

RAVENEL'S MAY SPECTRAL SEQUENCE COLLAPSES IMMEDIATELY AT LARGE PRIMES.

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ABSTRACT. At large primes, the height n Ravenel-May spectral sequence takes as input the cohomology of a certain solvable Lie \mathbb{F}_p -algebra, and produces as output the mod p cohomology of the height n strict Morava stabilizer group scheme. We construct simultaneous integral deformations of the height n Morava stabilizer algebras and related objects, and we use them to prove that, for fixed n , the height n Ravenel-May spectral sequence collapses for all sufficiently large primes p . Consequently, for large p , the mod p cohomology of the strict Morava stabilizer group scheme is the cohomology of a finite-dimensional solvable Lie algebra, and is computable algorithmically.

1. INTRODUCTION.

Let n be a positive integer. Let p be a prime number satisfying $p > n + 1$. In the 1977 paper [14], Ravenel used the methods of May's thesis [7] to construct a spectral sequence

$$(1) \quad E_1^{s,t,u} \cong H^{s,t,u}(L(n, n); \mathbb{F}_p) \Rightarrow H_c^{s,t}(\mathrm{str} \mathrm{Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p),$$

which we call the *p -primary height n Ravenel-May spectral sequence*, or RMSS for short. Here $L(n, n)$ is a particular n^2 -dimensional solvable Lie \mathbb{F}_p -algebra, and $\mathrm{str} \mathrm{Aut}(\mathbb{G}_{1/n})$ is the height n strict Morava stabilizer group scheme¹, i.e., the strict automorphism group scheme of a² height n one-dimensional formal group law over \mathbb{F}_p . The E_1 -page of the RMSS is the cohomology of the Lie algebra $L(n, n)$, while the abutment of the RMSS is the continuous cohomology of the profinite group scheme $\mathrm{str} \mathrm{Aut}(\mathbb{G}_{1/n})$. The latter is the input for many spectral sequences which are used to calculate stable homotopy groups. For example:

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¹Some readers may be much less familiar with the Morava stabilizer group *scheme* than with the Morava stabilizer *group*. We give some background about the Morava stabilizer group scheme in section 2, but the upshot is the following three points: 1. The Morava stabilizer group scheme is the prime spectrum of the Hopf algebra $S(n)$, the Morava stabilizer algebra. In this paper, we usually write $S(n, p)$ rather than the more common notation $S(n)$, to emphasize the role of the prime p . The cohomology of the Morava stabilizer group scheme is simply Cotor over $S(n, p)$. 2. In every situation in stable homotopy in which one considers the cohomology of the Morava stabilizer group, it is at least as good to consider the cohomology of the Morava stabilizer group scheme—and sometimes it is even a bit better, in the sense that it sometimes allows us to skip a descent spectral sequence for a $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ -action. 3. The main results of this paper remain true after replacing the Morava stabilizer group scheme with the Morava stabilizer group, throughout.

²For concreteness, throughout this paper we take $\mathbb{G}_{1/n}$ to be the Honda one-dimensional formal group law of height n over \mathbb{F}_p , defined in Conventions 1.2. However, this is not an essential point: similar constructions and results are possible for other height n one-dimensional formal group laws over \mathbb{F}_p .

- From $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p)$, one runs a sequence of n Bockstein spectral sequences to get the height n line in the chromatic spectral sequence E_1 -page. The chromatic spectral sequence converges to the Adams-Novikov E_2 -page, and the Adams-Novikov spectral sequence converges to the stable homotopy groups of spheres. See chapters 4-6 of [15] for this material.
- From $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p)$, one runs a sequence of n Bockstein spectral sequences (which are dual, in a certain sense, of the n Bockstein spectral sequences mentioned just above) to get to $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); E(\mathbb{G}_{1/n})_*)$, where $E(\mathbb{G}_{1/n})_*$ is the Morava E -theory spectrum of $\mathbb{G}_{1/n}$. The continuous cohomology $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); E(\mathbb{G}_{1/n})_*)$ is the input for spectral sequences converging to $\pi_*(L_{K(n)}S^0)$, the $K(n)$ -local stable homotopy groups of spheres. See [3] for this material.
- If the p -local Smith-Toda complex $V(n-1)$ exists, then $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_n^{\pm 1}]$ is the input for a spectral sequence converging to the $K(n)$ -local stable homotopy groups of $V(n-1)$. If $p > \frac{n^2+2}{2}$, then this spectral sequence collapses immediately with no differentials, yielding an isomorphism $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_n^{\pm 1}] \cong \pi_*(L_{K(n)}V(n-1))$. See Corollary 5.4 for further explanation. At sufficiently large primes it is also true that there exists a spectrum which plays the role of $K(n)$ -local $V(n-1)$ in this spectral sequence, even if $V(n-1)$ doesn't itself exist.

Because of the applications of $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p)$ in computational stable homotopy theory, it is of great interest to know an answer to the following question:

Question 1.1. For which values of p and n does the RMSS collapse immediately?

When the RMSS collapses immediately, $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p)$ is simply the cohomology of a certain solvable finite-dimensional Lie algebra, computable (algorithmically!) via a Chevalley-Eilenberg complex [2].

Since the Ravenel-May spectral sequence only has the stated³ form (1) when $p > n + 1$, the most optimistic possible answer for Question 1.1 would be: the Ravenel-May spectral sequence collapses immediately as long as $p > n + 1$. Let us call this the *optimistic answer* to Question 1.1.

Indeed, in Theorem 1.7 of [14], Ravenel claims that this optimistic answer is true. However, Ravenel later pointed out [15, section 6.3] that the argument given for the optimistic answer in [14] is incorrect. In section 2, below, we review Ravenel's original argument for the optimistic answer. In [15], Ravenel also describes a possibly nonzero differential in the $p = 11, n = 9$ case of the Ravenel-May spectral sequence which he suggests may yield a counterexample to the optimistic answer.

Question 1.1 remains wide open. In particular, it is today not even known whether the optimistic answer is true for $n > 3$. From explicit calculation, it is known that the optimistic answer is indeed correct for $n = 1, 2$ [15, section 6.3] and $n = 3$ [19, section 5] (see also [4] for a simplification of the argument from [19], at $p \geq 7$). In each of those cases, one verifies the correctness of the optimistic answer by explicitly calculating $H^*(L(n, n); \mathbb{F}_p)$, then checking that $H^*(L(n, n); \mathbb{F}_p)$ is generated as a ring by elements which, by routine inspection of tridegrees, cannot

³When $p \leq n + 1$, a Ravenel-May spectral sequence still exists, and still converges to $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p)$, but its E_1 -page is more complicated than described above in (1). Its E_1 -page is the *restricted* cohomology of a certain *restricted* Lie algebra which is larger than $L(n, n)$. See Theorem 1.5 of [14] or Theorem 6.3.4 of [15].

support a nonzero Ravenel-May differential. This strategy for answering Question 1.1 does not generalize well to arbitrary values of n , since the computational cost of calculating $H^*(L(n, n); \mathbb{F}_p)$ increases rapidly⁴ as n grows.

In this paper, we give an *asymptotic* answer to Question 1.1. The main results are:

Theorem 5.2. Fix a positive integer n . Then there exists some integer N_n such that, for all $p > N_n$, the p -primary height n Ravenel-May spectral sequence has no nonzero differentials.

Theorem 5.3. Fix a positive integer n . Then there exists some integer N_n such that, for all $p > N_n$, the mod p continuous cohomology $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p)$ of the p -primary height n strict Morava stabilizer group scheme is isomorphic, as a graded ring, to the cohomology of the solvable Lie \mathbb{F}_p -algebra $L(n, n)$.

The most obvious corollary is that the mod p cohomology of the height n strict Morava stabilizer group scheme is algorithmically computable for $p \gg n$. We offer a few other corollaries as well. If the Smith-Toda complex $V(n-1)$ exists for all sufficiently large p , then Corollary 5.4 gives us that the $K(n)$ -local stable homotopy groups of $V(n-1)$ are isomorphic to the Lie algebra cohomology $H^*(L(n, n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_n^{\pm 1}]$, for $p \gg n$.

Another corollary of our main theorems is Corollary 5.5: suppose that n is a positive integer, and suppose that p is a prime larger than the integer N_n described in the statement, given above, of Theorem 5.2. Suppose that X is an $E(n-1)$ -acyclic finite CW-complex. Then there exist strongly convergent spectral sequences

$$(2) \quad E_1^{*,*,*} \cong H^*(L(n, n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} E_0 E(n)_*(X) \Rightarrow \text{Cotor}_{E(n)_* E(n)}^{*,*}(E(n)_*, E(n)_*(X)),$$

$$(3) \quad E_2^{*,*} \cong \text{Cotor}_{E(n)_* E(n)}^{*,*}(E(n)_*, E(n)_*(X)) \Rightarrow \pi_*(L_{E(n)} X),$$

⁴We offer a little bit of data about the difficulty of calculating $H^*(L(n, n); \mathbb{F}_p)$. The brute-force approach is to simply calculate the cohomology of the Chevalley-Eilenberg complex $\Lambda^*(L(n, n)^*)$ of $L(n, n)$. This is a finite calculation, and in principle a computer can attempt this, and it can succeed for low values of n . However, since $L(n, n)$ is n^2 -dimensional, the Chevalley-Eilenberg complex is $2^{(n^2)}$ -dimensional. This grows extremely quickly as n grows, and exhausts any feasible supply of RAM if $n > 5$.

The cohomology of $L(n, n)$ also grows quickly as n grows. For $p > n + 1$, the total \mathbb{F}_p -linear dimension of $H^*(L(n, n); \mathbb{F}_p)$ and of the Chevalley-Eilenberg complex is given by the following table:

n	$\dim_{\mathbb{F}_p} H^*(L(n, n); \mathbb{F}_p)$	$\dim_{\mathbb{F}_p} \Lambda^*(L(n, n)^*)$
1	2	2
2	12	16
3	152	512
4	3440	65536

The cohomology $H^*(L(n, n); \mathbb{F}_p)$ is not known for $n > 4$. The $n = 4$ case is in the preprint [17], where it is also observed that these \mathbb{F}_p -linear dimensions agree with the coefficients in a particular generating function. If the coefficients of that generating function continue (for higher n) to agree with the \mathbb{F}_p -linear dimension of $H^*(L(n, n); \mathbb{F}_p)$, then, for example, $H^*(L(5, 5); \mathbb{F}_p)$ will be 128512-dimensional, while $H^*(L(6, 6); \mathbb{F}_p)$ will be 7621888-dimensional.

