STRUCTURE AND COHOMOLOGY OF MODULI OF FORMAL MODULES.

A. SALCH

Abstract. Given a commutative ring A, a "formal A-module" is a formal group equipped with an action of A. There exists a classifying ring L^A of formal A-modules. This paper proves structural results about L^A and about the moduli stack $\mathcal{M}_{fmA}$ of formal A-modules. We use these structural results to aid in explicit calculations of flat cohomology groups of $\mathcal{M}_{fmA}^{2-buds}$, the moduli stack of formal A-module 2-buds. For example, we find that a generator of the group $H^1_{fl}(\mathcal{M}_{fm\mathbb{Z}};\omega)$, which also generates (via the Adams-Novikov spectral sequence) the first stable homotopy group of spheres, also yields a generator of the A-module $H^1_{fl}(\mathcal{M}_{fmA}^{2-buds};\omega)$ for any torsion-free Noetherian commutative ring A. We show that the order of the A-modules $H^1_{fl}(\mathcal{M}_{fmA}^{2-buds};\omega)$ and $H^2_{fl}(\mathcal{M}_{fmA}^{2-buds};\omega\otimes\omega)$ are each equal to 2^{N_1}, where N_1 is the leading coefficient in the 2-local zeta-function of $\operatorname{Spec} A$. We also find that the cohomology of $\mathcal{M}_{fmA}^{2-buds}$ is closely connected to the delta-invariant and syzygetic ideals studied in commutative algebra: $H^0_{fl}(\mathcal{M}_{fmA}^{2-buds};\omega\otimes\omega)$ is the delta-invariant of the largest ideal of A which is in the kernel of every ring homomorphism $A \to \mathbb{F}_2$, and consequently $H^0_{fl}(\mathcal{M}_{fmA}^{2-buds};\omega\otimes\omega)$ vanishes if and only if A is a ring in which that ideal is syzygetic.

Contents

1. Introduction and review of some known facts.

1.1. Introduction.

1.1.1. *Formal modules.* This paper is about the classifying ring L^A and classifying Hopf algebroid $(L^A, L^A B)$ of formal A-modules; or, from another point of view, the moduli stack $\mathcal{M}_{fmA}$ of formal A-modules. We ought to explain what this means. When A is a commutative ring, a *formal A-module* is a formal group law F over a commutative A-algebra R, which is additionally equipped with a ring map $\rho : A \to \mathrm{End}(F)$ such that $\rho(a)(X) \equiv aX$ modulo X^2. An excellent introductory reference for formal A-modules is [8]. Higher-dimensional formal modules exist, but *all formal modules in this paper will be implicitly understood to be one-dimensional.*

Formal modules arise in algebraic and arithmetic geometry, for example, in Lubin and Tate's famous theorem [16] on the abelian closure of a p-adic number field, in Drinfeld's generalizations of results of class field theory in [5], and in Drinfeld's p-adic symmetric domains, which are (rigid analytic) deformation spaces of certain formal modules; see [6] and [25]. Formal A-modules also arise in algebraic topology, by using the natural map from the moduli stack of formal A-modules to the moduli stack of formal groups to detect certain classes in the cohomology of the latter, particularly in order to resolve certain differentials in spectral sequences used to compute the Adams-Novikov E_2-term and stable homotopy groups of spheres; for example, see [30].

More to the point for the present paper: it is easy to show (see [5]) that there exists a classifying ring L^A for formal A-modules, i.e., a commutative A-algebra L^A such that $\hom_{A-alg}(L^A, R)$ is in natural bijection with the set of formal A-modules over R. It is not so easy to calculate L^A, however. Here is a summary of known results.

- The pioneer in this area was M. Lazard, who, in the case $A = \mathbb{Z}$, proved in [15] that $L^{\mathbb{Z}} \cong \mathbb{Z}[x_1, x_2, \dots]$, a polynomial algebra on countably infinitely many generators. The ring $L^{\mathbb{Z}}$ is consequently often called the Lazard ring.
- Next, in [5], Drinfeld handled the case in which A is the ring of integers in a local nonarchimedean field (e.g. a p-adic number field). In that case Drinfeld proved that $L^A \cong A[x_1, x_2, \dots]$, again a polynomial algebra.
- In [8], Hazewinkel proved that the same result holds for discrete valuation rings, as well as for global number rings of class number one. That is, for all such rings A, the classifying ring L^A of formal A-modules is a polynomial A-algebra on countably infinitely many generators.
- Hazewinkel also makes the observation, in 21.3.3A of [8], that the same result cannot possibly hold for arbitrary global number rings. Specifically, when A is the ring of integers in $\mathbb{Q}(\sqrt[4]{-18})$, then Hazewinkel shows that the sub-A-module of L^A consisting of elements of grading degree 2 (see Theorem 1.2.1 for where this grading comes from) is not a free A-module, which could

not occur if L^A were polynomial. Hazewinkel does not, however, attempt to compute L^A for such rings A.

- The preprint [28] contains calculations of L^A for various Dedekind domains A, including cases with nontrivial class group. For a Dedekind domain A of characteristic zero, it is shown that L^A is always a symmetric A-algebra on a certain projective A-module, but fails to be a polynomial A-algebra when the relevant projective module is not free.

The moduli stack $\mathcal{M}_{fmA}$ admits a natural presentation by Spec of the Hopf algebroid (L^A, L^AB), and the flat cohomology of $\mathcal{M}_{fmA}$ coincides with the derived functors of the cotensor product (i.e., Cotor) in the category of L^AB-comodules. Consequently some understanding of the ring L^A and the Hopf algebroid (L^A, L^AB) is a great help in calculating the cohomology of $\mathcal{M}_{fmA}$.

When formulated in terms of the Adams-Novikov spectral sequence, Adams's famous calculation of the image of the stable J-homomorphism, from [1], establishes that for a positive integer n, the flat cohomology group $H^1_{fl}(\mathcal{M}_{fm\mathbb{Z}}; \omega^{\otimes n})$ is a finite abelian group of order equal to the denominator of the Riemann zeta-value $\zeta(1-n)$ times a power of 2. Here ω is the line bundle of invariant differentials on $\mathcal{M}_{fm\mathbb{Z}}$. One would like to know if this result generalizes to rings other than $\mathbb{Z}$. In the paper [26], Ravenel remarks that this result does "not appear to generalize to other number fields. For example if the field is not totally real its Dedekind zeta function vanishes at all negative integers."

This paper has two purposes:

(1) to carry out a structural study of the Hopf algebroid (L^A, L^AB) *for a general ring* A, aimed particularly at establishing structural properties of (L^A, L^AB) which would support cohomology calculations, again *for a general ring* A, not just $\mathbb{Z}$, and not just a number ring or a Dedekind domain.
(2) Subsequently, to use those structural properties, and some new computational tools, to make some cohomology calculations and demonstrate that the flat cohomology of the moduli of formal A-modules do in fact carry some zeta-function-theoretic data, *for a general ring* A.

1.1.2. *Summary of structural results.*

Colimits: The functor sending A to L^A commutes with filtered colimits and with coequalizers, but not coproducts. The same is true for the functor sending A to the Hopf algebroid (L^A, L^AB). This is Proposition 2.1.1.

