BP)}^{*,*}(BP_*, v_n^{-1} BP_*(X)) \cong 0.$$

Now since $BP_*(X)$ is assumed to be v_{n-1}-power-torsion, we have that $v_{n-1}^{-1} BP_*(X) \cong 0$, and hence Theorem 3.2 gives us that $BP_*(L_{E(n)}X) \cong v_n^{-1} BP_*(X)$. Using triviality of the Ext-group 4.0.7 together with another Morava-Miller-Ravenel change-of-rings (since $BP_*(X)$ is I_n-nil, i.e., v_i-power-torsion for all $i < n$), we have

$$0 \cong \mathrm{Ext}_{(BP_*,BP_*BP)}^{*,*}(BP_*, v_n^{-1} BP_*(X))$$

$$\cong \mathrm{Ext}_{(BP_*,BP_*BP)}^{*,*}(BP_*, BP_*(L_{E(n)}X))$$

$$\cong \mathrm{Ext}_{(E(n)_*,E(n)_*E(n))}^{*,*}(E(n)_*, BP_*(L_{E(n)}X) \otimes_{BP_*} E(n)_*)$$

(4.0.8) $$\cong \mathrm{Ext}_{(E(n)_*,E(n)_*E(n))}^{*,*}(E(n)_*, E(n)_*(L_{E(n)}X))$$

(4.0.9) $$\cong \mathrm{Ext}_{(E(n)_*,E(n)_*E(n))}^{*,*}(E(n)_*, E(n)_*(X)),$$

with isomorphism 4.0.8 due to $E(n)_*$ being a Landweber-exact BP_*-module. The bigraded $\mathbb{Z}_p$-module 4.0.9 is the E_2-term of the $E(n)$-Adams spectral sequence converging to $\hat{X}_{E(n)}$, the $E(n)$-nilpotent completion of X. Hence $\hat{X}_{E(n)}$ is contractible. Consequently, by Theorem 3.3, $0 \simeq \hat{X}_{E(n)} \simeq L_{E(n)}X$, as claimed. Since the $E(n)$-localization of X is contractible, Lemma 3.5 gives us that $BP_*(X)$ is v_n-power-torsion.   □

Lemma 4.7. *Let $K/\mathbb{Q}_p$ be a finite, totally ramified extension of degree greater than one. Then, for each integer $n > 1$, the element $v_{n+1} \in BP_*$ acts with eventual division by v_n on V^A modulo I_n. Furthermore, if the degree $[K : \mathbb{Q}_p]$ of the field extension is greater than two, then $v_2 \in BP_*$ acts with eventual division by v_1 on V^A modulo I_1.*

Proof. Let $e = [K : \mathbb{Q}_p]$. Let $\gamma : BP_* \to V^A$ denote the usual ring map (i.e., the one classifying the underlying p-typical formal group law of the universal A-typical formal A-module) by which we regard V^A as a BP_*-module. Let $\ell_1, \ell_2, \ldots$ denote the logarithm coefficients for the universal p-typical formal group law, and let $\ell_1^A, \ell_2^A, \ldots$ denote the logarithm coefficients for the universal A-typical formal A-module. Then $\gamma(\ell_i) = \ell_i^A$ for all i, since $K/\mathbb{Q}_p$ is totally ramified; see Remark 2.1. Then, by the definition of the Hazewinkel generators of BP_* and of V^A, we have the equations

$$p\ell_{n+1} = \sum_{j=0}^{n} (v_{n+1-j}^{p^j})\ell_j \text{ and}$$

$$\pi\ell_{n+1}^A = \sum_{j=0}^{n} (v_{n+1-j}^{p^j})\ell_j^A.$$

Applying γ, dividing by p and by π, setting $\gamma(\ell_{n+1})$ equal to ℓ_{n+1}^A, and solving for $\gamma(v_{n+1})$ yields the equation

$$(4.0.10) \qquad \gamma(v_{n+1}) = \frac{p}{\pi}v_{n+1}^A + \sum_{j=1}^{n}\left(\left(\frac{p}{\pi}(v_{n+1-j}^A)^{p^j} - \gamma(v_{n+1-j})^{p^j}\right)\ell_j^A\right).$$

We now break into two cases:

- If $n = 1$, then, by Proposition 2.4, we have:

$$(4.0.11) \qquad \gamma(v_2) = \frac{p}{\pi}v_2^A + \frac{p}{\pi^2}(v_1^A)^{p+1} - \frac{p^p}{\pi^{p+1}}\gamma(v_1)^{p+1}.$$

Let m be a power of p such that $m > e$. Then

$$\gamma(v_2^m) \equiv \left(\frac{p}{\pi}v_2^A\right)^m + \left(\frac{p}{\pi^2}(v_1^A)^{p+1}\right)^m + \left(\frac{-p^p}{\pi^{p+1}}\gamma(v_1)^{p+1}\right)^m$$

$$\equiv \frac{p^m}{\pi^{2m}}(v_1^A)^{m(p+1)},$$

which is zero modulo π if the ramification degree of the extension $K/\mathbb{Q}_p$ is greater than two. Hence, if $[K : \mathbb{Q}_p] > 2$, then v_2 acts with eventual division by v_1 on V^A modulo $I_1 = (p)$, since being zero is a trivial case of being divisible by v_2.

- If $n > 1$, then, for any $i < n$, both $\gamma(v_i)$ and any integral expression in V^A divisible by ℓ_i^A in $\mathbb{Q} \otimes_{\mathbb{Z}} V^A$ are zero modulo $\gamma(I_n)V^A$. Consequently 4.0.10 reduces to

$$\gamma(v_{n+1}) \equiv \frac{p}{\pi}v_{n+1}^A + \frac{p}{\pi}(v_1^A)^{p^n}\ell_n^A \mod \gamma(I_n)V^A.$$

Now let m be any power of p satisfying $m > e$. Since $\frac{p}{\pi}$ has positive π-adic valuation, $\left(\frac{p}{\pi}\right)^m \equiv 0$ modulo p, and consequently $\gamma(v_{n+1})^m \equiv \left(\frac{p}{\pi}v_{n+1}^A\right)^m + \left(\frac{p}{\pi}(v_1^A)^{p^n}\ell_n^A\right)^m \equiv 0$ modulo $\gamma(I_n)V^A$. Hence v_{n+1} acts with eventual division by v_n on V^A modulo I_n (since, again, being zero is a trivial case of being divisible by v_n).

□

Beginning in the proof of Theorem 4.8, we refer (when convenient) to section 6, the appendix on basic ideas from local cohomology at the end of this paper.