In the preprint [17], a “height-shifting” approach is used to calculate $H^*(L(4, 4); \mathbb{F}_p)$. This approach is much more efficient and illuminating than brute-force calculation of the cohomology of the Chevalley-Eilenberg complex, but it is still quite demanding, increasingly so as n grows.

where $E_0E(n)_*(X)$ is the associated graded abelian group of the p -adic filtration

$$E(n)_*(X) \supseteq pE(n)_*(X) \supseteq p^2E(n)_*(X) \supseteq p^3E(n)_*(X) \supseteq \dots$$

Of course (3) is not new: it is simply the $E(n)$ -Adams spectral sequence of X . The new part is spectral sequence (2), which together with (3), gives us a means to pass from the cohomology of a solvable Lie algebra to the $E(n)$ -local stable homotopy groups of a finite $E(n-1)$ -acyclic CW-complex.

We think that the methods used to prove our main results are themselves of interest. The basic strategy is to find *integral deformations* of the objects involved in the construction of the Ravenel-May spectral sequence. Here is a sketch of the steps involved:

- For each positive integer n , in section 3 we construct a Lie \mathbb{Z} -algebra $L_{int}(n)$, free over \mathbb{Z} , whose mod p reduction for each p is equal to the Lie \mathbb{F}_p -algebra $L(n, n)$.
- The (graded) \mathbb{Z} -linear dual of the universal enveloping \mathbb{Z} -algebra of $L_{int}(n)$ is a divided power \mathbb{Z} -algebra. Inside this divided power algebra, there is a natural polynomial Hopf \mathbb{Z} -algebra, which we call $\mathcal{Z}(n)$.
- We filter $\mathcal{Z}(n)$ in such a way that, for $p \gg n$, upon reducing modulo p and modulo all the p th powers of the polynomial generators, we recover Ravenel's filtration on $S(n, p)$. This filtration, and the definition of $\mathcal{Z}(n)$ itself, are in section 4.
- In section 4 we also show that the resulting spectral sequence

$$(4) \quad \text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} E_0 \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$$

collapses immediately.

- In section 5, we construct a set of cocycles in the cobar complex of $E_0 \mathcal{Z}(n)$ which map, for $p \gg n$, to an \mathbb{F}_p -linear basis for $\text{Cotor}_{E_0 S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p) \cong H^*(L(n, n); \mathbb{F}_p)$.
- We then show that these cocycles survive the integral Ravenel-May spectral sequence (4), and that their images in the Ravenel-May spectral sequence (1) must also survive. Hence, for $p \gg n$, all elements of $H^*(L(n, n); \mathbb{F}_p)$ must survive the Ravenel-May spectral sequence, i.e., the spectral sequence collapses immediately with no differentials.

One corollary is that the cohomology of our n^2 -dimensional Lie \mathbb{Z} -algebra $L_{int}(n)$ has, upon reduction modulo p , the same cohomology as $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p)$ for $p \gg n$. In that sense, if one makes the single calculation of the Lie algebra cohomology of $L_{int}(n)$, then one knows the mod p cohomology of the height n strict Morava stabilizer group scheme for all but finitely many p . We demonstrate with explicit integral calculations for $n = 1$ and $n = 2$ in section 6.

We also define an integral deformation $S_{int}(n)^\bullet$ of the cobar complex of the Morava stabilizer algebra $S(n, p)$ itself. However, to prove the main results, the smaller object $\mathcal{Z}(n)$ (whose cobar complex is a sub-DGA of $S_{int}(n)^\bullet$) suffices. Since it is not used in the proofs of the main results, we consign the definition of $S_{int}(n)^\bullet$ to an appendix.

We remark that Theorems 5.2 and 5.3 are used in our joint work with Mohammad Behzad Kang, currently in preparation, in which we show that the cohomology of the height n *extended* Morava stabilizer group, with trivial mod p coefficients, is isomorphic to the cohomology $H^*(U(n); \mathbb{F}_p)$ of the unitary group for all $p \gg n$.

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Conventions 1.2.

- The symbol C_n denotes a cyclic group of order n . Throughout, we fix a choice of generator σ for C_n .
- The symbol $[i]$ denotes the integer floor of a rational number i .
- For each prime p and each positive integer n , the height n Morava stabilizer algebra is a certain Hopf \mathbb{F}_p -algebra. It is denoted $S(n)$ in the standard reference, [15], but this suppresses the choice of prime p from the notation. It will be convenient to include the choice of p in the notation for the Morava stabilizer algebra, so we will write $S(n, p)$ rather than $S(n)$.
- Throughout, all formal group laws will be implicitly understood to be one-dimensional.
- Given a prime number p and a positive integer n , we use the symbol $\mathbb{G}_{1/n}$ for the p -typical height n formal group law over \mathbb{F}_p classified by the ring map $BP_* \rightarrow \mathbb{F}_p$ sending the Hazewinkel generator $v_n \in BP_*$ to 1 and sending the other Hazewinkel generators v_i to zero. (Sometimes this formal group law is called the “height n Honda formal group law.”) The symbol $\mathbb{G}_{1/n}$ comes from the Dieudonné-Manin classification [5] of p -divisible groups over a separably closed perfect field.
- In this paper we will need to consider the cohomology of restricted Lie algebras and also of their underlying unrestricted Lie algebras over fields of positive characteristic. We write $H_{res}^*(\mathfrak{g}, \rho)$ and $H^*(\mathfrak{g}, \rho)$ for the restricted Lie algebra cohomology $\text{Ext}_{V_{\mathfrak{g}}}^*(k, \rho)$ and the unrestricted Lie algebra cohomology $\text{Ext}_{U_{\mathfrak{g}}}^*(k, \rho)$, respectively, with coefficients in a representation ρ of \mathfrak{g} .

2. REVIEW OF RAVENEL'S LIE ALGEBRAS AND ASSOCIATED SPECTRAL SEQUENCES.

This section is a review of known results for readers unfamiliar with Ravenel's computational approach to the cohomology of the Morava stabilizer algebras. Readers already familiar with the ideas from chapter 6 of [15] can skip ahead to section 3.

The p -primary height n Morava stabilizer algebra $S(n, p)$ is isomorphic, as a $\mathbb{Z}/2(p^n - 1)\mathbb{Z}$ -graded \mathbb{F}_p -algebra, to

$$(5) \quad \mathbb{F}_p[t_1, t_2, \dots] / (t_i^{p^n} - t_i \ \forall i),$$

with t_i in degree $2(p^i - 1)$ [13]. The algebra $S(n, p)$ is a C_n -equivariant commutative Hopf algebra over \mathbb{F}_p , where C_n acts by the Frobenius map. The prime spectrum $\text{Spec } S(n, p)$ represents a profinite group scheme, the strict automorphism group scheme of the p -height n formal group law $\mathbb{G}_{1/n}$. This group scheme is pro-étale, i.e., after a separable base change⁵, the Hopf algebra $S(n, p)$ becomes isomorphic to the continuous linear dual of the group ring of a profinite group—an actual *group*, not just a group scheme. This profinite group is a pro- p -group, isomorphic to a maximal pro- p -subgroup of the group of units in the maximal order in an

⁵In fact, base change to \mathbb{F}_{p^n} suffices.

invariant $1/n$ central division algebra over \mathbb{Q}_p , and isomorphic also to the strict automorphism group of any height n formal group law over the extension field.

In section 1 of [14] (see chapter 6 of [15] for a textbook reference), Ravenel puts an increasing filtration on the Morava stabilizer algebra $S(n, p)$ as follows:

- Let $d_{n,i}$ be the integer defined recursively by the rule

$$d_{n,i} = \begin{cases} 0 & \text{if } i \leq 0 \\ \max\{i, pd_{n,i-n}\} & \text{if } i > 0. \end{cases}$$

- Let the filtration degree of $t_i^{p^j}$ be $d_{n,i}$ for each j . More generally, let the filtration degree of t_i^k be $d_{n,i}$ times the sum of the coefficients in the base p expansion of k . The filtration is otherwise multiplicative, e.g. the filtration degree of $t_i t_j$ is equal to $d_{n,i}$ plus $d_{n,j}$, if $i \neq j$.

The associated graded Hopf algebra $E_0S(n, p)$ is isomorphic, as a bigraded Hopf algebra, to $\mathbb{F}_p[t_{i,j} : i \geq 1, j \in \mathbb{Z}/n\mathbb{Z}]/t_{i,j}^p$, with $t_{i,j}$ in internal degree $2(p^i - 1)p^j$ and Ravenel degree $d_{n,i}$. The element $t_{i,j}$ of $E_0S(n, p)$ represents the element $t_i^{p^j}$ of $S(n, p)$.

Ravenel's filtration has the agreeable property that its associated graded Hopf algebra $E_0S(n, p)$ is the linear dual of a primitively generated Hopf algebra, so the methods of Milnor–Moore [11] and May's thesis [7] can be applied. In particular, $E_0S(n, p)$ is dual to the restricted enveloping algebra $VP(E_0S(n, p)^*)$ of the restricted Lie \mathbb{F}_p -algebra $P(E_0S(n, p)^*)$ of primitives in the dual Hopf algebra $E_0S(n, p)^*$. In [6],[7], and [8], May constructed a spectral sequence

$$(6) \quad \begin{aligned} E_1^{*,*,*} &\cong H^*(\mathfrak{g}, k) \otimes_k k[\mathfrak{g}^*] \Rightarrow H_{res}^*(\mathfrak{g}, k) \\ d_r &: E_r^{s,t,u} \rightarrow E_r^{s+1,t+r,u} \end{aligned}$$

for every graded restricted Lie algebra \mathfrak{g} , concentrated in even degrees, over a field k of characteristic p . Here $k[\mathfrak{g}^*]$ is the free commutative k -algebra on the dual k -vector space of \mathfrak{g} . The grading is as follows: elements of $H^s(\mathfrak{g}, k)$ in internal⁶ degree u are in tridegree $(s, 0, u)$. Nonzero elements of $\mathfrak{g}^* \subseteq k[\mathfrak{g}^*]$ of internal degree u are in tridegree $(2, 1, pu)$. Elements in tridegree (s, t, u) contribute, in the abutment, to elements of $H_{res}^s(\mathfrak{g}, k)$ of internal degree u .