Localization: If A is a commutative ring and S a multiplicatively closed subset of A, then the homomorphism of graded rings $L^A[S^{-1}] \to L^{A[S^{-1}]}$ is an isomorphism. Furthermore, the homomorphism of graded Hopf algebroids

$$(L^A[S^{-1}], L^AB[S^{-1}]) \to (L^{A[S^{-1}]}, L^{A[S^{-1}]}B)$$

is an isomorphism. This is Theorem 2.2.1. This particular result is not new: it appears also in Hazewinkel's book [9], although we think the proof we offer in this paper is a useful addition to the literature.

Localization and cohomology: Let A be a commutative ring and let S be a multiplicatively closed subset of A. Let f denote the stack homomorphism $f : \mathcal{M}_{fmA[S^{-1}]} \to \mathcal{M}_{fmA}$ classifying the underlying formal A-module of the universal formal $A[S^{-1}]$-module. Then, for all quasicoherent $\mathcal{O}_{\mathcal{M}_{fmA}}$-modules $\mathcal{F}$, we have an isomorphism

$$H^s_{fl}(\mathcal{M}_{fmA};\omega^{\otimes t}\otimes\mathcal{F})[S^{-1}]\cong H^s_{fl}(\mathcal{M}_{fmA[S^{-1}]};\omega^{\otimes t}\otimes f^*\mathcal{F})$$

for all integers s,t. This is the stack-theoretic formulation of Corollary 2.2.2. It is also not new, having already appeared in [26] and in [21]. Examples of its application in the course of cohomological calculations occur in Propositions 3.2.2 and 3.5.2.

Finite generation: Section 2.3 contains a variety of finiteness results which establish that, for a wide class of commutative rings A, the graded rings L^A and L^AB are finitely generated A-modules in each degree.

Completion: Theorem 2.3.12 establishes that, for a wide class of commutative rings A, the functor sending A to the Hopf algebroid (L^A, L^AB) commutes with completion at a maximal ideal I. As a consequence, Corollary 2.3.14 establishes that the spectral sequence obtained from the I-adic filtration on the cobar complex of (L^A, L^AB) converges to $H^*_{fl}(\mathcal{M}_{fm\hat{A}_I};\mathcal{F})$, the cohomology of the moduli stack of formal $\hat{A}_I$-modules. The resulting spectral sequence is useful for making explicit calculations: see section 4.1 and section 4.2 for examples.

1.1.3. *Summary of cohomological results.* Extensive partial calculations of the flat cohomology of $\mathcal{M}_{fm\mathbb{Z}}$ have been made in stable homotopy theory, since $H^*_{fl}(\mathcal{M}_{fm\mathbb{Z}};\omega^{\otimes *})$ is the E_2-term of the Adams-Novikov spectral sequence which converges to the stable homotopy groups of spheres. Partial calculations of $H^*_{fl}(\mathcal{M}_{fmA};\omega^{\otimes *})$ for A a (local or global) number ring can be found in [26], [29], and [30], and for one particular number ring A, in [14]. That is the extent of calculations of the cohomology of $\mathcal{M}_{fmA}$ to be found in the literature.

In particular, there are no existing calculations of $H^*_{fl}(\mathcal{M}_{fmA};\omega^{\otimes *})$ for rings A of global dimension greater than one. There is good reason for this state of affairs: there are great difficulties[1] in calculating the ring L^A when A has global dimension > 1. As a consequence, we begin our cohomological investigation of $\mathcal{M}_{fmA}$ with calculations of the flat cohomology of the moduli stack $\mathcal{M}^{2-buds}_{fmA}$ of *formal A-module 2-buds*, i.e., power series in $R[[X,Y]]/(X,Y)^3$ which are required only to satisfy the formal group law axioms modulo $(X,Y)^3$, and which are equipped with an action of A which again is only required to be unital and associative modulo $(X,Y)^3$. Compared to $\mathcal{M}_{fmA}$, the Artin stack $\mathcal{M}^{2-buds}_{fmA}$ is much simpler and easier to work with. Nevertheless it already enjoys remarkable cohomological properties in low degrees. Here are our results:

In degree zero: $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes *})$, i.e., the sections of the tensor powers of the line bundle ω of invariant differentials, is described as follows. Let A be a torsion-free commutative ring. In Theorem 3.4.1 we obtain an isomorphism of A-modules:

$$H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes n})\cong\begin{cases}0 & \text{if } n<0\\ A & \text{if } n=0\\ 0 & \text{if } n=1\\ \ker\tau_n^{I_2} & \text{if } n>1,\end{cases}$$

[1] See the footnote at the start of section 3.1 for an indication of where these difficulties lie.

where I_2 is the ideal of A generated by 2 and by $a^2 - a$ for all $a \in A$, and where $\tau_n^{I_2}$ is the A-module map

$$\begin{aligned} \tau_n^{I_2} : \mathrm{Sym}_A^n(I_2) &\to \mathrm{Sym}_A^{n-1}(I_2) \\ x_1 \cdot \dots \cdot x_n &\mapsto \iota(x_1)x_2 \cdot \dots \cdot x_n \\ &\quad + \iota(x_2)x_1 \cdot x_3 \cdot \dots \cdot x_n \\ &\quad + \dots + \iota(x_n)x_1 \cdot \dots \cdot x_{n-1}. \end{aligned}$$

Here $\mathrm{Sym}_A^n(I_2)$ is the nth symmetric power of the ideal I_2, and ι denotes the inclusion of the ideal I_2 into A, regarded as an A-module morphism $\iota : I_2 \to A$. See the discussion immediately preceding Theorem 3.4.1 for some intuition behind the morphism $\tau_n^{I_2}$. See also section 2.4 for discussion of the ideal I_2, including a universal property which it enjoys: it is the "universal $\mathbb{F}_2$-point-detecting ideal," meaning it is the largest ideal of A which is in the kernel of every ring homomorphism $A \to \mathbb{F}_2$.

Suppose A is Noetherian. Then, in the particular case $n = 2$, the kernel $\ker \tau_2^{I_2}$ coincides with the *delta-invariant* of the ideal I_2, which has enjoyed some attention in commutative algebra: see for example Micali-Roby in [20], and Simis-Vasconcelos in [31]. The delta-invariant $\delta(I)$ is known (see [3]) to agree with the second Andre-Quillen homology group $H_2(A, A/I; A/I)$ of A/I regarded as an A-algebra, with coefficients in A/I.

Finitely generated ideals I whose delta-invariant vanishes are called *syzygetic* in commutative algebra. We have Corollary 3.4.3: if A is a Noetherian integral domain of characteristic zero, then $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega \otimes \omega)$ vanishes if and only if the universal $\mathbb{F}_2$-point-detecting ideal of A is syzygetic.

If A is a Cohen-Macaulay integral domain of characteristic zero, then Theorem 3.4.4 shows that $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes n})$ is trivial for all $n \neq 0$.

The EFM spectral sequence: In Theorem 5.2.1, we construct a spectral sequence which converges to $H^*_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes *})$. Its E_1-term is a tensor product of $H^*_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}}; \omega^{\otimes *})$ and symmetric powers of a certain module of twisted Kähler differentials in the sense of [11]. Because this spectral sequence allows us to pass from the cohomology of the moduli of formal groups (i.e., formal $\mathbb{Z}$-modules) to the cohomology of the moduli of formal groups equipped with a larger ring (namely, A) of formal multiplications, we call it the "extension of of formal multiplications spectral sequence," or "EFM spectral sequence" for short.