Theorem 4.8. *Let $K/\mathbb{Q}_p$ be a finite, totally ramified extension of degree greater than one. Let X be a spectrum such that $BP_*(X)$ is the underlying graded BP_*-module of a graded V^A-module. Then the following statements are all true:*

- *For all nonnegative integers n, $E(n)_*(X)$ is a $\mathbb{Q}$-vector space.*

- *For all nonnegative integers n, $L_{E(n)}X$ is a rational spectrum, i.e., a wedge of suspensions of $H\mathbb{Q}$.*
- *For all positive integers n, $K(n)_*(X) \cong 0$.*
- *X is an extension of a rational spectrum by a dissonant spectrum. That is, X sits in a homotopy fiber sequence*

(4.0.12) $$cX \to X \to LX,$$

  where LX is a rational spectrum, that is, a wedge product of suspensions of copies of the Eilenberg-Mac Lane spectrum $H\mathbb{Q}$, and cX is a dissonant spectrum, that is, $E(n) \wedge cX$ for all $n > 0$. (Here LX is the usual rationalization of X, i.e., $LX \simeq X \wedge H\mathbb{Q}$.)

Proof. Let cX be as in the statement of the theorem, i.e., cX is the homotopy fiber of the rationalization map $X \to LX$. Since $BP_*(LX) \cong p^{-1}BP_*(X)$ and the map induced in BP-homology by the rationalization agrees up to isomorphism with the natural localization map $i : BP_*(X) \to p^{-1}BP_*(X)$, we have a short exact sequence of graded BP_*-modules

(4.0.13) $$0 \to \operatorname{coker} \Sigma i \to BP_*(cX) \to \ker i \to 0,$$

i.e., using Propositions 6.2 and 6.3, a short exact sequence of graded BP_*-modules

(4.0.14) $$0 \to H^1_{(p)}(\Sigma BP_*(X)) \to BP_*(cX) \to H^0_{(p)}(BP_*(X)) \to 0.$$

Even better, the short exact sequences 4.0.13 and 4.0.14 are short exact sequences of graded V^A-modules, since p is not just an homogeneous element of BP_* but also a homogeneous element of V^A.

In general, for any commutative graded ring R, any homogeneous element $r \in R$, and any graded R-module M, the R-module $H^1_{(r)}(M) \cong (r^{-1}M)/M$ is r-power-torsion, since any homogeneous element $m \in r^{-1}M$ can be multiplied by some power of r to get an element of M, which represents zero in $H^1_{(r)}(M)$. Consequently $H^1_{(p)}(BP_*(X))$ is a p-power-torsion graded V^A-module. Proposition 6.3 also gives us that $H^0_{(p)}(BP_*(X))$ is a p-power-torsion graded V^A-module. Now Lemma 4.4 gives us that $BP_*(cX)$ is also a p-power-torsion graded V^A-module. By Proposition 2.4, $\gamma(v_1) = \frac{p}{\pi}v_1^A$, so $BP_*(cX)$ being p-power-torsion implies that $BP_*(cX)$ is also v_1-power-torsion. So Lemma 3.5 implies that $L_{E(1)}cX$ is contractible.

This is the initial step in an induction: if n is at least 2 and we have already shown that $L_{E(n-1)}cX$ is contractible and $BP_*(cX)$ is v_i-power-torsion for all $i \leq n - 1$, then by Lemma 4.7 and Lemma 4.5, v_{n+1} acts with eventual division by v_n on $BP_*(cX)$ modulo I_n. Then Lemma 4.6 implies that $BP_*(cX)$ is v_n-power-torsion and that the Bousfield localization $L_{E(n)}cX$ is contractible, completing the inductive step. So cX is acyclic with respect to all Johnson-Wilson theories, hence cX is dissonant, hence X is an extension of a rational spectrum by a dissonant spectrum, as claimed.

Consequently, $L_{E(n)}cX$ is contractible for all $n \geq 0$, and $BP_*(cX)$ is v_n-power-torsion for all $n \geq 0$. Consequently cX is $E(n)$-acyclic, and smashing fiber sequence 4.0.12 with $E(n)$ yields the weak equivalence $E(n) \wedge X \xrightarrow{\simeq} E(n) \wedge LX$. Since LX is a rational spectrum, it splits as a wedge of suspensions of copies of $H\mathbb{Q}$, and hence $E(n) \wedge LX$ splits as a wedge of suspensions of copies of $E(n) \wedge HQ$. Hence $E(n)_*(X)$ is a $\mathbb{Q}$-vector space, as claimed.

Smashing the fiber sequence 4.0.12 with the $E(n)$-local sphere $L_{E(n)}S$, and now using Theorem 3.3, and the fact that $L_{E(n)}L_{E(0)}X \simeq L_{E(0)}X$, we get the homotopy fiber sequence

$$L_{E(n)}cX \to L_{E(n)}X \to L_{E(0)}X.$$

We have shown that $L_{E(n)}cX$ is contractible, so the localization map $L_{E(n)}X \to L_{E(0)}X$ is an equivalence, so $L_{E(n)}X$ is a rational spectrum as claimed.

Furthermore, recall that, for all positive integers n, the notation $\mu_n X$ is used to denote the "nth monochromatic layer of X," i.e., $\mu_n X$ sits in the homotopy fiber sequence

$$\mu_n X \to L_{E(n)}X \to L_{E(n-1)}X.$$

So $\mu_n X$ is contractible for all $n > 0$. Recall also (from e.g. [10]) that there exists natural isomorphisms in the stable homotopy category $L_{K(n)} \circ \mu_n \simeq L_{K(n)}$ and $\mu_n \circ L_{K(n)} \simeq \mu_n$; consequently the contractibility of $\mu_n X$ implies the contractibility of $L_{K(n)}X$. So X is $K(n)$-acyclic for all $n > 0$, so $K(n)_*(X) \cong 0$ for all $n > 0$, as claimed. $\square$

Corollary 4.9. *Let $K/\mathbb{Q}_p$ be a finite, totally ramified extension of degree greater than one. Let X be a space such that $BP_*(X)$ is the underlying graded BP_*-module of a graded V^A-module. Then X is stably rational, that is, its stable homotopy groups are all $\mathbb{Q}$-vector spaces.*

Proof. By [9], suspension spectra are harmonic, hence $\Sigma^\infty X$ does not admit nontrivial maps from dissonant spectra. By Theorem 4.8 $\Sigma^\infty X$ is rational. $\square$

Corollary 4.10. *Let $K/\mathbb{Q}_p$ be a finite, totally ramified extension of degree greater than one, and with ring of integers A. Let X be a spectrum such that $BP_*(X)$ is the underlying graded BP_*-module of a graded V^A-module, and suppose that $BP_*(X)$ is torsion-free as an abelian group, i.e., $BP_*(X)$ is p-torsion-free. Then the local cohomology module $H^1_{(p)}(BP_*(X))$ and the mod p reduction $BP_*(X)/pBP_*(X)$ are both v_n-power-torsion BP_*-modules for all $n \geq 0$.*

Proof. Since $BP_*(X)$ is assumed to be p-torsion-free, its maximal p-power-torsion sub-module $\Gamma_{(p)}(BP_*(X))$ is trivial, so from Propositions 6.2 and 6.3 we get the short exact sequence of graded V^A-modules

$$0 \to BP_*(X) \to p^{-1}BP_*(X) \to H^1_{(p)}(BP_*(X)) \to 0,$$

and the localization map $BP_*(X) \to p^{-1}BP_*(X)$ coincides with the localization map in homotopy $BP_*(X) \to BP_*(L_{E(0)}X)$. Consequently $H^1_{(p)}(BP_*(X)) \cong BP_*(cX)$, the BP-homology of the $E(0)$-acyclization cX of X. Now Theorem 4.8 tells us that $BP_*(cX)$ is v_n-power-torsion for all $n \geq 0$.