Spectral sequence (6) was one of two new spectral sequences studied in May's thesis. In the literature, both spectral sequences have at times been called “the May spectral sequence.” To resolve the ambiguity in naming, we will refer to (6) as the *Lie-May spectral sequence*, since it is the spectral sequence from May's thesis which relates restricted Lie algebra cohomology to unrestricted Lie algebra cohomology. (The other spectral sequence studied in May's thesis was the spectral sequence of a Hopf algebra, and particularly the 2-primary Steenrod algebra, filtered by powers of its augmentation ideal.)

In Theorem 1.6 of [14] (cf. [15, Theorem 6.3.5]), Ravenel shows that spectral sequence (6) admits a tensor splitting in the case that $\mathfrak{g} = P(E_0S(n, p)^*)$. One tensor factor converges to \mathbb{F}_p concentrated in tridegree $(0, 0, 0)$, while the other tensor factor has E_1 -term isomorphic to

$$H^* \left(L \left(n, \lfloor \frac{pn}{p-1} \rfloor \right); \mathbb{F}_p \right) \otimes_{\mathbb{F}_p} \mathbb{F}_p \left[b_{i,j} : 1 \leq i \leq \lfloor \frac{n}{p-1} \rfloor, j \in \mathbb{Z}/n\mathbb{Z} \right],$$

⁶We follow the standard convention of referring to the grading on $H^*(\mathfrak{g}, k)$ induced by the grading on \mathfrak{g} itself as the *internal* grading.

where the notation and gradings are as follows:

- $L\left(n, \lfloor \frac{pn}{p-1} \rfloor\right)$ is a certain $n\lfloor \frac{pn}{p-1} \rfloor$ -dimensional quotient Lie algebra of $P(E_0S(n, p)^*)$,
- elements of $H^s\left(L\left(n, \lfloor \frac{pn}{p-1} \rfloor\right); \mathbb{F}_p\right)$ in internal grading u are in tridegree $(s, 0, u)$,
- and the polynomial generator $b_{i,j}$ is in tridegree $(2, 1, 2p^{j+1}(p^i - 1))$.

Consequently Ravenel gets a spectral sequence

$$(7) \quad E_1^{*,*,*} \cong H^*\left(L\left(n, \lfloor \frac{pn}{p-1} \rfloor\right); \mathbb{F}_p\right) \otimes_{\mathbb{F}_p} \mathbb{F}_p \left[b_{i,j} : 1 \leq i \leq \lfloor \frac{n}{p-1} \rfloor, j \in \mathbb{Z}/n\mathbb{Z} \right]$$

$$\Rightarrow H_{res}^*(P(E_0S(n, p)^*); \mathbb{F}_p) \cong \text{Cotor}_{E_0S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p)$$

$$d_r : E_r^{s,t,u} \rightarrow E_r^{s+1,t+r,u}.$$

We will refer to (7) as the *Ravenel-Lie-May spectral sequence*, or RLMSS for short.

It will be useful to have an explicit description of the Lie \mathbb{F}_p -algebra $L(n, m)$ for positive integers n, m . In Theorem 1.4 of [14] (cf. [15, Theorem 6.3.3]), Ravenel shows that the restricted Lie \mathbb{F}_p -algebra $P(E_0S(n, p)^*)$ has \mathbb{F}_p -linear basis $\{x_{i,j} : i \geq 1, j \in \mathbb{Z}/n\mathbb{Z}\}$, with Lie bracket

$$(8) \quad [x_{i,j}, x_{k,\ell}] = \begin{cases} \delta_{i+j}^\ell x_{i+k,j} - \delta_{k+\ell}^j x_{i+k,\ell} & \text{if } i+k \leq \lfloor \frac{pn}{p-1} \rfloor \\ 0 & \text{otherwise.} \end{cases}$$

The symbol δ denotes the *mod* n Kronecker symbol, i.e., δ_i^j is equal to 1 if $i \equiv j$ modulo n , and equal to zero otherwise. The Lie \mathbb{F}_p -algebra $L(n, m)$ is defined to be the quotient of $P(E_0S(n, p)^*)$ by the linear span of the elements $x_{i,j}$ satisfying $i > m$.

We also have the spectral sequence

$$(9) \quad E_1^{*,*,*} \cong H_{res}^*(P(E_0S(n, p)^*); \mathbb{F}_p)$$

$$\cong \text{Cotor}_{E_0S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p)$$

$$\Rightarrow \text{Cotor}_{S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p)$$

$$(10) \quad d_r : E_r^{s,t,u} \rightarrow E_r^{s+1,t,u-r}$$

arising from applying Ravenel's filtration to the cobar complex of $S(n, p)$. Here s is the cohomological degree, t the internal degree, and u the Ravenel filtration degree. As in section 1, we will call (9) the *Ravenel-May spectral sequence*, or RMSS for short.

If $p > n + 1$, then $\lfloor \frac{pn}{p-1} \rfloor = n$, and consequently the polynomial factor in the E_1 -term of the RLMSS is trivial. Hence the RLMSS collapses immediately, yielding an isomorphism of graded rings $H^*(L(n, n); \mathbb{F}_p) \cong H_{res}^*(P(E_0S(n, p)^*); \mathbb{F}_p)$. This is why, for $p > n + 1$, the E_1 -term of the RMSS is simply the cohomology of the Lie algebra $L(n, n)$, as described above in (1).

The RMSS is known to have nonzero differentials for some values of p and n satisfying $p \leq n + 1$. For example, in the case $p = 2$, $n = 3$, we have the nonzero Ravenel-May differential $d_1(b_{20}) = h_{11}b_{11} + h_{12}b_{10}$, as in⁷ [15] (see the sentence beginning with "By 4.3.22" in the proof of Theorem 6.3.14). However, it is not known whether there exist any nonzero RMSS differentials for $p > n + 1$.

⁷To be clear, in [15], the indexing on the RMSS is slightly different from the indexing given above in (10). In the indexing used in [15], the differential on b_{20} is a d_2 rather than a d_1 .

Here is some explanation of how Ravenel–May differentials arise. The coproduct in $S(n, p)$ is given by

$$(11) \quad \Delta(t_i) = \sum_{k=0}^i t_k \otimes t_{i-k}^{p^k}$$

for $i \leq n$, while the coproduct in $E_0S(n, p)$ is given by

$$(12) \quad \Delta(t_{i,j}) = \sum_{k=0}^i t_{k,j} \otimes t_{i-k,j+k}$$

for $i \leq n$. The element $t_{i,j}$ in $E_0S(n, p)$ represents the element $t_i^{p^j}$ in $S(n, p)$, so (12) is a simple consequence of (11). If $i > n$, then the coproduct on t_i in $S(n, p)$, and on $t_{i,j}$ in $E_0S(n, p)$, is more complicated than the formulas (11) and (12). However, if $p > n + 1$, then it is the coproduct on t_i for $i \leq n$ which is responsible for the Cotor-groups, in the sense that the cohomology of $L(n, n)$ agrees with Cotor over $E_0S(n, p)$, and the Lie bracket (8) in $L(n, n)$ is dual to the coproduct on elements $t_{i,j}$ in $E_0S(n, p)$ with $i \leq n$. By the existence of the Ravenel–May spectral sequence $\text{Cotor}_{E_0S(n,p)}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \text{Cotor}_{S(n,p)}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p)$, all the Cotor-groups of $S(n, p)$ come from the Cotor-groups of $E_0S(n, p)$, and in that sense the generators $t_i^{p^j}$ of $S(n, p)$ for $i \leq n$ are also “responsible” for $\text{Cotor}_{S(n,p)}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p)$, just as the generators $t_{i,j}$ of $E_0S(n, p)$ for $i \leq n$ are “responsible” for $\text{Cotor}_{E_0S(n,p)}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p)$.

To summarize: the Hopf algebra $E_0S(n, p)$ can be generated by elements $\{t_{i,j}\}$ which are representable by generators $\{t_i^{p^j}\}$ for $S(n, p)$, and on the generators whose coproducts determine the Cotor groups, the coproduct in $E_0S(n, p)$ agrees with the coproduct in $S(n, p)$. At a glance, this makes it seem like $S(n, p)$ and $E_0S(n, p)$ must have the same Cotor-groups. In fact, this was Ravenel’s argument from [14], which he later [15] pointed out was not correct. The trouble is the difference between the relations in $E_0S(n, p)$ and the relations in $S(n, p)$. In $S(n, p)$, the p^n th power of each generator t_i is equal to t_i itself. Meanwhile, in $E_0S(n, p)$, the p th power of each generator $t_{i,j}$ is zero. As a consequence, a cocycle in the cobar complex for $E_0S(n, p)$ can involve linear combinations of tensor products of polynomials in the generators $t_{i,j}$ with the property that, upon applying the cobar complex differential d , each term involves a p th power of some generator, hence is zero. If we lift such an element to the cobar complex of $S(n, p)$ by replacing each instance of $t_{i,j}$ with $t_i^{p^j}$, it is an *a priori* possibility that applying d to the resulting cochain does not yield zero, since p th powers of generators are no longer zero: for example in $S(1, p)$ we have $t_i^p = t_i$ instead of $t_i^p = 0$.

This phenomenon—a cocycle in the cobar complex of $S(n, p)$ lifting to a non-cocycle in the cobar complex of $E_0S(n, p)$ due to the difference between the multiplicative relations in $S(n, p)$ and in $E_0S(n, p)$ —would occur whenever there is a nonzero differential in the RMSS for $p > n + 1$, if such differentials indeed exist.