Cohomology of certain twists: Assume that A is a torsion-free commutative Noetherian ring. Using the EFM spectral sequence, in Theorem 5.2.2 we calculate the flat cohomology in all degrees with coefficients in the first three tensor powers of ω:

$$H^*_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes n}) \cong 0 \text{ if } n < 0,$$

$$H^s_{fl}(\mathcal{M}^{2-buds}_{fmA};\mathcal{O}) \cong \begin{cases} A & \text{if } s = 0 \\ 0 & \text{if } s \neq 0, \end{cases}$$

$$H^s_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega) \cong \begin{cases} A/I_2 & \text{if } s = 1 \\ 0 & \text{if } s \neq 1, \end{cases}$$

$$H^s_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes 2}) \cong \begin{cases} \delta(I_2) & \text{if } s = 0 \\ A/I_2^2 & \text{if } s = 1 \\ A/I_2 & \text{if } s = 2 \\ 0 & \text{otherwise,} \end{cases}$$

Consequently we get Corollary 5.2.3: $H^1_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega)$ is a finite abelian group of order equal to 2^{N_1}, where N_1 is the number of $\mathbb{F}_2$-points of $\operatorname{Spec} A$, i.e., the logarithmic derivative of the 2-local zeta-function $Z(\operatorname{Spec} A, t)$ of the affine scheme $\operatorname{Spec} A$, evaluated at $t = 0$. See Remark 2.4.3 for a very brief discussion of the local zeta-function of an affine variety.

The local zeta-function is, when evaluated at $t = p^{-s}$, an Euler factor in the Hasse-Weil zeta-function of a variety. The point here is that, even when restricting the scope of our calculations to the Artin stack of formal A-module 2-buds, we still recover some zeta-function-theoretic information about A, not only for number rings A (as Ravenel remarked about in [26]), but for characteristic zero integral domains quite generally.

Some generalization to the number of k-points of $\operatorname{Spec} A$ for larger finite fields k is possible, but requires calculations of the flat cohomology of $\mathcal{M}^{n-buds}_{fmA}$ for $n > 2$, or the flat cohomology of $\mathcal{M}_{fmA}$. We have some preliminary results in this direction, but we regard them as beyond the scope of this already-too-long paper.

Comparison to stable homotopy: The EFM spectral sequence calculates $H^*_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes *})$ in such a way that, along the way, it also calculates the homomorphism

$$H^*_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}};\omega^{\otimes *}) \to H^*_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes *}) \tag{1.1}$$

induced by the map of Artin stacks $\mathcal{M}^{2-buds}_{fmA} \to \mathcal{M}^{2-buds}_{fm\mathbb{Z}}$ classifying the underlying formal group law 2-bud of the universal formal A-module 2-bud. There is also a stack map $\mathcal{M}_{fm\mathbb{Z}} \to \mathcal{M}^{2-buds}_{fm\mathbb{Z}}$ classifying the underlying 2-bud of the universal formal group law. The flat cohomology of $\mathcal{M}_{fm\mathbb{Z}}$ is the input for the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres. In section 5.2 we calculate that the element η in $H^1_{fl}(\mathcal{M}_{fm\mathbb{Z}};\omega)$ which yields the generator of the first stable homotopy group $\mathbb{Z}/2\mathbb{Z} \in \pi_1^{st}(S^0)$ is the image, under the map

$$H^1_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}};\omega) \to H^1_{fl}(\mathcal{M}_{fm\mathbb{Z}};\omega),$$

of a unique element $H^1_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}};\omega)$. The image of this element in $H^1_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega)$ is an A-module generator. The A-module $H^1_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{A}};\omega)$ has order equal to 2^{N_1}, as already explained. In this sense, the same element in the flat cohomology of $\mathcal{M}_{fm\mathbb{Z}}$ which is responsible for the first stable homotopy group of spheres is also responsible for the first coefficient in the 2-local zeta function for any torsion-free commutative Noetherian ring.

The author of this paper is an algebraic topologist, but is optimistic that this paper may be interesting to readers from various mathematical backgrounds. I have made an effort to write this paper so that it will be readable by researchers in other subjects. However, in computational algebraic topology, there is a tradition of engaging in detailed and meticulous spectral sequence calculations, and to some extent that tradition is reflected in the second half of this paper. I apologize to readers who do not care for spectral sequence calculations, and I am grateful for their patience with section 4 and section 5, and with the length of this paper.

Conventions 1.1.1.

- We often deal with graded rings in this paper, but we will always set up the gradings on the classifying rings L^A and on L^AB so that all elements are in even degrees. Consequently the Koszul graded-commutativity sign relation does not occur in our discussions of the structure of L^A and L^AB. We often use the phrase "commutative graded ring" rather than "graded-commutative" to emphasize that the graded ring in question is assumed to be *strictly* commutative, not merely commutative up to sign.
- We call a ring *torsion-free* when its underlying abelian group is torsion-free.

I am grateful to an anonymous referee for helpful comments on this paper.

1.2. **Review of standard facts about L^A and L^AB.** Nothing in this subsection is new, but we think it may be helpful to the reader to have many of the basic ideas and known results on formal modules and their classifying Hopf algebroids collected in one place.

1.2.1. *Formal modules and formal module n-buds.* If A is a commutative ring and R is a commutative A-algebra, then a (one-dimensional) *formal A-module*[2] *over R* is a formal group law F over R, together with a ring homomorphism $\rho : A \to \operatorname{End}(F)$ such that $\rho(a) \in \operatorname{End}(F) \subseteq R[[X]]$ is congruent to aX modulo X^2. Here $\operatorname{End}(F)$ is a ring in which:

- addition is given by formal addition, i.e., the sum of $f(X)$ and $g(X)$ is $F(f(X), g(X))$, *not* the ordinary componentwise addition of power series,
- and multiplication is given by composition of power series, *not* the usual multiplication of power series in $R[[X]]$.

If n is a positive integer, a *formal A-module n-bud over R* is a formal group law n-bud over R, i.e., an element $F(X,Y) \in R[[X,Y]]/(X,Y)^{n+1}$ which satisfies the unitality, associativity, commutativity, and existence of inverses axioms modulo $(X,Y)^{n+1}$, together with a ring homomorphism $\rho : A \to \operatorname{End}(F)$ such that the endomorphism $\rho(a) \in \operatorname{End}(F) \subseteq R[[X]]/(X^{n+1})$ is congruent to aX modulo X^2.

[2]Morally, a formal A-module F is a "formal group law with complex multiplication by A." This perspective was taken already by Lubin and Tate in [16], and suggests the close connection between formal modules and abelian varieties with particular endomorphism rings. The turn of phrase "formal group law with complex multiplication" is not as clear as might be hoped, however: if A is the ring of integers in a totally real extension of $\mathbb{Q}$, then an abelian variety with a suitable action by A would be called an abelian variety with *real* multiplication, rather than *complex* multiplication. Consequently we prefer to write "formal module" rather than "formal group law with complex muliplication." Rather than complex multiplication or real multiplication, we will refer to the action of A on F as *formal multiplication*.