Similarly, since $BP_*(X)$ is p-torsion-free, the BP-homology of X smashed with the mod p Moore spectrum is $BP_*(X)/pBP_*(X)$:

$$BP_*(X \wedge S/p) \cong BP_*(X)/pBP_*(X).$$

Of course the mod p reduction of a graded V^A-module is still a graded V^A-module, so $X \wedge S/p$ is a spectrum with the property that $BP_*(X \wedge S/p)$ is a graded V^A-module, hence by Theorem 4.8 $X \wedge S/p$ is an extension of a rational spectrum by a dissonant spectrum. However, the homotopy groups of $X \wedge S/p$ are all p-torsion, hence rationally trivial, hence $X \wedge S/p$ is $E(0)$-acyclic and does not map nontrivially to a rational spectrum; so $X \wedge S/p$ is dissonant. Now Proposition 3.7 gives us that $BP_*(X \wedge S/p)$ is v_n-power-torsion for all $n \geq 0$. $\square$

Proposition 4.11. (Hazewinkel.) *Let A be the ring of integers in a finite field extension of $\mathbb{Q}$. Let π be a uniformizer for A and let $\mathbb{F}_q$ be the residue field of A. Then the logarithm*

coefficients $\ell_1^A, \ell_2^A, \ldots$ *of the universal A-typical formal A-module satisfy the relation*

$$
\begin{aligned}
\pi \ell_h^A &= \ell_{h-1}^A (v_1^A)^{q^{h-1}} + \cdots + \ell_1^A (v^A)_{h-1}^q + v_h^A, \\
\ell_h^A &= \sum_{i_1 + \cdots + i_r = h} \pi^{-r} v_{i_1}^A (v_{i_2}^A)^{q^{i_1}} \ldots (v_{i_r}^A)^{q^{i_1 + \cdots + i_{r-1}}},
\end{aligned}
$$

where all i_j are positive integers, and where $v_1^A, v_2^A, \ldots$ are Hazewinkel's generators for V^A.

Proof. This is formula 21.5.4 of [7]. □

Proposition 4.12. *Let $K/\mathbb{Q}_p$ and $L/\mathbb{Q}_p$ be finite field extensions, and let L/K be a totally ramified extension of degree $n > 1$. Fix a positive integer j. Let A, B be the rings of integers of K, L, respectively, and let $\mathbb{F}_q$ be their residue field. Let π_A, π_B be uniformizers for A, B, respectively. Let $\gamma : V^A \to V^B$ denote the ring map classifying the underlying A-typical formal A-module of the universal B-typical formal B-module. The least h such that $\gamma(v_h^A)$ is nonzero modulo $(\pi_B, v_1^B, \ldots, v_{j-1}^B)$ is $h = jn$. For $h = jn$ we have*

$$(4.0.15) \qquad \gamma(v_{jn}^A) \equiv \frac{\pi_A}{\pi_B^n}(v_j^B)^{\frac{q^{jn}-1}{q^j-1}} \mod (\pi_B, v_1^B, \ldots, v_{j-1}^B).$$

Proof. Write I_j^B for the ideal $(\pi_B, v_1^B, \ldots, v_{j-1}^B)$ in V^B. Clearly $\gamma(v_0^A) = \gamma(\pi_A)$ is zero modulo I_j^B. That was the initial step in an induction. For the inductive step, let h be some positive integer, $h \le jn$, and suppose that we have already shown that $\gamma(v_{h'}^A) \in I_j^B$ for all $h' < h$. Using Proposition 4.11 we have

$$
\gamma(v_h^A) = \frac{\pi_A}{\pi_B} v_h^B + \sum_{i_1 + \cdots + i_r = h \text{ and } r > 1} \left(\frac{\pi_A}{\pi_B} v_{i_1}^B \prod_{m=2}^{r} \frac{(v_{i_m}^B)^{q^{\sum_{s=1}^{m-1} i_s}}}{\pi_B} - \gamma(v_{i_1}^A) \prod_{m=2}^{r} \frac{\gamma(v_{i_m}^A)^{q^{\sum_{s=1}^{m-1} i_s}}}{\pi_A} \right),
$$

which is zero in V^B/I_j^B unless $j \mid h$. Assume $j \mid h$; then we have

$$
\begin{aligned}
\gamma(v_h^A) &\equiv \frac{\pi_A}{\pi_B} v_j^B \prod_{m=2}^{h/j} \frac{(v_j^B)^{q^{(m-1)j}}}{\pi_B} \mod I_j^B \\
&\equiv \frac{\pi_A}{(\pi_B)^{h/j}} (v_j^B)^{\frac{q^h-1}{q^j-1}} \mod I_j^B
\end{aligned}
$$

and $v_{\pi_B}\left(\frac{\pi_A}{(\pi_B)^{h/j}}\right) > 0$ if and only if $\frac{h}{j} < n$; since, in this case, $r = \frac{h}{j} > 1$ and $n > 1$, we cannot have $\frac{h}{j} > n$, so the lowest h such that $\gamma(v_h^A) \notin I_j^B$ is $h = jn$. This gives us equation 4.0.15. □

Theorem 4.13. *Let $K/\mathbb{Q}_p$ be a finite, totally ramified extension of degree greater than one, and with ring of integers A. Let X be a spectrum such that $BP_*(X)$ is the underlying graded BP_*-module of a graded V^A-module. Suppose that either $BP_*(X)$ is p-power-torsion, or that $BP_*(X)$ is torsion-free as an abelian group, i.e., $BP_*(X)$ is p-torsion-free. Then $BP_*(X)$ is not finitely presented as a BP_*-module, and $BP_*(X)$ is not finitely presented as a V^A-module.*

Proof. • Suppose first that $BP_*(X)$ is p-power-torsion. Then by Lemma 3.5, $L_{E(0)}X$ is contractible, so X is $E(0)$-acyclic and does not map nontrivially to a rational spectrum. So, by Theorem 4.8, X is dissonant, and $BP_*(X)$ is v_n-power-torsion for all $n \ge 0$. Now Lemma 3.8 gives us that $BP_*(X)$ is not finitely presented as a BP_*-module.

Similarly, if $[K : \mathbb{Q}_p] = d$, then v_{dn} acts by a power of v_n^A on $V^A/(\pi, v_1^A, \ldots, v_{n-1}^A)$, by Proposition 4.12. Consequently a finitely presented V^A-module cannot be v_{dn}-power-torsion for all $n \geq 0$. So $BP_*(X)$ cannot be finitely presented as a V^A-module.