3. A SIMULTANEOUS INTEGRAL LIFT OF RAVENEL’S HEIGHT n LIE ALGEBRAS FOR ALL PRIMES.

Fix a positive integer n . For each prime p , we have the Lie \mathbb{F}_p -algebra $L(n, n)$. There exists a simultaneous integral lift of all these Lie algebras, for fixed n :

Definition 3.1. Let $L_{int}(n, n)$ denote the C_n -equivariant Lie \mathbb{Z} -algebra with \mathbb{Z} -linear basis

$$\{x_{i,j} : i \in \{1, \dots, n\}, j \in \mathbb{Z}/n\mathbb{Z}\}$$

where C_n acts freely by letting $\sigma x_{i,j} = x_{i,j+1}$, and where the Lie bracket is defined by the rule (8), above.

While $L_{int}(n, n)$ is n^2 -dimensional as a Lie \mathbb{Z} -algebra, it is only n -dimensional as a Lie $\mathbb{Z}[C_n]$ -algebra. To be clear, Lie algebra cohomology *does* depend on the choice of ground ring⁸, and throughout, whenever we speak of the cohomology of $L_{int}(n, n)$, we shall always mean its cohomology as a Lie \mathbb{Z} -algebra, not as a Lie $\mathbb{Z}[C_n]$ -algebra.

The Lie \mathbb{Z} -algebra $L_{int}(n, n)$ is free and finite-dimensional over \mathbb{Z} , and its reduction modulo a prime p is Ravenel's Lie \mathbb{F}_p -algebra $L(n, n)$. We recall the relationship in cohomology between a finite-dimensional \mathbb{Z} -free Lie \mathbb{Z} -algebra and its mod p reduction. The author has never seen this relationship in print, nor heard it mentioned, so we offer a proof, but the proof is easy, and the result must surely be well-known. This relationship is used later, in the proof of Lemma 5.1.

Proposition 3.2. Let L be a Lie \mathbb{Z} -algebra which is finite-dimensional and free over \mathbb{Z} . Then, for each prime p and each integer m , the natural map $H^m(L; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow H^m(L \otimes_{\mathbb{Z}} \mathbb{F}_p; \mathbb{F}_p)$ fits into a short exact sequence

$$(13) \quad 0 \rightarrow H^m(L; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow H^m(L \otimes_{\mathbb{Z}} \mathbb{F}_p; \mathbb{F}_p) \rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H^{m+1}(L; \mathbb{Z}), \mathbb{F}_p) \rightarrow 0.$$

Proof. Recall that the Chevalley-Eilenberg complex of L is the \mathbb{Z} -linear dual of the exterior \mathbb{Z} -algebra on L . Exterior powers commute with base-change, so we have isomorphisms

$$(14) \quad \begin{aligned} \mathrm{hom}_{\mathbb{Z}}(\Lambda_{\mathbb{Z}}^{\bullet}(L), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p &\cong \mathrm{hom}_{\mathbb{Z}}(\Lambda_{\mathbb{Z}}^{\bullet}(L), \mathbb{F}_p) \\ &\cong \mathrm{hom}_{\mathbb{F}_p}(\Lambda_{\mathbb{Z}}^{\bullet}(L) \otimes_{\mathbb{Z}} \mathbb{F}_p, \mathbb{F}_p) \end{aligned}$$

$$(15) \quad \cong \mathrm{hom}_{\mathbb{F}_p}(\Lambda_{\mathbb{F}_p}^{\bullet}(L \otimes_{\mathbb{Z}} \mathbb{F}_p), \mathbb{F}_p).$$

The cohomology of cochain complex (15) is $H^*(L \otimes_{\mathbb{Z}} \mathbb{F}_p; \mathbb{F}_p)$, while the cohomology of cochain complex (14) is describable by the universal coefficient sequence relating the cohomology of a base-changed cochain complex to the base-changed cohomology of the cochain complex. The resulting exact sequence is (13). \square

4. THE HOPF ALGEBRA $\mathcal{Z}(n)$.

To introduce the Hopf algebra $\mathcal{Z}(n)$, it is convenient to begin with some general, well-known observations about the relationship between Lie algebras, divided power algebras, and polynomial algebras. Suppose that L is a Lie \mathbb{Z} -algebra which is free and finite-dimensional over \mathbb{Z} . For any free \mathbb{Z} -module A , we will write A^* for its \mathbb{Z} -linear dual $\mathrm{hom}_{\mathbb{Z}}(A, \mathbb{Z})$. The dual $(UL)^*$ of the universal enveloping algebra UL

⁸Consider how this goes in the more familiar situation over \mathbb{F}_p , rather than over \mathbb{Z} . The Lie algebra $L(2, 2)$ is four-dimensional and unimodular, hence $H^4(L(2, 2); \mathbb{F}_p)$ is nontrivial, spanned by $h_{1,0}h_{1,1}h_{2,0}h_{2,1}$. However, if we consider $L(2, 2)$ as a Lie $\mathbb{F}_p[C_2]$ -algebra rather than as a Lie \mathbb{F}_p -algebra, then it is only two-dimensional, hence its Chevalley-Eilenberg complex (over $\mathbb{F}_p[C_2]$!) is trivial above cohomological dimension 2, and certainly does not have nonvanishing H^4 . It is the cohomology over \mathbb{F}_p , not over $\mathbb{F}_p[C_n]$, which is always considered in applications to stable homotopy.

is, as a ring⁹, the divided power \mathbb{Z} -algebra¹⁰ on any \mathbb{Z} -linear basis for L^* . We write $\Gamma_{\mathbb{Z}}(L^*)$ for this divided power \mathbb{Z} -algebra. Inside the divided power algebra, we have the subring generated by L^* , i.e., the free commutative \mathbb{Z} -algebra $\mathbb{Z}[L^*]$ on the \mathbb{Z} -module L^* . Put another way, $\mathbb{Z}[L^*]$ is the polynomial ring with one generator for each element in a fixed \mathbb{Z} -linear basis for L^* . The inclusion of Hopf \mathbb{Z} -algebras $\mathbb{Z}[L^*] \hookrightarrow \Gamma_{\mathbb{Z}}(L^*)$ induces a map

$$(16) \quad \text{Cotor}_{\mathbb{Z}[L^*]}^*(\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Cotor}_{\Gamma_{\mathbb{Z}}(L^*)}^*(\mathbb{Z}, \mathbb{Z})$$

which is generally not an isomorphism, although it becomes an isomorphism after rationalization. It is a nice exercise to calculate that the domain of (16) has a lot of torsion elements which map to zero in the codomain of (16).

Now we consider the special case in which n is a positive integer and L is the Lie \mathbb{Z} -algebra $L_{int}(n, n)$. We have an isomorphism of rings

$$(17) \quad (UL_{int}(n, n))^* \cong \Gamma_{\mathbb{Z}}(t_{i,j} : i \in \{1, \dots, n\}, j \in \mathbb{Z}/n\mathbb{Z}),$$

where $t_{i,j}$ denotes the dual element of $x_{i,j}$. It is a routine calculation (essentially the same as that of Theorem 6.3.2 of [15]) to show that, under the isomorphism (17), the coproduct on the Hopf \mathbb{Z} -algebra $(UL_{int}(n, n))^*$ yields the coproduct on $\Gamma_{\mathbb{Z}}(t_{i,j} : i \in \{1, \dots, n\}, j \in \mathbb{Z}/n\mathbb{Z})$ given by the formula $\Delta(t_{i,j}) = \sum_{k=0}^i t_{k,j} \otimes t_{i-k,j+k}$.

We have explained how, within the divided power algebra $(UL)^*$, there is the natural polynomial algebra $\mathbb{Z}[L^*]$. We now define the Hopf algebra $\mathcal{Z}(n)$ to be precisely that polynomial subalgebra of $(UL_{int}(n, n))^*$.

Definition 4.1. *Let $\mathcal{Z}(n)$ be the C_n -equivariant commutative Hopf \mathbb{Z} -algebra which, as a commutative ring, is the subring of $\Gamma_{\mathbb{Z}}(t_{i,j} : i \in \{1, \dots, n\}, j \in \mathbb{Z}/n\mathbb{Z})$ generated by the elements $t_{i,j}$.*

Consequently, $\mathcal{Z}(n)$ has the following presentation:

- *As a commutative $\mathbb{Z}[C_n]$ -algebra, $\mathcal{Z}(n)$ is free on generators t_1, t_2, \dots, t_n .*
- *The coproduct on $\mathcal{Z}(n)$ is given by $\Delta(t_i) = \sum_{j=0}^i t_j \otimes \sigma^j t_{i-j}$.*
- *The augmentation on $\mathcal{Z}(n)$ is given by $\epsilon(t_i) = 0$ for all i .*
- *The map $\mathcal{Z}(n) \hookrightarrow \Gamma_{\mathbb{Z}}(t_{i,j} : i \in \{1, \dots, n\}, j \in \mathbb{Z}/n\mathbb{Z})$ sends $\sigma^j t_i$ to $t_{i,j}$.*

Definition 4.2. *Given integers $n \geq 1$ and $q \geq 2$, let $d_{n,q,i}$ be the integer defined recursively by the rule*

$$d_{n,q,i} = \begin{cases} 0 & \text{if } i \leq 0 \\ \max\{i, qd_{n,q,i-n}\} & \text{if } i > 0. \end{cases}$$

The q -Ravenel filtration on $\mathcal{Z}(n)$ is the increasing filtration in which, for all j , the element $\sigma^j t_i$ is in degree $d_{n,q,i}$.

⁹The coproduct on $(UL)^*$ depends on the Lie bracket of L , but the product on $(UL)^*$ is totally insensitive to the bracket.

¹⁰This statement is not quite true if taken at surface value, since $(UL)^*$ will really be isomorphic to the product, not the direct sum, of the divided powers of L^* . The simplest way to make the statement honest is as follows. For reasonable finite-dimensional solvable Lie algebras L , one can equip UL with a grading so that it is finite-type. Then, by $(UL)^*$, one ought to mean the graded dual of UL , and then we get the isomorphism $\Gamma_{\mathbb{Z}}(L^*) \cong (UL)^*$, as in [1, chapter III section 11.5, discussion preceding Proposition 8]. This approach works for all Lie algebras considered in this paper, since if $L = L_{int}(n, n)$ or $L(n, n)$, we can grade UL by putting $x_{i,j}$ in degree $2i$, and this grading is finite-type. We are grateful to the anonymous referee for pointing out the need for clarity about this.