In this paper we will *always* write $\mathrm{End}(F)$ for the endomorphism ring of a formal group law F or formal group law n-bud F. That is, even if F has the additional structure of a formal module, by $\mathrm{End}(F)$ we will mean the endomorphism ring of F as a formal group law or formal group law n-bud, without regard to any additional structure.

1.2.2. *Hopf algebroids and stacks.* This paper is largely about certain graded Hopf algebroids, i.e., cogroupoid objects in commutative graded rings. We give only a cursory review of the basic theory here. For more detail, we refer readers to the standard reference for Hopf algebroids and their cohomology, Appendix 1 of [27]. Whenever convenient, we will use the common notations for structure maps of bialgebroids and Hopf algebroids: η_L for left unit, η_R for right unit, Δ for coproduct, and ϵ for augmentation.

Starting in section 3.4, we will make calculations of cohomology groups of Hopf algebroids. The cohomology groups of a graded Hopf algebroid (A, Γ) with coefficients in a graded Γ-comodule M can be defined in two (isomorphic) ways:

- as the right derived functors $\mathrm{Cotor}_{\Gamma}^{s,t}(A, M) = \mathrm{Cotor}_{\Gamma}^{s}(A, \Sigma^t M)$ of the cotensor product $A \,\square_{\Gamma} - : \mathrm{gr}\,\mathrm{Comod}(\Gamma) \to \mathrm{Ab}$ applied to M,
- or as the relative right derived functors $\mathrm{Ext}_{(A,\Gamma)}^{s,t}(A, M) = \mathrm{Ext}_{(A,\Gamma)}^{s}(A, \Sigma^t M)$ of $\hom_{\mathrm{gr}\,\mathrm{Comod}(\Gamma)}(A, -) : \mathrm{gr}\,\mathrm{Comod}(\Gamma) \to \mathrm{Ab}$ applied to M, relative to the allowable class generated by the comodules tensored up from A. This is a relative Ext-group, in the sense of relative homological algebra, as in Chapter IX of [17],
- A third description of the cohomology of (A, Γ) with coefficients in M is available whenever the unit maps η_L, η_R of the Hopf algebroid are smooth (respectively, formally smooth). In that case, the stackification $\mathcal{X}$ of the groupoid scheme $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ is an Artin (respectively, formally Artin) stack in the fpqc topology. Its category of quasicoherent modules is equivalent to the category of Γ-comodules. Under this equivalence, the Hopf algebroid cohomology group $\mathrm{Cotor}_{\Gamma}^{s}(A, M)$ is isomorphic to the flat stack cohomology group $H^s_{fl}(\mathcal{X}; \tilde{M})$, where $\tilde{M}$ is the quasicoherent module associated to the comodule M. These facts are standard; a nice reference is [23].

We also refer to "the moduli stack of formal A-modules" several times in this paper. This is slightly ambiguous for the following reason: formal A-modules have only a moduli prestack and not a moduli stack. This moduli prestack "stackifies" (as in [13]) to a stack which is a moduli stack for "coordinate-free" formal A-modules[3], a situation which parallels that of formal group laws and formal groups, as in [32].

1.2.3. *The Hopf algebroid* $(L^A, L^A B)$. Theorem 1.2.1 is the main foundational result about the Hopf algebroid $(L^A, L^A B)$. It gathers together many results proven in chapter 21 of [8], although parts of the theorem are older than Hazewinkel's book; for example, the computation of the ring L^A, when A is a field or the ring of integers in a nonarchimedean local field, is due to Drinfeld in [5].

Theorem 1.2.1. *Let A be a commutative ring.*

- *Then there exist commutative A-algebras L^A and $L^A B$ having the following properties:*
  - *For any commutative A-algebra R, there exists a bijection, natural in R, between the set of A-algebra homomorphisms $L^A \to R$ and the set of formal A-modules over R.*
  - *For any commutative A-algebra R, there exists a bijection, natural in R, between the set of A-algebra homomorphisms $L^A B \to R$ and the set of strict[4] isomorphisms of formal A-modules over R.*
- *The natural maps of sets between the set of formal A-modules over R and the set of strict isomorphisms of formal A-modules over R (sending a strict isomorphism to its domain or codomain; or sending a formal module to its identity strict isomorphism; or composing two strict isomorphisms; or sending a strict isomorphism to its inverse) are co-represented by maps of A-algebras between L^A and $L^A B$. Consequently $(L^A, L^A B)$ is a Hopf algebroid co-representing the functor sending a commutative A-algebra R to its groupoid of formal A-modules and their strict isomorphisms.*
- *If n is a positive integer, then the functor from commutative A-algebras to groupoids which sends a commutative A-algebra R to the groupoid of formal A-module n-buds over R and strict isomorphisms is also co-representable*

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[3]Here is a bit of detail about what a "coordinate-free" formal module is. These details are routine, not important for the rest of this paper, and can be safely skipped by the reader.

It is classical that a formal group *law* over R is a power series $F(X,Y) \in R[[X,Y]]$ satisfying associativity, commutativity, unitality, and inverse axioms, i.e., $F(X,Y)$ defines the structure of a commutative group object on the affine formal scheme $\mathrm{Spf}\, R[[X]]$, with identity element 0. A *formal group* is a commutative group structure on the formal affine line $\hat{\mathbb{A}}^1_R$ with identity element 0. Consequently a formal group law is a formal group together with a choice of isomorphism $\hat{\mathbb{A}}^1_R \cong \mathrm{Spf}\, R[[X]]$. Formal groups form a stack $\mathcal{M}_{fg}$, while formal group *laws* only form a prestack whose stackification is equivalent to $\mathcal{M}_{fg}$.

A similar story applies here. It is usual to say "formal A-module" to mean the power series $F(X,Y) \in R[[X,Y]]$ equipped with the action of A by further power series over R, as defined in section 1.2.1. Perhaps it would better if such objects were instead called "formal A-module laws," since such an object determines the structure of an A-module object on the affine formal scheme $\mathrm{Spf}\, R[[X]]$, with identity element 0. Then we could use the term "formal A-module" to mean the structure of an A-module object on $\hat{\mathbb{A}}^1_R$ with identity element 0, and the terminology would mirror the standard distinction between "formal group law" and "formal group." Using the terms in this way, formal A-module laws form a prestack, represented by the groupoid scheme $(\mathrm{Spec}\, L^A, \mathrm{Spec}\, L^A B)$. The stackification of that prestack is then equivalent to the stack of formal A-modules $\mathcal{M}_{fmA}$. Unfortunately the weight of tradition is against this distinction in terminology between "formal modules" and "formal module laws."