- Now suppose that $BP_*(X)$ is p-torsion-free and finitely presented as a BP_*-module. Then $BP_*(X)/pBP_*(X)$ is also finitely presented as a BP_*-module, and by Corollary 4.10, it is also v_n-power-torsion for all $n \geq 0$. This contradicts Lemma 3.8.

  Suppose instead that $BP_*(X)$ is p-torsion-free and finitely presented as a V^A-module. Then $BP_*(X)/pBP_*(X)$ is also finitely presented as a V^A-module, and by Corollary 4.10, it is also v_n-power-torsion for all $n \geq 0$. Let $d = [K : \mathbb{Q}_p]$, and let $\gamma : BP_* \to V^A$ be the map classifying the underlying p-typical formal group law of the universal A-typical formal A-module. By Proposition 4.12, $\gamma(v_{dn})$ is congruent to a unit times a power of v_n^A modulo $(\pi, v_1^A, \ldots, v_{n-1}^A)$. I claim that this implies that $BP_*(X)/pBP_*(X)$ is v_n^A-power-torsion for all $n \geq 0$. The proof is as follows:

  - Since $BP_*(X)/pBP_*(X)$ is v_n-power-torsion for all $n \geq 0$, the graded V^A-module $\pi^i BP_*(X)/\pi^{i+1}BP_*(X)$ is also v_n-power-torsion for all $n \geq 0$ and all $i = 0, 1, \ldots, d-1$, by Lemma 4.1.
  - Since $\gamma(v_d)$ is congruent to a unit times v_1^A modulo π and since $\pi^i BP_*(X)/\pi^{i+1}BP_*(X)$ is v_d-power-torsion for all $i = 0, 1, \ldots, d-1$, we have that $\pi^i BP_*(X)/\pi^{i+1}BP_*(X)$ is also v_1^A-power torsion for all $i = 0, 1, \ldots, d-1$.
  - That was the initial step in an induction, which proceeds as follows: suppose we have already shown that $\pi^i BP_*(X)/\pi^{i+1}BP_*(X)$ is v_j^A-power-torsion for all $j = 1, \ldots, n-1$ and all $i = 0, 1, \ldots, d-1$. By Proposition 4.12,

$$\gamma(v_{dn}) \equiv \frac{p}{\pi^d}(v_n^A)^{\frac{p^{dn}-1}{p^n-1}} + v_1^A x_1 + v_2^A x_2 + \cdots + v_{n-1}^A x_{n-1} \mod \pi$$

for some $x_1, \ldots, x_{n-1} \in V^A$. For a given homogeneous $x \in \pi^i BP_*(X)/\pi^{i+1}BP_*(X)$, choose positive integers $\epsilon_1, \ldots, \epsilon_{n-1}$ such that $(v_i^A)^{\epsilon_i}x = 0$, and choose a positive integer ϵ such that $v_{dn}^\epsilon x = 0$. Let ϵ' be any positive integer such that $p^{\epsilon'} > \epsilon$ and such that $p^{\epsilon'} > \epsilon_j$ for all $j = 1, \ldots, n-1$. Then

$$0 = v_{dn}^{p^{\epsilon'}} x$$
$$= \left(\frac{p}{\pi^d}(v_n^A)^{\frac{p^{dn}-1}{p^n-1}}\right)^{p^{\epsilon'}} x + \sum_{j=1}^{n-1} \left(v_j^A x_j\right)^{p^{\epsilon'}} x$$
$$= \left(\frac{p}{\pi^d}\right)^{p^{\epsilon'}} (v_n^A)^{p^{\epsilon'}\frac{p^{dn}-1}{p^n-1}} x,$$

and since $\frac{p}{\pi^d} \in \mathbb{F}_p^\times$, we have that $(v_n^A)^{p^{\epsilon'}\frac{p^{dn}-1}{p^n-1}} x = 0$. Hence $\pi^i BP_*(X)/\pi^{i+1}BP_*(X)$ is v_n^A-power-torsion for all $i = 0, \ldots, d-1$, completing the inductive step. Hence $\pi^i BP_*(X)/\pi^{i+1}BP_*(X)$ is v_n^A-power-torsion for all $i = 0, \ldots, d-1$ and for all $n \geq 0$.

– Now repeated application of Lemma 4.4 to the finite tower of extensions of
graded V^A-modules

$$\pi^{d-1}BP_*(X)/\pi^d BP_*(X) \hookrightarrow BP_*(X)/pBP_*(X)$$

$$\pi^{d-2}BP_*(X)/\pi^{d-1}BP_*(X) \hookrightarrow BP_*(X)/\pi^{d-2}BP_*(X)$$

$$\vdots$$

$$\pi BP_*(X)/\pi^2 BP_*(X) \hookrightarrow BP_*(X)/\pi^2 BP_*(X)$$

$$BP_*(X)/\pi BP_*(X) \xrightarrow{\;\cong\;} BP_*(X)/\pi BP_*(X)$$

shows that $BP_*(X)/pBP_*(X)$ is v_n^A-power-torsion for all $n \geq 0$.
Again Lemma 3.8 then gives us a contradiction: $BP_*(X)$ cannot be finitely pre-
sented as a V^A-module.

$\square$

Corollary 4.14. *Let $K/\mathbb{Q}_p$ be a finite field extension of degree greater than one, and with
ring of integers A. Then there does not exist a spectrum X such that $BP_*(X) \cong V^A$ as
graded BP_*-modules.*

Proof. Immediate from Corollary 3.10 together with Theorem 4.13. $\square$

5. Conclusions.

Theorem 5.1. (Nonexistence of Ravenel's algebraic spheres, local case.) *Let K be a
finite extension of $\mathbb{Q}_p$ of degree > 1, with ring of integers A. Then there does not exist a
spectrum X such that $MU_*(X) \cong L^A$.*

Proof. This is a proof by contrapositive: suppose that a spectrum X exists so that $MU_*(X) \cong
L^A$. Let q denote the cardinality of the residue field of A. We have an isomorphism of
graded V^A-modules

$$L^A \cong \coprod_{0 \leq n \neq q^k - 1} \Sigma^{2n} V^A$$

given by Cartier typification (see 21.7.17 of [7] for this). Now, since $MU_*(X)$ is already
p-local, we have the isomorphism of graded MU_*-modules

$$MU_*(X) \cong \coprod_{0 \leq n \neq p^k - 1} \Sigma^{2n} BP_*(X).$$

Let Y be the coproduct spectrum $Y = \coprod_{0 \leq n \neq p^k - 1} \Sigma^{2n} X$, so that we have isomorphisms of graded BP_*-modules

$$BP_*(Y) \cong \coprod_{0 \leq n \neq p^k - 1} \Sigma^{2n} BP_*(X)$$
$$\cong MU_*(X)$$
$$\cong L^A$$
$$\cong \coprod_{0 \leq n \neq q^k - 1} \Sigma^{2n} V^A.$$

So the graded V^A-module $\coprod_{0 \leq n \neq q^k - 1} \Sigma^{2n} V^A$ is the BP-homology of the spectrum Y. But $\coprod_{0 \leq n \neq q^k - 1} \Sigma^{2n} V^A$ is a free V^A-module.