If $q > n$, then $d_{n,q,i} = d_{n,q+1,i} = d_{n,q+2,i} = \dots$, i.e., the q -Ravenel filtration on $\mathcal{Z}(n)$ does not in fact depend on q , as long as q is sufficiently large.

For a prime p , write $E_0^p \mathcal{Z}(n)$ for the associated graded Hopf \mathbb{Z} -algebra of the p -Ravenel filtration on $\mathcal{Z}(n)$.

Observation 4.3. For all n and p , the product and coproduct on $\mathcal{Z}(n)$ each strictly preserve the p -Ravenel filtration. Consequently $E_0^p \mathcal{Z}(n) = \mathcal{Z}(n)$.

We have a map of C_n -equivariant Hopf algebras $\mathcal{Z}(n) \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow S(n, p)$ given by sending $\sigma^j t_i$ to $t_i^{p^j}$. This map sends the p -Ravenel filtration on $\mathcal{Z}(n)$ to the Ravenel filtration on $S(n, p)$. Each of these two filtrations yields a May-type spectral sequence. Hence we have a map of spectral sequences

$$(18) \quad \begin{array}{ccc} \text{Cotor}_{E_0^p(\mathcal{Z}(n) \otimes_{\mathbb{Z}} \mathbb{F}_p)}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p) & \Longrightarrow & \text{Cotor}_{\mathcal{Z}(n) \otimes_{\mathbb{Z}} \mathbb{F}_p}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) \\ \downarrow & & \downarrow \\ \text{Cotor}_{E_0 S(n,p)}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p) & \Longrightarrow & \text{Cotor}_{S(n,p)}^{*,*}(\mathbb{F}_p, \mathbb{F}_p). \end{array}$$

5. THE RAVENEL-MAY SPECTRAL SEQUENCE COLLAPSES IMMEDIATELY AT LARGE PRIMES.

Finally, in this section we use the definitions in the preceding sections to actually prove something new.

Lemma 5.1. *Fix a positive integer n . Then there exists an integer N_n such that, if $p > N_n$, then the left-hand vertical map in diagram (18) is surjective.*

Proof. Choose a \mathbb{Q} -linear basis \mathcal{B} for the rational Lie algebra cohomology $H^*(\mathbb{Q} \otimes_{\mathbb{Z}} L_{int}(n, n); \mathbb{Q})$. For each element $b \in \mathcal{B}$, choose a cocycle representative \tilde{b} for b in the cobar complex of the \mathbb{Q} -linear dual $U(\mathbb{Q} \otimes_{\mathbb{Z}} L_{int}(n, n))^*$ of the universal enveloping \mathbb{Q} -algebra of $\mathbb{Q} \otimes_{\mathbb{Z}} L_{int}(n, n)$. By exactness of rationalization, that cobar complex is the rationalization of the cobar complex of the \mathbb{Z} -linear dual $(UL_{int}(n, n))^*$ of the universal enveloping \mathbb{Z} -algebra of $L_{int}(n, n)$.

Since \mathcal{B} is finite, we may clear denominators to pass from the rationalization of the cobar complex of $(UL_{int}(n, n))^*$ to the cobar complex of $(UL_{int}(n, n))^*$ itself. Write D for the least common multiple of the denominators of the chosen elements $\{\tilde{b} : b \in \mathcal{B}\}$ of the cobar complex of $(\mathbb{Q} \otimes_{\mathbb{Z}} UL_{int}(n, n))^* \cong \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Z}}(L_{int}(n, n))^*$. Write $D \cdot \tilde{\mathcal{B}}$ for the subset $\{D \cdot \tilde{b} : b \in \mathcal{B}\}$ of $\Gamma_{\mathbb{Z}}(L_{int}(n, n))^*$.

Finally, for each element $D \cdot \tilde{b}$, there exist only finitely many denominators in the expression of $D \cdot \tilde{b}$ in terms of divided powers of the elements of $L_{int}(n, n)^*$. Hence, again using the finiteness of \mathcal{B} , we may also clear the denominators in all those divided powers. That is, there exists a positive integer E such that

$$\begin{aligned} E \cdot D \cdot \tilde{\mathcal{B}} &:= \{E \cdot D \cdot \tilde{b} : b \in \mathcal{B}\} \\ &\subseteq \mathbb{Z}[L_{int}(n, n)^*] \\ &\subseteq \Gamma_{\mathbb{Z}}(L_{int}(n, n)^*). \end{aligned}$$

For any prime $p > n + 1$, Ravenel proved (see section 2, above) that the map of Hopf \mathbb{F}_p -algebras

$$\begin{aligned} (UL(n, n))^* &\rightarrow E_0 S(n, p) \\ t_{i,j} &\mapsto t_{i,j} \end{aligned}$$

induces an isomorphism

$$(19) \quad \text{Cotor}_{(UL(n,n))^*}^*(\mathbb{F}_p, \mathbb{F}_p) \rightarrow \text{Cotor}_{E_0S(n,p)}^*(\mathbb{F}_p, \mathbb{F}_p).$$

The domain of (19) is the Lie algebra cohomology $H^*(L(n, n); \mathbb{F}_p)$. We now make use of Proposition 3.2: since $L_{int}(n, n)$ is a finite-dimensional Lie \mathbb{Z} -algebra, its cohomology $H^*(L_{int}(n, n); \mathbb{Z})$ is a finitely generated abelian group, hence has ℓ -torsion for only finitely many primes ℓ . Hence, if we choose p sufficiently large, then the Tor_1 -group in Proposition 3.2 vanishes, yielding an isomorphism

$$H^*(L_{int}(n, n); \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\cong} H^*(L(n, n); \mathbb{F}_p).$$

The set of cocycles $E \cdot D \cdot \tilde{\mathcal{B}}$ in the cobar complex of $\mathcal{Z}(n) = \mathbb{Z}[L_{int}(n, n)^*]$ is finite and represents a rational basis for $H^*(L_{int}(n, n); \mathbb{Z})$, hence a basis after inverting finitely many primes. Hence it represents a basis in

$$H^*(L_{int}(n, n); \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong H^*(L(n, n); \mathbb{F}_p) \cong \text{Cotor}_{E_0S(n,p)}^*(\mathbb{F}_p, \mathbb{F}_p)$$

for $p \gg n$, by Proposition 3.2. (We are grateful to the anonymous referee for suggesting this phrasing.) Consequently, for $p \gg n$, the classes in $\text{Cotor}_{\mathcal{Z}(n) \otimes_{\mathbb{Z}} \mathbb{F}_p}^*(\mathbb{F}_p, \mathbb{F}_p)$ represented by the members of $E \cdot D \cdot \tilde{\mathcal{B}}$ surject on to $\text{Cotor}_{E_0S(n,p)}^*(\mathbb{F}_p, \mathbb{F}_p)$. Using Observation 4.3 to identify $\mathcal{Z}(n) \otimes_{\mathbb{Z}} \mathbb{F}_p$ and $E_0^p(\mathcal{Z}(n) \otimes_{\mathbb{Z}} \mathbb{F}_p) \cong (E_0^p \mathcal{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{F}_p$, the left-hand vertical map in (18) is surjective. \square

Theorem 5.2. *Fix a positive integer n . Then there exists some integer N_n such that, for all $p > N_n$, the Ravenel-May spectral sequence*

$$E_1^{*,*,*} \cong \text{Cotor}_{E_0S(n,p)}^*(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \text{Cotor}_{S(n,p)}^*(\mathbb{F}_p, \mathbb{F}_p)$$

has no nonzero differentials.

Proof. Using Lemma 5.1, choose p large enough that the left-hand vertical map in (18) is surjective. The spectral sequence drawn along the top horizontal edge of diagram (18) collapses immediately with no differentials, by Observation 4.3. It is elementary to see that, if a map of spectral sequence is surjective on some page, and the domain spectral sequence has no nonzero differentials on and after that page, then the codomain spectral sequence also has no nonzero differentials on and after that page. Hence the spectral sequence drawn along the bottom horizontal edge of (18)—i.e., the Ravenel-May spectral sequence—collapses with no nonzero differentials at the E_1 -page. \square

Theorem 5.3. *Fix a positive integer n . Then there exists some integer N_n such that, for all $p > N_n$, the mod p continuous cohomology $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p)$ of the height n strict Morava stabilizer group scheme is isomorphic, as a graded ring, to the cohomology $H^*(L(n, n); \mathbb{F}_p)$ of the solvable Lie \mathbb{F}_p -algebra $L(n, n)$.*

Proof. Choose a minimal presentation for the graded \mathbb{Q} -algebra $H^*(\mathbb{Q} \otimes_{\mathbb{Z}} L_{int}(n, n); \mathbb{Q})$. Let \mathcal{G} denote the generating set in that presentation, and let \mathcal{R} denote the set of relations. Choose a set $\tilde{\mathcal{G}}$ of cocycles in the cobar complex for $(U(\mathbb{Q} \otimes_{\mathbb{Z}} L_{int}(n, n)))^*$ representing the elements of \mathcal{G} , and choose a set $\tilde{\mathcal{R}}$ of cochain representatives for \mathcal{R} in the same cobar complex. To be clear, by “cochain representatives for \mathcal{R} ,” we mean that the coboundaries of the elements of $\tilde{\mathcal{R}}$ are polynomials in the cocycles in $\tilde{\mathcal{G}}$, and those polynomials are precisely the relations \mathcal{R} between the cohomology classes \mathcal{G} represented by the members of $\tilde{\mathcal{G}}$.

By clearing denominators using the same method as in the proof of Lemma 5.1, there exists some pair of nonzero integers E, D such that:

- $E \cdot D \cdot \tilde{\mathcal{G}} := \{e \cdot d \cdot g : g \in \tilde{\mathcal{G}}\}$ lifts to a set $\hat{\mathcal{G}}$ of cocycles in the cobar complex of $\mathcal{Z}(n)$ which represent a set of generators for the graded \mathbb{Q} -algebra $\mathbb{Q} \otimes_{\mathbb{Z}} H^*(L_{int}(n, n); \mathbb{Z})$,
- and $E \cdot D \cdot \tilde{\mathcal{R}} := \{e \cdot d \cdot r : r \in \tilde{\mathcal{R}}\}$ lifts to a set $\hat{\mathcal{R}}$ of cochains in the cobar complex of $\mathcal{Z}(n)$ whose coboundaries enforce the same relations between those generators which hold in $\mathbb{Q} \otimes_{\mathbb{Z}} H^*(L_{int}(n, n); \mathbb{Z})$.