[4]Recall that an isomorphism $f(X)$ of formal groups, or of formal A-modules, is said to be *strict* if $f(X) \equiv X \mod X^2$.

by a Hopf algebroid $(L^A_{n-buds}, L^A_{n-buds}B)$*. Since the groupoid of formal A-modules over R is the inverse limit over n of the groupoid of formal A-module n-buds over R, we have that*

$$(L^A, L^A B) \cong \left(\mathrm{colim}_{n\to\infty} L^A_{n-buds}, \mathrm{colim}_{n\to\infty} L^A_{n-buds}B\right).$$

For example, $L^A_{\leq 1} \cong A$ *as commutative A-algebras.*

The filtration of L^A *and* $L^A B$ *by* L^A_{n-buds} *and* $L^A_{n-buds}B$ *induces a grading on* L^A *and on* $L^A B$*, in which the* indecomposable *homogeneous grading degree* $2n$ *elements in* L^A *are the parameters for deforming (i.e., extending) a formal A-module n-bud to a formal A-module* $(n+1)$*-bud. The summands of* L^A *and of* $L^A B$ *of odd grading degree are trivial.*

- *If A is a field of characteristic zero or a discrete valuation ring or a global number ring of class number one, then we have isomorphisms of graded A-algebras*

$$\begin{aligned} L^A_{n-buds} &\cong A[x_1^A, x_2^A, x_3^A, \dots, x_n^A], \\ L^A_{n-buds}B &\cong L^A_{n-buds}[t_1^A, t_2^A, t_3^A, \dots, t_n^A], \quad \textit{and consequently} \\ L^A &\cong A[x_1^A, x_2^A, x_3^A, \dots], \\ L^A B &\cong L^A[t_1^A, t_2^A, t_3^A, \dots], \end{aligned}$$

with each x_i^A *and each* t_i^A *homogeneous of grading degree* $2i$*. (However, the natural map* $L^A \to L^B$ *induced by a ring homomorphism* $A \to B$ *does not necessarily send each* x_i^A *to* x_i^B*!)*

The factor of 2 in the gradings in Theorem 1.2.1 is due to the graded-commutativity sign convention in algebraic topology and the fact that $L^{\mathbb{Z}}$, with the above grading, is isomorphic to the graded ring of homotopy groups $\pi_*(MU)$ of the complex bordism spectrum MU, while $L^{\mathbb{Z}}B$ with the above grading is isomorphic to the graded ring $\pi_*(MU \wedge MU)$ of stable co-operations in complex bordism. In fact $(L^{\mathbb{Z}}, L^{\mathbb{Z}}B) \cong (\pi_*(MU), \pi_*(MU \wedge MU))$ as graded Hopf algebroids. See [24] for these ideas. In the base case $A = \mathbb{Z}$, one often writes L and LB rather than $L^{\mathbb{Z}}$ and $L^{\mathbb{Z}}B$.

Proposition 1.2.2 appears as Proposition 1.1 in [5].

Proposition 1.2.2. *Let A be a commutative ring, let n be an integer, and let* D^A *denote the homogeneous ideal in* L^A *generated by all products of elements xy with* $x, y \in L^A$ *each homogeneous of positive degree. Let* $\overline{L}^A$ *denote the quotient ring* L^A/D^A*. The ring* $\overline{L}^A$ *is graded, so we may consider its degree n summand* $\overline{L}^A_n$ *for various integers n. The ring* L^A *is concentrated in even degrees, so* $\overline{L}^A$ *is as well. If* $n \geq 2$*, then* $\overline{L}^A_{2n-2}$ *is isomorphic to the A-module generated by symbols* γ *and* $\{c_a : a \in A\}$*, that is, one generator* c_a *for each element a of A along with one additional generator* γ*, modulo the relations:*

$$(a^n - a)\gamma = \nu(n) c_a \quad \textit{for all } a \in A \tag{1.2}$$

$$c_{a+b} - c_a - c_b = \gamma C_n(a,b) \quad \textit{for all } a, b \in A \tag{1.3}$$

$$a c_b + b^n c_a = c_{ab} \quad \textit{for all } a, b \in A, \tag{1.4}$$

where:

- *$\nu(n)$ is defined to be the integer 1 if n is not a prime power, while $\nu(n) = p$ if n is a power of a prime number p,*
- *and where $C_n(x,y)$ is the polynomial $\frac{(x+y)^n - x^n - y^n}{\nu(n)} \in \mathbb{Z}[x,y]$.*

We will call this Drinfeld's presentation for $\overline{L}^A_{2n-2}$.

The grading degrees in Proposition 1.2.2 are twice what they are in Drinfeld's statement of the result in [5], for the sake of compatibility with the grading conventions in algebraic topology.

1.3. **Change of A.** Proposition 1.3.1 is a standard tool in Hopf algebroids (see A1.3.12 of [27]), and we omit the proof.

Proposition 1.3.1. *Let (R,Γ) be a commutative bialgebroid over a commutative ring A, and let S be a right Γ-comodule algebra, such that the following diagram commutes:*

$$\begin{tikzcd} R \arrow[r, "\eta_R"] \arrow[d, "f"] & \Gamma \arrow[d, "f\otimes_R \mathrm{id}_\Gamma"] \\ S \arrow[r, "\psi"] & S\otimes_R \Gamma \end{tikzcd} \tag{1.5}$$

where f is the R-algebra structure map $R \xrightarrow{f} S$. Then we have a bialgebroid $(S, S\otimes_R \Gamma)$, with right unit $S \to S\otimes_R \Gamma$ equal to the comodule structure map ψ on S. The map

$$(R,\Gamma) \to (S, S\otimes_R \Gamma), \tag{1.6}$$

with components f and ψ, is a morphism of bialgebroids.

If (R,Γ) is a Hopf algebroid (respectively, graded Hopf algebroid), then so is $(S, S\otimes_R \Gamma)$, and (1.6) is a map of Hopf algebroids (respectively, graded Hopf algebroids).

If, furthermore, the following conditions are also satisfied:

- *(R,Γ) is a graded Hopf algebroid which is connected (i.e., the grading degree zero summand Γ^0 of Γ is exactly the image of $\eta_L : R \to \Gamma$, equivalently $\eta_R : R \to \Gamma$), and*
- *S is a graded R-module concentrated in degree zero, and*
- *N is a graded left $S\otimes_R \Gamma$-comodule which is flat as an S-module, and*
- *M is a graded right Γ-comodule,*

then we have an isomorphism

$$\mathrm{Ext}^{s,t}_{(R,\Gamma)}(M,N) \cong \mathrm{Ext}^{s,t}_{(S,S\otimes_R\Gamma)}(S\otimes_R M, N)$$

for all nonnegative integers s and all integers t.

Proposition 1.3.2 appeared originally in the unpublished doctoral thesis [21] of A. Pearlman:

Proposition 1.3.2. *Let $f : A \to A'$ be a homomorphism of commutative rings. Then $L^{A'}$ admits a canonical L^AB-comodule structure satisfying the conditions of Proposition 1.3.1. Consequently we have an isomorphism of graded Hopf algebroids*

$$(L^{A'}, L^{A'}B) \cong (L^{A'}, L^{A'} \otimes_{L^A} L^AB),$$

and an isomorphism in cohomology

$$\mathrm{Cotor}^{s,t}_{L^A B}(M,N) \cong \mathrm{Cotor}^{s,t}_{L^{A'} B}(M \otimes_{L^A} L^{A'}, N)$$

for all nonnegative integers s, all integers t, any graded right $L^A B$-comodule M, and any graded $L^{A'} B$-comodule N which is flat as a $L^{A'}$-module.

2. Generalities on L^A and $L^A B$.

2.1. Colimits.

Proposition 2.1.1. *Let $\mathcal{L}, \mathcal{LB}$ be the functors*

$$\mathcal{L} : \mathrm{Comm\,Rings} \to \mathrm{Comm\,Rings}$$
$$\mathcal{L}(A) = L^A$$
$$\mathcal{LB} : \mathrm{Comm\,Rings} \to \mathrm{Comm\,Rings}$$
$$\mathcal{LB}(A) = L^A B.$$

Then $\mathcal{L}$ and $\mathcal{LB}$ each commute with filtered colimits, and $\mathcal{L}$ and $\mathcal{LB}$ each commute with coequalizers.