- By Theorem 3.9, if $K/\mathbb{Q}_p$ is not totally ramified, then Y must be dissonant, and in particular, $BP_*(Y)$ must be p-power-torsion; but this contradicts $BP_*(Y)$ being a free V^A-module, since V^A is p-torsion free.
- On the other hand, if $K/\mathbb{Q}_p$ is totally ramified, then by Theorem 4.8, the fiber cY of the rationalization map $Y \to LY$ is dissonant. Since $BP_*(Y)$ is free and $BP_*(LY) \cong p^{-1} BP_*(Y)$, we have a commutative diagram of graded BP_*-modules with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & BP_*(Y) & \longrightarrow & BP_*(LY) & \longrightarrow & BP_*(\Sigma cY) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \cong} & & \downarrow \\
0 & \longrightarrow & BP_*(Y) & \longrightarrow & p^{-1} BP_*(Y) & \longrightarrow & H^1_{(p)}(BP_*(Y)) & \longrightarrow & 0
\end{array}
$$

and, again since $BP_*(Y)$ is a free V^A-module, $BP_*(\Sigma cY)$ is isomorphic to a direct sum of copies of $H^1_{(p)}(V^A) \cong p^{-1} V^A / V^A$. Let $n = [K : \mathbb{Q}_p]$. Then Proposition 4.12 implies that v_n acts on V^A as $\frac{p}{\pi^n}(v_1^A)^{\frac{p^n - 1}{p - 1}}$ modulo π. Since $\frac{p}{\pi^n} \in \mathbb{F}_p^\times$, we now have that $p^{-1} V^A / V^A$ is v_n-torsion-free, consequently $BP_*(\Sigma cY)$ is v_n-torsion-free. But by Proposition 3.7, $BP_*(\Sigma cY)$ then cannot be dissonant, a contradiction.

So the spectrum X such that $MU_*(X) \cong L^A$ cannot exist.   □

The following lemma is classical (and stronger results along these lines are probably known to those who know more number theory than I do!):

Lemma 5.2. *Let $K/\mathbb{Q}$ be a finite extension of degree > 1, with ring of integers A. Then there exist infinitely many prime numbers p which neither ramify nor split completely in A.*

If we furthermore assume that either $K/\mathbb{Q}$ is cyclic of prime power order or that $K/\mathbb{Q}$ is Galois, then there exists some prime number p which neither ramifies nor splits completely in A, and such that each prime of A over p is of the same degree. (That is, there exists some positive integer n such that every prime of A over p is of degree d.)

Proof. See Corollary 5 of Proposition 7.16 of [15] for the fact that, in any finite extension of $\mathbb{Q}$, there are infinitely many primes of $\mathbb{Z}$ which do not split completely. (On the other hand, unless $K/\mathbb{Q}$ is cyclic, then there are only finitely many primes of $\mathbb{Z}$ which do not split *at all* in A; see Exercise VI.3.1 of [16].) Only finitely many such primes can ramify in A; the rest neither ramify nor split completely.

If $K/\mathbb{Q}$ is cyclic of prime power order, then there exist infinitely many primes of $\mathbb{Z}$ which do not split *at all* in A; see Corollary VI.3.7 of [16]. Only finitely many such primes ramify

in A, and the rest neither ramify nor split at all; any one of those primes will satisfy the claim made in the statement of the lemma.

On the other hand, if $K/\mathbb{Q}$ is Galois, then the Galois group $\text{Gal}(K/\mathbb{Q})$ acts transitively on the set of primes of A over p. Hence every prime of A over p has the same degree, and any prime p which neither ramifies nor splits completely in A will satisfy the claim made in the statement of the lemma. $\square$

Lemma 5.3 is another easy one and which is definitely not new, but for which I do not know a reference in the literature.

Lemma 5.3. *Let A be a Noetherian commutative ring, and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be a finite set of maximal ideals of A. Let $\mathfrak{m}$ denote the product of these ideals,*

$$\mathfrak{m} = \mathfrak{m}_1 \cdot \mathfrak{m}_2 \cdot \cdots \cdot \mathfrak{m}_n \subseteq A.$$

Suppose that the ring $A/\mathfrak{m}$ is Artinian. Then there is an isomorphism of A-algebras

$$\hat{A}_{\mathfrak{m}} \cong \prod_{i=1}^{n} \hat{A}_{\mathfrak{m}_i}$$

between the $\mathfrak{m}$-adic completion of A and the product of the $\mathfrak{m}_i$-adic completions of A for all i between 1 and n.

Furthermore, for any finitely generated A-module M, the $\mathfrak{m}$-adic completion of M is isomorphic to the product of the $\mathfrak{m}_i$-adic completions of M for all i between 1 and n, i.e.,

$$\hat{M}_{\mathfrak{m}} \cong \prod_{i=1}^{n} \hat{M}_{\mathfrak{m}_i}.$$

This isomorphism is natural in the choice of finitely generated A-module M.

Proof. Since M is finitely generated and A is Noetherian, we have an isomorphism

$$(5.0.1) \qquad\qquad \hat{M}_I \cong \hat{A}_I \otimes_A M$$

for any choice of ideal I in A, and isomorphism 5.0.1 is natural in the choice of finitely generated A-module M. (This is classical; see e.g. Proposition 10.13 of [1], or consult any number of other algebra textbooks.)

Now, for any positive integer i, the kernel of the A-algebra homomorphism

$$A/\mathfrak{m}^j \to A/\mathfrak{m}$$

is $\mathfrak{m}/\mathfrak{m}^j$, a nilpotent ideal in $A/\mathfrak{m}_j$; hence $A/\mathfrak{m}^j$ is a nilpotent extension of an Artinian ring for all $j \in \mathbb{N}$. Hence $A/\mathfrak{m}^j$ is Artinian for all $j \in \mathbb{N}$.