Now we will begin to use the zeroth algebraic Steenrod power P^0 . The operation P^0 acts on the Cotor-groups of Hopf algebras over fields of characteristic p . For an introduction to the basic ideas, we refer the reader to [15, appendix 1 section 5]. The main fact we will use is that the algebraic Steenrod operation P^0 is the map on Cotor induced by the endomorphism of the cobar complex induced by the Frobenius endomorphism of a Hopf \mathbb{F}_p -algebra. To put it more concretely: P^0 acts on the Cotor-groups $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ by taking the p th power of each cocycle, using the *internal* product on the cobar complex, in the sense described in appendix A.1. For example, the cohomology class $h_{10} \in \text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ is represented by the cobar complex 1-cocycle t_1 , so $P^0 h_{10}$ is the cohomology class of the 1-cocycle t_1^p , i.e., h_{11} .

We have an \mathbb{F}_p -coalgebra splitting of $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n) \cong \mathbb{F}_p[C_n][t_1, t_2, \dots, t_n]$ as the tensor product of the subcoalgebras

$$\mathbb{F}_p[C_n][t_1^{p^i}, t_2^{p^i}, \dots, t_n^{p^i}] / (t_j^{p^i})^p \cong \mathbb{F}_p[C_n][t_1, t_2, \dots, t_n] / t_j^p$$

for $i = 0, 1, 2, \dots$. This is only a coalgebra splitting, not a Hopf algebra splitting—clearly it fails to respect the ring structure!—and in particular, the action of P^0 sends the element named $t_j^{p^i}$ in the i th factor to the element named $t_j^{p^{i+1}}$ in the $(i+1)$ th factor.

Hence P^0 isomorphically maps each factor in this splitting to the next factor. If we choose p to be sufficiently large, then the sets $\hat{\mathcal{G}}$ and $\hat{\mathcal{R}}$ each live in the cobar complex of the bottommost factor in the tensor splitting. That is, choosing p sufficiently large will ensure that whenever a power of a generator t_j of $\mathcal{Z}(n)$ appears in $\hat{\mathcal{G}}$ or in $\hat{\mathcal{R}}$, the exponent on t_j is less than p .

Similarly, by the finiteness of $L_{int}(n, n)$ and of the chosen presentation for $H^*(\mathbb{Q} \otimes_{\mathbb{Z}} L_{int}(n, n); \mathbb{Q})$, for $p \gg n$ the images of $\hat{\mathcal{G}}$ and of $\hat{\mathcal{R}}$ in the cobar complex of $(U(\mathbb{F}_p \otimes_{\mathbb{Z}} L_{int}(n, n)))^*$ yield a minimal presentation for $H^*(\mathbb{F}_p \otimes_{\mathbb{Z}} L_{int}(n, n); \mathbb{F}_p)$ as a graded \mathbb{F}_p -algebra. Combined with the above observations about P^0 , for large p we get that $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ is isomorphic—at least as a graded vector space—to a tensor product of countably infinitely many copies of $H^*(L(n, n); \mathbb{F}_p)$, which are permuted by the action of P^0 .

We will now argue that the tensor splitting respects the multiplication in Cotor at least well enough for the bottommost tensor factor to be a subalgebra of $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$. Our generating set $\hat{\mathcal{G}}$ is finite, and for $p \gg n$, we explained how its elements represent cohomology classes which live in the bottommost tensor factor. Hence, if we choose p to be sufficiently large, then the relations which hold in $E_0 S(n, p)$ but not in $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)$ —namely, $(\sigma^j t_i)^p = 0$ for all $i = 1, \dots, n$ and all j —cannot impose additional algebraic relations in $\text{Cotor}_{E_0 S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p)$ between the cohomology classes in $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ represented by the members

of $\hat{\mathcal{G}}$. Consequently, if we choose p to be sufficiently large, then the bottommost tensor factor $H^*(L(n, n); \mathbb{F}_p)$ in $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ is in fact a *subalgebra* of $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$.

The Hopf algebra map $\mathcal{Z}(n) \rightarrow E_0 S(n, p)$ induces the map

$$(20) \quad \text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p) \rightarrow \text{Cotor}_{E_0 S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p)$$

which is, for all $p \gg n$, simply the projection map to the bottommost tensor factor in the splitting. (One way to see this is to know, from the general theory of Steenrod operations [9], that the zeroth Steenrod power is always zero on Lie algebra cohomology in positive cohomological degrees. Alternatively and more directly, one simply observes that all the p th powers of the generators are zero in $E_0 S(n, p)$. Either way, the copies of $H^*(L(n, n); \mathbb{F}_p)$ in $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ corresponding to *positive* powers of P^0 have all their positive-degree elements set to zero by the map (20).)

Meanwhile, we also have a Hopf algebra map $\mathcal{Z}(n) \rightarrow S(n, p)$, and this map sends $\sigma^j t_i$ to $t_i^{p^j}$. Hence the induced map in Cotor ,

$$\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p) \rightarrow \text{Cotor}_{S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p),$$

is the map that identifies P^0 with the action of σ . Since $\sigma^n = id$ on $\text{Cotor}_{S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p)$ and since the bottommost copy of $\text{Cotor}_{E_0 S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p) \cong H^*(L(n, n); \mathbb{F}_p)$ generates $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ under addition, multiplication, and the action of P^0 on $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$, the composite map

$$(21) \quad \text{Cotor}_{E_0 S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p) \hookrightarrow \text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p) \rightarrow \text{Cotor}_{S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p)$$

is surjective, where the left-hand map in (21) is the inclusion of the bottommost factor in the tensor splitting. To be clear, the right-hand map in (21) is induced by a Hopf algebra morphism and hence commutes with the action of P^0 , while the left-hand map in (21) does *not* commute with the action of P^0 : instead, we have already shown that it is the inclusion of a subring of $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ which generates $\text{Cotor}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{Z}(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ as a $\mathbb{F}_p[P^0]$ -algebra.

The composite map (21) is a surjective algebra map. The domain and the codomain of the composite map (21) are each finite-dimensional \mathbb{F}_p -vector spaces. By Theorem 5.2, they have the same dimension. Hence the surjective \mathbb{F}_p -algebra composite map (21) must in fact be an isomorphism. \square

The proof of Theorem 5.3 has a certain kind of indirectness: we chose a presentation for a Cotor -algebra, and we used the presentation together with cocycle-level arguments to deduce the existence of an algebra map

$$(22) \quad H^*(L(n, n); \mathbb{F}_p) \rightarrow \text{Cotor}_{S(n, p)}^*(\mathbb{F}_p, \mathbb{F}_p),$$

i.e., the composite map (21). It is natural to dislike this situation, and to ask for a more structured map, that is, for a map of Hopf algebras which directly induces the map (22) in Cotor , without any need to choose a presentation or to lift cocycles from one cobar complex to another. Such a map of Hopf algebras, however, is not possible, because the isomorphism of rings whose existence is asserted by Theorem 5.3 cannot possibly be compatible with algebraic Steenrod operations. In more detail, consider the following argument:

- the algebraic Steenrod operation P^0 is zero in all positive cohomological degrees in the domain of (22),
- but P^0 satisfies $(P^0)^n = id$ on the codomain of (22).
- Hence the map (22) cannot possibly preserve P^0 , hence cannot be induced by a map of Hopf algebras, or even by a map of differential graded \mathbb{F}_p -algebras.

Corollary 5.4. *Let n be a positive integer. Suppose that the Smith-Toda complex $V(n-1)$ exists for all $p \gg n$. Equip $H^*(L(n, n); \mathbb{F}_p)$ with the Adams grading, i.e., the grading in which an element in cohomological degree d and internal degree i is in Adams degree $i-d$. Then, for $p \gg n$, the $K(n)$ -local homotopy groups $\pi_*(L_{K(n)}V(n-1))$ of $V(n-1)$ are isomorphic as a graded abelian group to $H^*(L(n, n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_n^{\pm 1}]$, with v_n in degree $2(p^n-1)$ and with the Adams degree as the grading on $H^*(L(n, n); \mathbb{F}_p)$.*

If it is furthermore true that $V(n-1)$ is a homotopy-associative ring spectrum for $p \gg n$, then the isomorphism $\pi_(L_{K(n)}V(n-1)) \cong H^*(L(n, n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_n^{\pm 1}]$ for $p \gg n$ is an isomorphism of graded rings.*

Proof. Whenever $V(n-1)$ exists, we have a spectral sequence

$$(23) \quad E_2^{s,t} \cong \left(\text{Cotor}_{S(n,p)}^{s,*}(\mathbb{F}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} K(n)_* \right)^t \Rightarrow \pi_{t-s}(L_{K(n)}V(n-1))$$

$$d_r : E_r^{s,t} \rightarrow E_r^{s+r, t+r-1}.$$

Spectral sequence (23) is most simply constructed as the $E(n)$ -Adams spectral sequence of $V(n-1)$, which converges to $\pi_*(L_{E(n)}V(n-1)) \cong \pi_*(L_{K(n)}V(n-1))$. The Morava-Miller-Ravenel change-of-rings isomorphism (see [10] or [12] or section 6.1 of [15]) identifies the E_2 -page $\text{Cotor}_{E(n)_*E(n)}^{*,*}(E(n)_*, E(n)_*V(n-1))$ of the $E(n)$ -Adams spectral sequence for $V(n-1)$ with the continuous cohomology $H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); K(n)_*) \cong H_c^*(\text{str Aut}(\mathbb{G}_{1/n}); \mathbb{F}_p) \otimes_{\mathbb{F}_p} K(n)_*$.