Proof. Let $\mathcal{D}$ be a small category. Suppose that either $\mathcal{D}$ is filtered or $\mathcal{D}$ is the category indexing a parallel pair, i.e., the Kronecker quiver

$$\bullet \rightrightarrows \bullet.$$

Let $G : \mathcal{D} \to \mathrm{Comm\,Rings}$ be a functor, let R be a commutative ring, and suppose we are given a cone $\mathcal{L} \circ G \to R$. Then R has the natural structure of a commutative $\mathrm{colim}\, G$-algebra, since the grading degree zero subring of each $\mathcal{L}(G(d))$ is isomorphic to the ring $G(d)$ itself. Since $\mathbb{Z}$ is initial in commutative rings, there is a unique cocone $\mathbb{Z} \to G$ and hence a canonical cocone $L^{\mathbb{Z}} \to \mathcal{L} \circ G$. Hence the cone $\mathcal{L} \circ G \to R$ describes a choice of formal group law F over the commutative $\mathrm{colim}\, G$-algebra R, together with a choice of ring map $\rho_d : G(d) \to \mathrm{End}(F)$ for each $d \in \mathrm{ob}\, \mathcal{D}$, compatible with the morphisms in $\mathcal{D}$, and such that $\rho_d(r)(X) \equiv rX$ modulo $(X^2) \subseteq (\mathrm{colim}\, G)[[X]]$ for all $r \in G(d)$.

The colimit $\mathrm{colim}\, G$ here is computed in commutative rings. However, the ring $\mathrm{End}(F)$ is typically not commutative, so the universal property of $\mathrm{colim}\, G$ does not automatically yield a ring map $\mathrm{colim}\, G \to \mathrm{End}(F)$. We need one extra step before we get such a ring map: we observe that the image $\mathrm{im}\, \rho_d$ of each ρ_d is a commutative subring of $\mathrm{End}(F)$, so the union of the family of subrings $\cup_{d \in \mathrm{ob}\, \mathcal{D}} \mathrm{im}\, \rho_d$ is a commutative subring of $\mathrm{End}(F)$ since $\mathcal{D}$ is either filtered or is the category indexing parallel pairs[5]. Hence we have a cone $G \to \cup_{d \in \mathrm{ob}\, \mathcal{D}} \mathrm{im}\, \rho_d$ in the category of commutative rings, hence a canonical map $\rho : \mathrm{colim}\, G \to \cup_{d \in \mathrm{ob}\, \mathcal{D}} \mathrm{im}\, \rho_d$ such that $\rho(r)(X) \equiv rX$ modulo X^2 for all $r \in \mathrm{colim}\, G$, hence F is a formal $\mathrm{colim}\, G$-module over R. Clearly if we began instead with a formal $\mathrm{colim}\, G$-module over R, by neglect of structure we get a cone $\mathcal{L} \circ G \to R$, and the two operations (sending such a cone to its $\mathrm{colim}\, G$-module, and sending the $\mathrm{colim}\, G$-module to its cone) are mutually inverse. So $\mathrm{colim}(\mathcal{L} \circ G) \cong \mathcal{L}(\mathrm{colim}\, G)$.

[5] This is the only part of the argument that uses the assumption that $\mathcal{D}$ is either filtered or the Kronecker quiver. The argument fails if $\mathcal{D}$ is only assumed to be an arbitrary small category. As far as I know there is no reason to believe that the conclusion of Proposition 2.1.1 holds for coproducts, precisely because of the distinction between coproducts in commutative rings and coproducts in associative rings.

For $\mathcal{LB}$: we have already seen in Proposition 1.3.2 that $\mathcal{LB}$ is naturally equivalent to the functor $\mathcal{L} \otimes_{L^{\mathbb{Z}}} L^{\mathbb{Z}}B$. Since base change commutes with arbitrary colimits of commutative rings, the fact that $\mathcal{L}$ commutes with filtered colimits and coequalizers implies the same for $\mathcal{LB}$. □

Remark 2.1.2. Proposition 2.1.1 provides, at least in principle, a means of computing L^A and $L^A B$ for all commutative rings A: first, represent A as the coequalizer of a pair of maps

$$\mathbb{Z}[G] \leftleftarrows \mathbb{Z}[R] \tag{2.7}$$

where G is a set of generators and R a set of relations, and $\mathbb{Z}[G], \mathbb{Z}[R]$ are the free commutative algebras generated by the sets G and R, respectively. Then L^A is just the coequalizer, in commutative rings, of the two resulting maps $L^{\mathbb{Z}[R]} \to L^{\mathbb{Z}[G]}$.

Consequently, if one can compute L^A for polynomial rings A, then one can (at least in principle) compute L^A for all commutative rings A. Unfortunately, the computation of L^A for polynomial rings A is quite difficult, and since the functor $\mathcal{L}$ does not commute with coproducts, it is not as simple as computing $L^{\mathbb{Z}[x]}$ and then taking an n-fold tensor power to get $L^{\mathbb{Z}[x_1,\dots,x_n]}$.

2.2. **Localization.** In the proof of Theorem 21.3.5 of Hazewinkel's excellent book [8], also appearing in the second edition [9], one finds the following statement:

> "By the very definition of L_A (as the solution of a certain universal problem) we have that $(L_A)_{\mathfrak{p}} = L_{A_{\mathfrak{p}}}$ for all prime ideals $\mathfrak{p}$ of A."

It is true that $(L_A)_{\mathfrak{p}}$ is isomorphic to $L_{A_{\mathfrak{p}}}$. However, we prefer to give a few more lines of proof, since the universal properties of these rings do not obviously imply that every formal A-module over a commutative $A_{\mathfrak{p}}$-algebra extends to a formal $A_{\mathfrak{p}}$-module, as the endomorphism ring $\mathrm{End}(F)$ of a formal group law defined over a ring R is typically not an R-algebra. This is clear from the famous example of the endomorphism ring of a height n formal group law over $\mathbb{F}_{p^n}$ being the maximal order in the invariant $1/n$ central division algebra over $\mathbb{Q}_p$, which is certainly not an $\mathbb{F}_{p^n}$-algebra. The claimed isomorphism also does not follow from Proposition 2.1.1, the fact that $A \mapsto L^A$ commutes with coequalizers and filtered colimits, since although localizations of *modules* can be defined as colimits in that category of modules, a localization of a *commutative ring* is not usually expressible as a colimit in the category of commutative rings. The morphisms in the diagram whose colimit computes the localization of the underlying module typically fail to be ring homomorphisms.