Now every Artinian ring splits as the Cartesian product of its localizations at its maximal ideals (this is Theorem 8.7 in [1], for example), so we have an isomorphism of rings $A/\mathfrak{m}^j \cong \prod_{i=1}^{n} A/\mathfrak{m}_i^j$, and on taking limits, isomorphisms of rings

$$\hat{A}_{\mathfrak{m}} \cong \lim_{j \to \infty} A/\mathfrak{m}^j$$

$$\cong \lim_{j \to \infty} \left(\prod_{i=1}^{n} A/\mathfrak{m}_i^j \right)$$

$$\cong \prod_{i=1}^{n} \lim_{j \to \infty} A/\mathfrak{m}_i^j$$

$$\cong \prod_{i=1}^{n} \hat{A}_{\mathfrak{m}_i},$$

hence isomorphisms of A-modules

$$\hat{M}_{\mathfrak{m}} \cong \hat{A}_{\mathfrak{m}} \otimes_A M$$

$$\cong \left(\prod_{i=1}^{n} \hat{A}_{\mathfrak{m}_i}\right) \otimes_A M$$

$$\cong \left(\coprod_{i=1}^{n} \hat{A}_{\mathfrak{m}_i}\right) \otimes_A M$$

$$\cong \coprod_{i=1}^{n} \left(\hat{A}_{\mathfrak{m}_i} \otimes_A M\right)$$

$$\cong \coprod_{i=1}^{n} \left(\hat{M}_{\mathfrak{m}_i}\right)$$

$$\cong \prod_{i=1}^{n} \left(\hat{M}_{\mathfrak{m}_i}\right),$$

all natural in M. (The switch from $\times$ to $\oplus$ is because coproducts commute with the tensor product of modules; and the product in question happens to be a finite product of modules, hence it is isomorphic to a finite coproduct of modules.) □

Lemma 5.4. *Let A be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and with finite residue field. Let g, h be the natural maps of commutative graded rings*

$$(L^A) \otimes_A \hat{A}_{\mathfrak{m}} \xrightarrow{h} (L^A)\hat{}_{\mathfrak{m}} \xrightarrow{g} L^{\hat{A}_{\mathfrak{m}}},$$

where h is the usual completion comparison map (as in Proposition 10.13 of [1]), and g is the map classifying the underlying formal A-module of the universal formal $\hat{A}_{\mathfrak{m}}$-module. Let g', h' be the natural maps of commutative graded rings

$$(V^A) \otimes_A \hat{A}_{\mathfrak{m}} \xrightarrow{h'} (V^A)\hat{}_{\mathfrak{m}} \xrightarrow{g'} V^{\hat{A}_{\mathfrak{m}}},$$

where h' is the usual completion comparison map, and g' is the map classifying the underlying A-typical formal A-module of the universal $\hat{A}_{\mathfrak{m}}$-typical formal $\hat{A}_{\mathfrak{m}}$-module.

Then g, h, g', and h' are all isomorphisms.

Proof. For any commutative ring A and any integer $n > 2$, Drinfeld proved in [5] that the module of indecomposables L^A_{n-1}/D^A_{n-1} of L^A in grading degree $2(n-1)$ is the A-module with one generator c_a for each $a \in A$ and one additional generator d, and subject to the relations

$$d(a - a^n) = v(n)c_a \text{ for all } a \in A$$

$$c_{a+b} - c_a - c_b = d\frac{a^n + b^n - (a+b)^n}{v(n)} \text{ for all } a, b \in A$$

$$ac_b + b^n c_a = c_{ab} \text{ for all } a, b \in A.$$

Here $v(n)$ is defined to be p if n is a power of a prime number p, and $v(n) = 1$ if n is not a prime power. Hazewinkel then proved (this is Proposition 21.3.1 of [7]) that, if A is a discrete valuation ring with finite residue field, then L^A_{n-1}/D^A_{n-1} is isomorphic to a free A-module on one generator. Clearly this is true when $v(n) = 1$, since then L^A_{n-1}/D^A_{n-1} is generated by d. Suppose instead that $v(n) \neq 1$. Let I^A denote the ideal in A generated by all elements of the form $a - a^n, a \in A$. Then, as Hazewinkel observes, there are two possibilities:

If $I^A = A$: Then there exists an $a \in A$ such that $a - a^n$ is a unit in A, and consequently c_a generates L^A_{n-1}/D^A_{n-1}.

If $I^A \neq A$: Then, writing π for a generator of $\mathfrak{m}$, the element c_π generates L^A_{n-1}/D^A_{n-1}. Hazewinkel observes that this case only occurs when $A/\mathfrak{m}$ is finite and n is a power of the characteristic of $A/\mathfrak{m}$.

Since the completion map $f : A \to \hat{A}_\mathfrak{m}$ is injective, if $I^A = A$, then there exists some finite collection of elements $a_1, b_1, a_2, b_2, \ldots, a_m, b_m \in A$ such that $\sum_{i=1}^{m} b_m(a_m - a_m^n) = 1$, and consequently $\sum_{i=1}^{m} f(b_m)(f(a_m) - f(a_m)^n) = 1$ in $\hat{A}_\mathfrak{m}$. So $I^{\hat{A}_\mathfrak{m}} = \hat{A}_\mathfrak{m}$. Choose an element $a \in A$ such that $a - a^n \in A^\times$, and now the map

$$A\{c_a\} \xrightarrow{\cong} L^A_{n-1}/D^A_{n-1} \to L^{\hat{A}_\mathfrak{m}}_{n-1}/D^{\hat{A}_\mathfrak{m}}_{n-1} \xrightarrow{\cong} \hat{A}_\mathfrak{m}\{c_{f(a)}\}$$

is simply the $\mathfrak{m}$-adic completion map.

On the other hand, if $I^A \neq A$, I claim that $I^{\hat{A}_\mathfrak{m}} \neq \hat{A}_\mathfrak{m}$ as well. We prove this by contra-positive: suppose that $I^A \neq A$ but $I^{\hat{A}_\mathfrak{m}} = \hat{A}_\mathfrak{m}$. Then there exists an element $a \in \hat{A}_\mathfrak{m}$ such that $a - a^n \in \hat{A}_\mathfrak{m}^\times$, and consequently the residue $\overline{a} \in \hat{A}_\mathfrak{m}/\mathfrak{m}\hat{A}_\mathfrak{m}$ also has the property that $\overline{a} - \overline{a}^n$ is a unit, and in particular, $\overline{a}$ is nonzero. Since

$$(5.0.2) \qquad\qquad\qquad \hat{A}_\mathfrak{m}/\mathfrak{m}\hat{A}_\mathfrak{m} \cong A/\mathfrak{m},$$

we can choose an element $\tilde{a} \in A$ such that the residue of $\tilde{a}$ in $A/\mathfrak{m}$ is (under the isomor-phism 5.0.2) $\overline{a}$. Now $\tilde{a}$ is an element of A such that $\tilde{a} - \tilde{a}^n$ is nonzero modulo $\mathfrak{m}$, hence $I^A = A$ since every ideal in the discrete valuation ring A is a power of $\mathfrak{m}$, so the only ideal containing an element which is nonzero modulo $\mathfrak{m}$ is A itself and so $\tilde{a} - \tilde{a}^n$ is a unit in A. But this contradicts our assumption that $I^A \neq A$.

So if $I^A \neq A$, then $I^{\hat{A}_\mathfrak{m}} \neq \hat{A}_\mathfrak{m}$. Writing π for a generator of the maximal ideal $\mathfrak{m}$ of A, we now have that the composite

$$A\{c_\pi\} \xrightarrow{\cong} L^A_{n-1}/D^A_{n-1} \to L^{\hat{A}_\mathfrak{m}}_{n-1}/D^{\hat{A}_\mathfrak{m}}_{n-1} \xrightarrow{\cong} \hat{A}_\mathfrak{m}\{c_\pi\}$$

is simply the $\mathfrak{m}$-adic completion map.