The s -line in the E_2 -term of spectral sequence (23) is concentrated in Adams degrees congruent to $-s$ modulo $2p-2$. If $p > \frac{n^2+2}{2}$, then an elementary bidegree argument shows that any $E(n)$ -Adams differential would have to cross the horizontal vanishing line at $s = n^2$, hence must be zero. The same bidegree argument also rules out multiplicative filtration jumps in the $E(n)$ -Adams spectral sequence when $p > \frac{n^2+2}{2}$. (This argument for collapse of spectral sequence (23) for $p > \frac{n^2+2}{2}$ is folklore, and certainly not new. What is new here is that Theorems 5.2 and 5.3 identify the input for (23) in terms of Lie algebra cohomology.) \square

We report one more corollary of Theorems 5.2 and 5.3:

Corollary 5.5. *Let n be a positive integer, and let p be a prime satisfying $p > N_n$. Suppose that X is an $E(n-1)$ -acyclic finite CW-complex. Then there exist strongly convergent spectral sequences*

$$(24) \quad E_1^{*,*,*} \cong H^*(L(n, n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} E_0E(n)_*(X) \Rightarrow \text{Cotor}_{E(n)_*E(n)}^{*,*}(E(n)_*, E(n)_*(X)),$$

$$(25) \quad E_2^{*,*} \cong \text{Cotor}_{E(n)_*E(n)}^{*,*}(E(n)_*, E(n)_*(X)) \Rightarrow \pi_*(L_{E(n)}X),$$

where $E_0E(n)_*(X)$ is the associated graded abelian group of the p -adic filtration on $E(n)_*(X)$.

Proof. Spectral sequence (25) is, of course, not new: it is simply the $E(n)$ -Adams spectral sequence for X , which converges strongly to $\pi_*(L_{E(n)}X)$ by the existence of a horizontal vanishing line at a finite page in the $E(n)$ -Adams spectral sequence of every finite CW-complex; see [16, Theorem A.6.11]. If $p > n + 1$, then $\text{Cotor}_{E(n)_*E(n)}^*(E(n)_*, -)$ vanishes above cohomological dimension $n^2 + n$, so this vanishing line occurs already at the E_2 -page.

Recall that $\Sigma(n)$ is the Hopf \mathbb{F}_p -algebra

$$\begin{aligned}\Sigma(n) &= K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_* \\ &= E(n)_*E(n)/(p, v_1, \dots, v_{n-1}, \eta_R(v_1), \dots, \eta_R(v_{n-1})) \\ &= E(n)_*E(n)/(p, v_1, \dots, v_{n-1}).\end{aligned}$$

Spectral sequence (24) arises from a four-step process:

- First, filter $E(n)_*(X)$ by powers of the ideal (p, v_1, \dots, v_{n-1}) in $E(n)_*$, so that the associated graded $E(n)_*E(n)$ -comodule is a $\Sigma(n)$ -comodule. This filtration is finite because we have assumed that X is finite and $E(n-1)$ -acyclic.
- Recall (e.g. from the beginning of section 6.2 of [15]) that \mathbb{Z} -graded $\Sigma(n)$ -comodules are equivalent to $\mathbb{Z}/2(p^n - 1)\mathbb{Z}$ -graded $S(n, p)$ -comodules, via the isomorphism $\Sigma(n)/(1 - v_n) \cong S(n, p)$.
- Now observe that the pro- p -group scheme represented by $\text{Spec } S(n, p)$ —i.e., the height n strict Morava stabilizer group scheme—is pro-unipotent. To see this, recall the presentation (5) for $S(n, p)$. For a positive integer m , let $S(n, p)_{\leq m}$ be the Hopf subalgebra $\mathbb{F}_p[t_1, \dots, t_m]/(t_i^{p^n} - t_i \ \forall i)$ of $S(n, p)$. Then $\text{Spec } S(n, p)_{\leq m}$ is the strict automorphism group scheme of the formal $(m+1)$ -bud truncation of the formal group law $\mathbb{G}_{1/n}$. The affine group scheme $\text{Spec } S(n, p)$ is the limit, over m , of the affine group schemes $\text{Spec } S(n, p)_{\leq m}$, so our task is to show that $\text{Spec } S(n, p)_{\leq m}$ is unipotent for each m .

We use the filtration criterion for unipotence: see [18, section 8.3] for an excellent textbook reference. The filtration criterion states that, if k is a field and A is a finitely generated commutative Hopf k -algebra, then $\text{Spec } A$ is unipotent if and only if there is a chain of k -linear subspaces $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ with $A_0 = k$ and with $\Delta(x) \in \sum_{i=0}^j A_i \otimes_k A_{j-i}$ for all $x \in A_j$. This condition is satisfied by $S(n, p)$, hence also¹¹ by $S(n, p)_{\leq m}$, by Ravenel's recursive formula [15, Theorem 4.3.13] for the coproduct in BP_*BP and hence also in $S(n, p)_{\leq m}$.

- Since $\text{Spec } S(n, p)$ is pro-unipotent, the only irreducible continuous representations of $\text{Spec } S(n, p)$ are one-dimensional. Hence the (p, \dots, v_{n-1}) -adic filtration on $E(n)_*(X)/(1 - v_n)$ admits a finite refinement by subcomodules whose filtration quotients are each one-dimensional and trivial. The relationship between $\Sigma(n)$ -comodules and $S(n, p)$ -comodules lets us lift this filtration to a finite filtration of $E(n)_*(X)$ by $E(n)_*E(n)$ -subcomodules

¹¹Both $S(n, p)$ and $S(n, p)_{\leq m}$ satisfy the filtration criterion for unipotence. We introduce $S(n, p)_{\leq m}$ because it is finitely generated, and as far as we know, unipotence is only studied when an affine group scheme is algebraic, i.e., when the representing Hopf algebra of the group scheme is finitely generated. In particular, the author does not know whether the filtration criterion for unipotence guarantees that the only irreducible comodules are one-dimensional, except when the representing Hopf algebra is finitely generated.

whose filtration quotients are each copies of the trivial $\Sigma(n)$ -comodule $K(n)_*$.

Consequently the spectral sequence arising from applying $\text{Cotor}_{E(n)_*E(n)}^{*,*}(E(n)_*, -)$ to this filtration has E_1 -page

$$\begin{aligned}
(26) \quad \text{Cotor}_{E(n)_*E(n)}^{*,*}(E(n)_*, E_0E(n)_*(X)) &\cong \text{Cotor}_{\Sigma(n)}^{*,*}(K(n)_*, E_0E(n)_*(X)) \\
&\cong \text{Cotor}_{\Sigma(n)}^{*,*}(K(n)_*, K(n)_*) \otimes_{K(n)_*} E_0E(n)_*(X) \\
&\cong (H^*(L(n, n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} K(n)_*) \otimes_{K(n)_*} E_0E(n)_*(X) \\
&\cong H^*(L(n, n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} E_0E(n)_*(X),
\end{aligned}$$

where (26) is from the Morava-Miller-Ravenel change-of-rings isomorphism. Here $E_0E(n)_*(X)$ is the associated graded of the finite filtration we have constructed on $E(n)_*(X)$ using pro-unipotence of $\text{Spec } S(n, p)$, but as an abelian group, it agrees with the associated graded of the p -adic filtration on $E(n)_*(X)$. Strong convergence of the spectral sequence is a routine consequence of finiteness of the filtration. \square

Remark 5.6. The proof of Lemma 5.1 is nonconstructive: given a positive integer n , there is no obvious way to write down an integer M such that the left-hand vertical map in (18) is surjective for all primes $p > M$. However, all existing calculations are consistent with the possibility that M can be chosen to be simply 1: that is, the left-hand vertical map in (18) is surjective for all n and all p .

Theorems 5.2 and 5.3 inherit their nonconstructiveness from Lemma 5.1. For a fixed choice of n , let N_n denote the least integer $N \geq n + 1$ satisfying the conditions described in the statement of Theorem 5.2. We would very much like to have some understanding of the sequence of integers N_1, N_2, N_3, \dots . In section 6.1, we see that $N_1 = 2$, and in section 6.2, we see that $N_2 = 3$. The ‘‘optimistic answer’’ described in section 1 is the proposition that $N_n = n + 1$ for all $n \geq 1$. All existing calculations are consistent with the possibility that the RMSS collapses for all $p > n + 1$, i.e., $N_n = n + 1$, although as Ravenel remarks following Theorem 6.3.5 of [15], Ravenel suspects that this possibility is dashed by a hypothetical RMSS differential for $n = 9, p = 11$.

6. EXPLICIT EXAMPLE CALCULATIONS.

The mod p reductions of the integral calculations in this section can be compared with those of section 3 of [14] or section 6.3 of [15], to see that we indeed recover known cases of the mod p cohomology of Morava stabilizer algebras.

6.1. Height 1. The Lie \mathbb{Z} -algebra $\mathcal{Z}(1)$ is the abelian Lie \mathbb{Z} -algebra on a single generator x_1 . Consequently we have $H^*(\mathcal{Z}(1); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(h_{10})$. Upon reduction modulo p for p sufficiently large, we recover the cohomology $\Lambda_{\mathbb{F}_p}(h_{10})$ of the height 1 strict Morava stabilizer group; in fact, $p > 2$ suffices.