However, suppose that r is a unit in a commutative A-algebra R, and suppose that F is a formal A-module over R. Then the power series $\rho(r)(X) \in R[[X]]$ admits a (unique) composition inverse, by the tangent condition $\rho(r)(X) \equiv rX \mod X^2$ on the formal group law. Consequently, if S is a set of non-zero-divisors in A, and if we write G for the universal formal A-module base-changed to $L^A \otimes_A A[S^{-1}]$, then the formal A-multiplication map $\rho : A \to \mathrm{End}(G)$ admits a unique extension to a ring homomorphism $A[S^{-1}] \to \mathrm{End}(G)$, which furthermore satisfies the tangent condition. Hence G is in fact a formal $A[S^{-1}]$-module. Consequently every formal A-module over a commutative $A[S^{-1}]$-algebra admits a unique compatible formal $A[S^{-1}]$-multiplication. From here it is routine to see how Hazewinkel's argument establishes the isomorphism $(L^A)[S^{-1}] \cong L^{A[S^{-1}]}$, and consequently the theorem:

Theorem 2.2.1. *Let A be a commutative ring and let S be a multiplicatively closed subset of A. Then the homomorphism of graded rings $L^A[S^{-1}] \to L^{A[S^{-1}]}$ is an isomorphism. Even better, the homomorphism of graded Hopf algebroids*

$$(L^A[S^{-1}], L^A B[S^{-1}]) \to (L^{A[S^{-1}]}, L^{A[S^{-1}]}B)$$

is an isomorphism of Hopf algebroids.

Corollary 2.2.2. *Let A be a commutative ring and let S be a multiplicatively closed subset of A. Then, for all graded left $L^A[S^{-1}]$-comodules M, we have an isomorphism*

$$\left(\mathrm{Cotor}^{s,t}_{L^A B}(L^A, M)\right)[S^{-1}] \cong \mathrm{Cotor}^{s,t}_{L^{A[S^{-1}]}B}(L^{A[S^{-1}]}, M)$$

for all nonnegative integers s and all integers t.

2.3. **Finiteness, separation, and completion properties.**

Lemma 2.3.1. *Let A be a commutative ring, and let R be a commutative graded A-algebra which is connective, i.e., the degree n grading summand R_n is trivial for all $n < 0$. Suppose that, for all integers n, the A-module R_n/D_n is finitely generated, where D_n is the sub-A-module of R_n generated by all elements of the form xy where x, y are homogeneous elements of R of grading degree $< n$.*

Then, for all integers n, R_n is a finitely generated A-module.

Proof. Routine. □

Proposition 2.3.2. *Let A be a commutative ring, and suppose that A is finitely generated as a commutative ring. Then, for each integer m, the degree m summand L^A_m of the classifying ring L^A of formal A-modules is a finitely generated A-module.*

Proof. Suppose that A is generated, as a commutative ring, by a finite set of generators $x_1, \dots, x_n$. Using Drinfeld's presentation for $\overline{L}^A_{2m-2}$, the Drinfeld relations (1.3) and (1.4) yield that $\overline{L}^A_{2m-2}$ is generated, as an A-module, by the $n+1$ elements $\gamma, c_{x_1}, \dots, c_{x_n}$. Now Lemma 2.3.1 implies that L^A_m is a finitely generated A-module for all integers m. □

In Proposition 2.3.2 it is important that L^A is typically not a finitely-generated A-module, nor even finitely generated as an A-algebra; rather, the summand in each individual degree is a finitely generated A-module.

Corollary 2.3.3. *Let A be a commutative ring, and suppose that A is finitely generated as a commutative ring. Let I be a maximal ideal of A, and let A_I denote A localized at I, i.e., A with all elements outside of I inverted. Then, for each integer m, the grading degree m summand $L^{A_I}_m$ of the classifying ring L^{A_I} of formal A_I-modules is a finitely-generated, I-adically separated A_I-module.*

Proof. By Propositions 2.2.1 and 2.3.2, $(L^A_m)_I \cong L^{A_I}_m$ is a finitely generated A_I-module for all integers m. The ring A_I is Noetherian and local, so the Krull intersection theorem (classical; see Corollary 10.20 in [2]) implies that every finitely generated A_I-module is I-adically separated. □

Proposition 2.3.4. *Let A be a local commutative ring with maximal ideal $\mathfrak{m}$. Suppose that $\mathfrak{m}$ can be generated by κ elements, where κ is some cardinal number.*

Then, for each positive integer n, the A-module $\overline{L}^A_{2n-2}$ can be generated by:

- $1+\kappa$ *elements, if the residue field* $A/\mathfrak{m}$ *is isomorphic to a finite field* $\mathbb{F}_q$ *and* n *is a power of* q,
- *and* 1 *element (i.e.,* $\overline{L}^A_{2n-2}$ *is a cyclic* A*-module) otherwise.*

Proof. For this theorem we use Drinfeld's presentation for $\overline{L}^A_{2n-2}$. Let p denote the characteristic of $A/\mathfrak{m}$. (We allow $p = 0$ as a possibility.) There are three cases to consider:

- **If n is not a power of p:** Then $\nu(n)$ is not divisible by p, so $\nu(n) \in (A/\mathfrak{m})^\times$, so $\nu(n)$ is a unit in A since A is local. So we can solve relation (1.2) to get
$$c_a = \frac{\gamma}{\nu(n)}(a^n - a)$$
for all $a \in A$. Hence $\overline{L}^A_{2n-2}$ is generated by γ.
- **If $n = p^t$ and either $A/\mathfrak{m}$ is infinite or $A/\mathfrak{m}$ has p^s elements and $s \nmid t$:** Then there exists some element $a \in A/\mathfrak{m}$ such that $a^{p^t} \neq a$. Hence a lifts to an element $\tilde{a} \in A$ such that $\tilde{a}^{p^t} - \tilde{a}$ is not in the maximal ideal in A. Consequently $\tilde{a}^{p^t} - \tilde{a} \in A^\times$ and hence we can solve relation (1.2) to get
$$\gamma = \frac{pc_{\tilde{a}}}{\tilde{a}^{p^t} - \tilde{a}}.$$
generated by c_a. We may also solve (1.4) to get
$$c_b = \frac{b^{p^t} - b}{\tilde{a}^{p^t} - \tilde{a}} c_{\tilde{a}}.$$
Hence $c_{\tilde{a}}$ generates $\overline{L}^A_{2n-2}$ as an A-module.
- **If $n = p^t$ and $A/\mathfrak{m}$ has p^s elements and $s \mid t$:** This first half of this argument was inspired by Hazewinkel's Proposition 21.3.1 in [8]. Let M denote the A-submodule of $\overline{L}^A_{2n-2}$ generated by γ and by all the elements c_m with $m \in \mathfrak{m}$. Solving relation (1.4), we get
$$(a^{p^t} - a)c_m = (m^{p^t} - m)c_a \tag{2.8}$$
for all $a, m \in A$. If $m \in \mathfrak{m}$, then $m^{p^t-1} - 1 \notin \mathfrak{m}$, hence $m^{p^t-1} - 1$ is a unit since A is local. Hence
$$\frac{a^{p^t} - a}{m^{p^t-1} - 1} c_m = mc_a \tag{2.9}$$
for all $m \in \mathfrak{m}$ and all $a \in A$ with $a \notin \mathfrak{m}$. Hence $\overline{L}^A_{2n-2}/M$ is an $A/\mathfrak{m}$-vector space.