In either case, the map $L^A_{n-1}/D^A_{n-1} \to L^{\hat{A}_\mathfrak{m}}_{n-1}/D^{\hat{A}_\mathfrak{m}}_{n-1}$ is simply (up to isomorphism) $\mathfrak{m}$-adic completion. Hazewinkel shows that, for any discrete valuation ring A, L^A is simply the symmetric A-algebra on the direct sum $\coprod_{n \geq 2} L^A_{n-1}/D^A_{n-1}$; and now,

- since A and $\hat{A}_\mathfrak{m}$ are both discrete valuation rings,
- since the map $\coprod_{n \geq 2} L^A_{n-1}/D^A_{n-1} \to \coprod_{n \geq 2} L^{\hat{A}_\mathfrak{m}}_{n-1}/D^{\hat{A}_\mathfrak{m}}_{n-1}$ is simply $\mathfrak{m}$-adic completion,
- since A is Noetherian and $\coprod_{n \geq 2} L^A_{n-1}/D^A_{n-1}$ is a finitely-generated A-module in each grading degree, and consequently $\mathfrak{m}$-adic completion coincides with tensor-ing over A with $\hat{A}_\mathfrak{m}$,
- and since taking symmetric algebras commutes with tensoring over A with $\hat{A}_\mathfrak{m}$,

we conclude that the map $L^A \to L^{\hat{A}_\mathfrak{m}}$ is simply, up to isomorphism, $\mathfrak{m}$-adic completion, and it also coincides with the base-change map from A to $\hat{A}_\mathfrak{m}$. The maps g, h in the statement of the lemma are hence isomorphisms.

Since the map of discrete valuation rings $A \to \hat{A}_\mathfrak{m}$ sends a uniformizer to a uniformizer and induces an isomorphism on residue fields, the splitting of L^A as an infinite direct sum of copies of V^A is compatible with the maps from L^A to $L^{\hat{A}_\mathfrak{m}}$ and from V^A to $V^{\hat{A}_\mathfrak{m}}$; see section 21.5 of [7].   □

We also need to use an easy result from the paper [24]:

Proposition 5.5. *Let A be the ring of integers in a finite field extension $K/\mathbb{Q}$. Then, for each integer n, the grading degree n summand L_n^A in the graded ring L^A is a finitely generated A-module.*

Proof. See [24]. (But, for the sake of self-containedness: one can easily prove this proposition from first principles by simply remembering that, to specify a formal A-module $n+1$-bud $(F(X,Y), \rho(X))$, one only needs to specify finitely many coefficients in the truncated power series $F(X,Y)$ and $\rho(X)$.) □

Theorem 5.6. (Nonexistence of Ravenel's algebraic spheres, global case.) *Let K be a finite extension of $\mathbb{Q}$ of degree > 1, with ring of integers A. Suppose that there exists a prime number p which neither ramifies nor splits completely in A, and such that each prime $\mathfrak{m}$ of A over p is of the same degree. (By Lemma 5.2, this condition is satisfied if $K/\mathbb{Q}$ is cyclic of prime power order, and it is also satisfied if $K/\mathbb{Q}$ is Galois.)*
Then there does not exist a spectrum X such that $MU_*(X) \cong L^A$.

Proof. This is a proof by contrapositive: suppose that a spectrum X exists so that $MU_*(X) \cong L^A$, and choose a prime number p such that p neither ramifies nor completely splits in A and such that every prime of A over p is of the same degree. It is classical that the homotopy limit $\mathrm{holim}_n S/p^n$ of the mod p^n Moore spectra has its homotopy groups $\pi_*(\mathrm{holim}_n S/p^n)$ isomorphic, as a $\pi_*(S)$-module, to the p-adic completion $(\pi_*(S))\hat{}_p$ of $\pi_*(S)$. Since $\pi_*(S)$ is finitely generated in each grading degree, $(\pi_*(S))\hat{}_p \cong \pi_*(S) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p$, and since $\hat{\mathbb{Z}}_p$ is flat as a $\mathbb{Z}$-module, the ring $(\pi_*(S))\hat{}_p$ is flat as an $\pi_*(S)$-module. Hence the E^2-term

$$\mathrm{Tor}_{*,*}^{\pi_*(S)}(MU_*(X), \pi_*(\mathrm{holim}_n S/p^n))$$

of the Künneth spectral sequence computing $\pi_*(MU \wedge X \wedge \mathrm{holim}_n S/p^n)$ collapses on to the $\mathrm{Tor}_0^{\pi_*(S)}$-line, hence

$$
\begin{aligned}
MU_*(X \wedge \mathrm{holim}_n S/p^n) &\cong MU_*(X) \otimes_{\pi_*(S)} \pi_*(S)\hat{}_p \\
&\cong L^A \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p \\
(5.0.3) \qquad &\cong (L^A)\hat{}_p,
\end{aligned}
$$

with isomorphism 5.0.3 due to L^A being finitely generated in each grading degree by Proposition 5.5. (Recall that, for a finitely generated module M over a Noetherian ring R, if I is an ideal in R then the natural map $M \otimes_A \hat{A}_I \to \hat{M}_I$ is an isomorphism; see e.g. Proposition 10.13 of [1].)

Now write $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ for the set of primes of A over p. By Lemma 5.3 and by the finite generation result in Proposition 5.5, $(L^A)\hat{}_p \cong \prod_{i=1}^n (L^A)\hat{}_{\mathfrak{p}_i}$.

Now, for each i, let q_i denote the cardinality of the residue field of $\hat{A}_{\mathfrak{p}_i}$. We have an isomorphism of graded $V^{\hat{A}_{\mathfrak{p}_i}}$-modules

$$L^{\hat{A}_{\mathfrak{p}_i}} \cong \coprod_{0 \le n \ne q_i^k - 1} \Sigma^{2n} V^{\hat{A}_{\mathfrak{p}_i}}$$

given by Cartier typification (see 21.7.17 of [7] for this).

Finally, putting all these isomorphisms together, we have a string of isomorphisms:

$$MU_*\left(X \wedge \mathrm{holim}_b\, S/p^b\right) \cong (L^A)\hat{}_p$$

$$\cong \prod_{i=1}^{m} (L^A)\hat{}_{\mathfrak{p}_i}$$

(5.0.4)
$$\cong \prod_{i=1}^{m} L^{\hat{A}_{\mathfrak{p}_i}}$$

(5.0.5)
$$\cong \prod_{i=1}^{m}\left(\coprod_{0 \le a \ne q_i^j - 1} \Sigma^{2a} V^{\hat{A}_{\mathfrak{p}_i}}\right),$$

with isomorphism 5.0.4 due to Lemma 5.4.