6.2. Height 2. The Lie \mathbb{Z} -algebra $\mathcal{Z}(2)$ has \mathbb{Z} -linear basis $x_{10}, x_{11}, x_{20}, x_{21}$, with Lie bracket determined by $[x_{10}, x_{11}] = x_{20} - x_{21}$ and by x_{20}, x_{21} being central in

$\mathcal{Z}(2)$. Running the Cartan-Serre spectral sequence for the extension of Lie \mathbb{Z} -algebras

$$1 \rightarrow \mathbb{Z}\{x_{20}, x_{21}\} \rightarrow \mathcal{Z}(2) \rightarrow \mathbb{Z}\{x_{10}, x_{11}\} \rightarrow 1,$$

one calculates easily that

$$\begin{aligned} H^*(\mathcal{Z}(2); \mathbb{Z}) &\cong \Lambda_{\mathbb{Z}}(\zeta_2) \otimes_{\mathbb{Z}} (\Lambda_{\mathbb{Z}}(h_{10}, h_{11}) \otimes_{\mathbb{Z}} \mathbb{Z}[h_{10}h_{20}, h_{11}h_{20}]) / \\ &\quad (h_{10} \cdot h_{11}, (h_{10}h_{20})^2, (h_{11}h_{20})^2, \\ &\quad (h_{10}h_{20}) \cdot h_{11} = -(h_{11}h_{20}) \cdot h_{10}, h_{10}h_{20} \cdot h_{11}h_{20}), \end{aligned}$$

where ζ_2 is the cohomology class of the Chevalley-Eilenberg cocycle $h_{20} + h_{21}$. The action of C_2 is by

$$\begin{aligned} \sigma(h_{10}) &= h_{11} & \sigma(h_{11}) &= h_{10} \\ \sigma(h_{10}h_{20}) &= h_{11}h_{20} - h_{11}\zeta_2 & \sigma(h_{11}h_{20}) &= h_{10}h_{20} - h_{10}\zeta_2 \\ \sigma(\zeta_2) &= \zeta_2. \end{aligned}$$

Upon reduction modulo p for p sufficiently large, we recover the cohomology of the height 2 strict Morava stabilizer group (compare [14, Theorem 3.2] or [15, Theorem 6.3.22]); in fact, $p > 3$ suffices.

APPENDIX A. SIMULTANEOUS INTEGRAL LIFT OF THE HEIGHT n MORAVA STABILIZER ALGEBRAS AT ALL PRIMES.

In this appendix, for each positive integer n , we define a differential graded \mathbb{Z} -algebra $S_{int}(n)^\bullet$ which, in a sense that we make precise, “specializes” at each prime p to the p -primary height n Morava stabilizer algebra $S(n, p)$. We do not need $S_{int}(n)^\bullet$ for the main theorems in this paper, because the smaller object $\mathcal{Z}(n)$ suffices for our purposes. This is because the main theorems in this paper are asymptotic: they are about the cohomology of the p -primary height n strict Morava stabilizer groups for $p \gg n$. For large p , the Hopf algebra $\mathcal{Z}(n)$ is an integral lift of a large enough piece of the Morava stabilizer algebras $S(n, p)$ to detect all the cohomology of $S(n, p)$. Readers with any interest in small-primary phenomena will perhaps want to know about an integral lift of the height n Morava stabilizer algebras which “gets the cohomology right” even at small primes. That integral lift is $S_{int}(n)^\bullet$.

A.1. The definition of the integral lift $S_{int}(n)^\bullet$. It would be nice to have a single Hopf \mathbb{Z} -algebra $S_{int}(n)$ whose mod p reduction, for each p , is isomorphic to $S(n, p)$. We are skeptical that such a Hopf \mathbb{Z} -algebra can exist. Instead we will construct a differential graded \mathbb{Z} -algebra $S_{int}(n)^\bullet$ whose reduction at each prime p is the cobar complex of $S(n, p)$. We give the construction in several steps, as follows:

- Let $R(n)$ denote the free commutative $\mathbb{Z}[C_n]$ -algebra on a set of generators $\{t_1, t_2, \dots\}$.
- Let $T(n)^\bullet$ denote the differential graded $\mathbb{Z}[C_n]$ -algebra whose degree i summand is the i -fold tensor power $R(n)^{\otimes_{\mathbb{Z}} i}$ of $R(n)$. We refer to $R(n)^{\otimes_{\mathbb{Z}} i}$ as the group of i -cochains in $T(n)^\bullet$.

We define the C_n -action on $T(n)^\bullet$ by letting the operator σ act diagonally on $R(n)^{\otimes_{\mathbb{Z}} i}$. Hence, for example, $\sigma(t_j \otimes t_k) = (\sigma t_j) \otimes (\sigma t_k)$.

The graded algebra $T(n)^\bullet$ has a certain feature in common with the cobar complex of a Hopf algebra (or Hopf algebroid), as defined in appendix

1 of [15]: it has *two* natural-looking multiplication operations, and unless care is taken to distinguish them, there will be confusion and errors. The degree i summand $R(n)^{\otimes_{\mathbb{Z}} i}$ of $T(n)^{\bullet}$ is, in itself, a ring, so we can multiply two i -cochains and get an i -cochain. We will call this multiplication the *internal product* on $T(n)^{\bullet}$. By contrast, we can instead take an element $x \in R(n)^{\otimes_{\mathbb{Z}} i}$ and an element $y \in R(n)^{\otimes_{\mathbb{Z}} j}$, and concatenate them across a tensor symbol to get an element $x \otimes y \in R(n)^{\otimes_{\mathbb{Z}} (i+j)}$. We will call this multiplication the *cup product* on $T(n)^{\bullet}$. For the sake of producing a differential graded algebra, it is the cup product on $T(n)^{\bullet}$ which is the relevant multiplication operation.

- Let $S_{int}(n)^{\bullet}$ be the graded $\mathbb{Z}[C_n]$ -algebra obtained from $T(n)^{\bullet}$ by freely (in the category of associative graded $\mathbb{Z}[C_n]$ -algebras) adjoining generators b_1, b_2, b_3, \dots in degree 2.
- We define a $\mathbb{Z}[C_n]$ -linear differential $d : S_{int}(n)^{\bullet} \rightarrow S_{int}(n)^{\bullet}$ of degree +1 by the following rules:

$$(27) \quad d(t_i) = \sum_{j=1}^{i-1} t_j \otimes \sigma^j t_{i-j} + \sum_{k=1}^{i/n} \sigma^{kn+1} b_{i-kn},$$

$$(28) \quad d(b_i) = -\sigma^{-1-kn} \left(\sum_{j=1}^{i+n-1} d(t_j \otimes \sigma^j t_{i+n-j}) + \sum_{k=1}^{\frac{i}{n}-1} d(b_{i-nk}) \right)$$

with the understanding that b_i is zero if $i \leq 0$.

The formula (28) is not a closed-form formula for $d(b_i)$, but it is enough to let us solve recursively for $d(b_i)$ by first calculating $d(b_1), d(b_2)$, and so on.

Note that (28) is a consequence of (27): since $d \circ d$ must be zero, apply d to both sides of (27) and solve for $d(b_{i-n})$, then re-index to arrive at formula (28). One can deduce formula (27) by setting v_n to 1, and setting v_i to zero for all $i \neq n$, in the formula for the coproduct $\Delta(t_n)$ in BP_*BP [15, Theorem 4.3.13].

Definition A.1. *For each positive integer n , the integral lift of the height n Morava stabilizer algebra is the C_n -equivariant differential graded \mathbb{Z} -algebra $S_{int}(n)^{\bullet}$ defined above.*

The cobar complex of $\mathcal{Z}(n)$ embeds into $S_{int}(n)^{\bullet}$ by the differential graded $\mathbb{Z}[C_n]$ -algebra homomorphism $\mathcal{Z}(n) \rightarrow S_{int}(n)^{\bullet}$ sending t_i to t_i and preserving both the internal product and the cup product.

A.2. How to specialize $S_{int}(n)^{\bullet}$ to a prime. We now need to explain the sense in which $S_{int}(n)^{\bullet}$ is an integral lift of the Morava stabilizer algebras $S(n, 2), S(n, 3), S(n, 5), \dots$. It is not a matter of simply reducing the \mathbb{Z} -algebra $S_{int}(n)^{\bullet}$ modulo a prime p to recover $S(n, p)$. Instead, to recover the cobar complex of $S(n, p)$, one must “specialize” $S_{int}(n)^{\bullet}$ in the following way:

Definition A.2. *Fix a positive integer n and a prime p . By the specialization of $S_{int}(n)^{\bullet}$ at p , we mean the C_n -equivariant differential graded \mathbb{F}_p -algebra obtained by:*

- setting p to zero,
- setting σ to the internal Frobenius endomorphism, i.e., setting $\sigma(x) = x^p$ for all homogeneous x , where the p th power is taken using the internal product, as defined in appendix A.1,

- and, for each positive integer i , setting b_i to -1 times the transpotent of t_i , i.e., setting b_i to the mod p reduction of the integral sum $-\frac{1}{p} \sum_{j=1}^{p-1} \binom{p}{j} t_i^j \otimes t_i^{p-j}$.

We will write $S_{int}(n)^\bullet \downarrow p$ for the specialization of $S_{int}(n)^\bullet$ at p .

Write $C_{S(n,p)}^\bullet$ for the cobar complex of $S(n,p)$. Then $C_{S(n,p)}^\bullet$ is a C_n -equivariant differential graded \mathbb{F}_p -algebra, with C_n acting by the internal Frobenius map $x \mapsto x^p$, i.e., the zeroth algebraic Steenrod operation P^0 . It is straightforward to use the calculations of sections 4.3 and 6.3 of [15] to verify that the map of C_n -equivariant differential graded algebras

$$\begin{aligned} S_{int}(n)^\bullet \downarrow p &\rightarrow C_{S(n,p)}^\bullet \\ t_i &\mapsto t_i \\ b_i &\mapsto b_{i,0} \end{aligned}$$

is an isomorphism.

A.3. The q -Ravenel filtration on $S_{int}(n)^\bullet$. To show that the Ravenel-May spectral sequence also has an integral lift, even at small primes, we define a lift of the Ravenel filtration to the integral lift $S_{int}(n)^\bullet$ of the Morava stabilizer algebras. In fact we will have a family of such lifts, one lift for each integer $q \geq 2$. It is defined in the same way as the q -Ravenel filtration on $\mathcal{Z}(n)$, from Definition 4.2:

Definition A.3. Equip the C_n -equivariant differential graded \mathbb{Z} -algebra $S_{int}(n)^\bullet$ with the increasing filtration in which, for all j , the element $\sigma^j t_i$ is in degree $d_{n,q,i}$ and the element $\sigma^j b_i$ is in degree $d_{n,q,i+n}$. We call this the q -Ravenel filtration on $S_{int}(n)^\bullet$.

The numbers $d_{n,q,1}, d_{n,q,2}, \dots$ were defined in Definition 4.2.

In the case $p = q$, the q -Ravenel filtration on $S_{int}(n)^\bullet$ specializes (in the sense of appendix A.2) to the Ravenel filtration on $S(n,p)$.

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