Now by assumption, each of the primes $\mathfrak{p}_i$ in A has the same residue degree; and since p was assumed to not ramify, this means that the extension of local fields $K(\hat{A}_{\mathfrak{p}_i})/\mathbb{Q}_p$ is unramified of some residue degree f_p, and this residue degree f_p does not depend on i. (I am writing $K(A)$ for the field of fractions of a ring A.) But up to isomorphism, there is only one unramified extension of $\mathbb{Q}_p$ of a given residue degree f_p. Consequently neither $K(\hat{A}_{\mathfrak{p}_i})$ nor its ring of integers $\hat{A}_{\mathfrak{p}_i}$ depends, up to isomorphism, on i. So from isomorphism 5.0.5 we get the isomorphism

$$MU_*\left(X \wedge \mathrm{holim}_b\, S/p^b\right) \cong \coprod_{0 \le a \ne q^j - 1} \Sigma^{2a}\left(V^{\hat{A}_{\mathfrak{p}}}\right)^{\times n}$$

where $\mathfrak{p}$ is any of the primes of A over p, and q is the cardinality of the residue field of $\hat{A}_{\mathfrak{p}}$.

Now we have the usual splitting of p-local (hence also p-complete) complex bordism as a sum of suspensions of BP-homology,

$$(MU_*(X))\hat{}_p \cong \coprod_{0 \le n \ne p^k - 1} \Sigma^{2n}(BP_*(X))\hat{}_p$$

$$\cong BP_*\left(\left(\coprod_{0 \le n \ne p^k - 1} \Sigma^{2n} X\right) \wedge \mathrm{holim}_b\, S/p^b\right),$$

so now we have isomorphisms

$$BP_*\left(\left(\coprod_{0 \le n \ne p^k - 1} \Sigma^{2n} X\right) \wedge \mathrm{holim}_b\, S/p^b\right) \cong (MU_*(X))\hat{}_p$$

$$\cong (L^A)\hat{}_p$$

$$\cong \left(\coprod_{0 \le n \ne q^k - 1} \Sigma^{2n} V^{\hat{A}_{\mathfrak{p}}}\right)^{\times m},$$

and since we assumed that p does not completely split in A, the residue degree f_p is greater than one. So Theorem 3.9 applies: $\left(\coprod_{0 \le n \ne q^k - 1} \Sigma^{2n} V^{\hat{A}_{\mathfrak{p}}}\right)^{\times f_p}$ must be a v_n-power-torsion BP_*-module for all $n \ge 0$, since the $V^{\hat{A}_{\mathfrak{p}}}$-module $\left(\coprod_{0 \le n \ne q^k - 1} \Sigma^{2n} V^{\hat{A}_{\mathfrak{p}}}\right)^{\times f_p}$ is the BP-homology of a spectrum, namely, the spectrum $\left(\coprod_{0 \le n \ne p^k - 1} \Sigma^{2n} X\right) \wedge \mathrm{holim}_b\, S/p^b$. But $\left(\coprod_{0 \le n \ne q^k - 1} \Sigma^{2n} V^{\hat{A}_{\mathfrak{p}}}\right)^{\times f_p}$ is a free $V^{\hat{A}_{\mathfrak{p}}}$-module, hence is not p-power-torsion, a contradiction.

Hence there cannot exist X such that $MU_*(X) \cong L^A$.   □

Corollary 5.7. *Except in the trivial case where $A = \hat{\mathbb{Z}}_p$, Ravenel's local and p-typical algebraic extensions of the sphere spectrum do not exist. In the case of an extension which is cyclic of prime power order or a Galois extension, Ravenel's global algebraic extensions of the sphere spectrum do not exist, except in the trivial case where $A = \mathbb{Z}$.*

That is, if $K/\mathbb{Q}$ is a finite extension of degree > 1, which is either cyclic of prime power order or Galois, and which has ring of integers A, there does not exist a spectrum S^A such that $MU_*(S^A) \cong L^A$ as graded MU_*-modules. If $K/\mathbb{Q}_p$ is a finite extension of degree > 1 with ring of integers A, then there does not exist a spectrum S^A such that $MU_*(S^A) \cong L^A$ as graded MU_*-modules, and there does not exist a spectrum S^A such that $BP_*(S^A) \cong V^A$ as graded BP_*-modules.

6. Appendix on basics of local cohomology.

In this short appendix I present some of the most basic definitions and ideas of local cohomology. The reader who wants something more substantial about this subject, or to read proofs of the results cited in this appendix, can consult the textbook of Brodmann and Sharp, [3].

First, recall that an ideal I in a commutative graded ring R is called *homogeneous* if there exists a generating set for I consisting of homogeneous elements.

Definition 6.1. *Let R be a commutative graded ring, and let I be a homogeneous ideal of R. Let* grMod(R) *denote the category of graded R-modules and grading-preserving R-module homomorphisms. We define two functors*

$$D_I, \Gamma_I : \mathrm{grMod}(R) \to \mathrm{grMod}(R)$$

as follows:

$$\Gamma_I(M) = \coprod_{m \in \mathbb{Z}} \mathrm{colim}_{n \to \infty} \mathrm{hom}_{\mathrm{grMod}(R)}(\Sigma^m R/I^n, M)$$

$$D_I(M) = \coprod_{m \in \mathbb{Z}} \mathrm{colim}_{n \to \infty} \mathrm{hom}_{\mathrm{grMod}(R)}(\Sigma^m I^n, M),$$

with natural transformations

(6.0.1) $$\Gamma_I \to \mathrm{id} \to D_I$$

induced by applying $\coprod_{m \in \mathbb{Z}} \mathrm{colim}_{n \to \infty} \mathrm{hom}_{\mathrm{grMod}(R)}(\Sigma^m -, M)$ to the short exact sequence of graded R-modules

$$0 \to I^n \to R \to R/I^n \to 0.$$

The right-derived functors $R^*\Gamma_I$ of the functor Γ_I are called local cohomology at I, *and written*

$$H_I^n(M) = (R^n(\Gamma_I))(M).$$

Proposition 6.2. *Let R be a commutative graded ring, and let I be a homogeneous ideal of R. The natural transformations 6.0.1 induce an exact sequence of R-modules*

(6.0.2) $$0 \to \Gamma_I(M) \to M \to D_I(M) \to H_I^1(M) \to 0,$$

natural in M, and an isomorphism

$$H_I^n(M) \cong (R^{n-1}(D_I))(M)$$

for all $n \geq 2$, also natural in M.

Proposition 6.3. *Let R be a commutative graded ring, and let $r \in R$ be a homogeneous element. Let (r) be the principal ideal generated by r.*

- *Then $D_{(r)}(M)$ is naturally isomorphic to $r^{-1}M$, and the natural transformation $\mathrm{id} \to D_{(r)}$ from 6.0.1 is naturally isomorphic to the standard localization map $M \to r^{-1}M$.*
- *Furthermore, $\Gamma_{(r)}(M)$ is naturally isomorphic to the sub-R-module of M consisting of the r-power-torsion elements, and the natural transformation $\Gamma_{(r)} \to \mathrm{id}$ from 6.0.1 is naturally isomorphic to the inclusion of that submodule.*
- *Finally, $H^n_{(r)}(M) \cong 0$ for all $n \geq 2$.*

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