  Now s divides t, and hence $x^{p^t} = x$ for all $x \in A/\mathfrak{m}$, so relation (1.4) becomes $c_{ab} = ac^b + bc^a$ in $\overline{L}^A_{2n-2}/M$. Similarly, since $\gamma \in M$, (1.3) becomes $c_{a+b} = c_a + c_b$ in $\overline{L}^A_{2n-2}/M$. Hence the map
$$\begin{aligned} c : A/\mathfrak{m} &\to \overline{L}^A_{2n-2}/M \\ a &\mapsto c_a \end{aligned}$$
is a $\mathbb{Z}$-linear derivation. But the relevant module of Kähler differentials $\Omega^1_{(A/\mathfrak{m})/\mathbb{Z}}$ vanishes, since $A/\mathfrak{m}$ is a field. Hence c factors through the zero module, i.e., c is the zero map. So $c_a = 0$ in $\overline{L}^A_{2n-2}/M$ for all $a \in A$ with

$a \notin \mathfrak{m}$, and $c_a = 0$ in $\overline{L}^A_{2n-2}/M$ for all $a \in \mathfrak{m}$ by the definition of M. So $\overline{L}^A_{2n-2}/M$ is trivial, i.e., $M = \overline{L}^A_{2n-2}$.

Now suppose we choose a set X of generators for the maximal ideal $\mathfrak{m}$ of A. We have already shown that $M = \{\gamma\} \cup \{c_m : m \in \mathfrak{m}\}$ is a set of A-module generators for $\overline{L}^A_{2n-2}$, but we still need to show that $Q := \{\gamma\} \cup \{c_x : x \in X\} \subseteq \overline{L}^A_{2n-2}$ is also a set of A-module generators for $\overline{L}^A_{2n-2}$. By the same argument as given two paragraphs ago, we may simplify the relations (1.2) through (1.4) in $\overline{L}^A_{2n-2}/Q$, yielding that $\overline{L}^A_{2n-2}/Q$ is an $A/\mathfrak{m}$-vector space in which $c_{a+b} = c_a + c_b$ and $ac_b + bc_a = c_{ab}$.

As a consequence, if we consider any element $m = \sum_{x \in X} m_x x \in \mathfrak{m}$, then in the quotient A-module $\overline{L}^A_{2n-2}/Q$ we have

$$\begin{aligned} c_\alpha &= \sum_{x \in X} c_{m_x x} \\ &= \sum_{x \in X} (m_x c_x + x c_{m_x}) \\ &= \sum_{x \in X} m_x c_x \in Q. \end{aligned}$$

That is, the A-submodule of $\overline{L}^A_{2n-2}$ generated by Q contains all the generators of M, which we already showed to be equal to $\overline{L}^A_{2n-2}$. Hence $Q = \overline{L}^A_{2n-2}$, as desired.

□

Corollary 2.3.5. *Let A be a local commutative ring with maximal ideal $\mathfrak{m}$, and suppose that $\mathfrak{m}$ is finitely generated. Then, for each positive integer n, $\overline{L}^A_{2n-2}$ is a finitely generated A-module.*

Corollary 2.3.6. *Let A be a Noetherian complete local commutative ring. Then, for each integer n, the grading degree n summand L^A_n of the classifying ring L^A of formal A-modules is $\mathfrak{m}$-adically separated and $\mathfrak{m}$-adically complete.*

Proof. Krull's intersection theorem implies that every finitely generated A-module is $\mathfrak{m}$-adically separated, and it is elementary that every finitely generated module over a Noetherian complete local ring with maximal ideal $\mathfrak{m}$ is $\mathfrak{m}$-adically complete. □

Lemma 2.3.7 is immediate, when R is Noetherian. The use of the lemma is when R is not Noetherian but R^0 is, e.g. $R \cong MU_* \cong L^{\mathbb{Z}}$.

Lemma 2.3.7. *Let R be a $\mathbb{Z}$-graded commutative ring which is connective, i.e., there exists some integer n such that $R^m \cong 0$ for all $m < n$. Assume furthermore that R^0 is Noetherian and that R^i is a finitely generated R^0-module for each integer i.*

Then, for any $\mathbb{Z}$-graded finitely generated R-module M and any ideal I of R generated by elements in R^0, the natural map

$$\hat{R}_I \otimes_R M \to \hat{M}_I$$

is an isomorphism of $\mathbb{Z}$-graded $\hat{R}_I$-modules.

Proof. Since M is finitely generated as an R-module, M^i is finitely generated as an R^0-module for any integer i, and since M^i is a finitely generated module over the Noetherian ring R^0, the map $\hat{R}^0_I \otimes_{R^0} M^i \to \hat{M}^i_I$ is an isomorphism for all i. □

Definition-Proposition 2.3.8. *Let A be a commutative ring and let I be an ideal in A. Let F be a formal group law defined over an A-algebra R. Equip the power series ring $R[[X]]$ with the $I+(X)$-adic filtration*[6]*, i.e., the decreasing filtration*

$$R[[X]] = F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots \tag{2.10}$$

in which F_n is the nth power of the sum of the ideals I and (X). Let $\mathfrak{c}_n$ be the intersection of F_n with $\operatorname{End}(F) \subseteq R[[X]]$*. Then*

$$\operatorname{End}(F) = \mathfrak{c}_0 \supseteq \mathfrak{c}_1 \supseteq \mathfrak{c}_2 \supseteq \dots \tag{2.11}$$

is a sequence of two-sided ideals of $\operatorname{End}(F)$*. Furthermore, if $x \in \mathfrak{c}_m$ and $y \in \mathfrak{c}_n$, then $xy \in \mathfrak{c}_{m+n}$.*

We refer to the filtration (2.11) *as the* $\mathfrak{c}$-adic filtration *on* $\operatorname{End}(F)$*. We call the resulting topology on* $\operatorname{End}(F)$*, in which the ideals* (2.11) *are a neighborhood basis of zero, the* $\mathfrak{c}$-adic topology.

Proof. Let $f(X), g(X) \in \operatorname{End}(F)$. It is routine to verify that

- if $f(X), g(X) \in \mathfrak{c}_n$, then $f+g \in \mathfrak{c}_n$,
- and if $f(X) \in \mathfrak{c}_m$ and $g(X) \in \mathfrak{c}_n$, then $fg \in \mathfrak{c}_{m+n}$ (note that this product in $\operatorname{End}(F)$ is the composition of power series).

The special cases $m=0$ and $n=0$ of the latter observation establish that each $\mathfrak{c}_n$ is a two-sided ideal of $\operatorname{End}(F)$. □

Lemma 2.3.9. *Let A be a Noetherian commutative ring and let I be an ideal in A. Let R be a commutative A-algebra, and let F be a formal A-module. If R is I-adically separated, then* $\operatorname{End}(F)$ *is $\mathfrak{c}$-adically separated. If R is I-adically complete, then* $\operatorname{End}(F)$ *is $\mathfrak{c}$-adically complete.*

Proof. If R is I-adically separated, then the filtration (2.10) on $R[[X]]$ is as well, so the intersection

$$\bigcap_n \mathfrak{c}_n = \cap_n (F_n \cap \operatorname{End}(F))$$

must be zero.

Now suppose instead that R is I-adically complete, and that

$$(\zeta_1(X), \zeta_2(X), \dots) \tag{2.12}$$

is a Cauchy sequence in the $\mathfrak{c}$-adic topology on $\operatorname{End}(F)$. Let $\zeta_{m,n}$ denote the nth coefficient in the power series $\zeta_m(X)$. For each n, the sequence $\zeta_{1,n}, \zeta_{2,n}, \dots$ is an I-adically Cauchy sequence in A, hence converges. Hence the limit $\lim_m \zeta_m(X)$ exists in the $\mathfrak{c}$-adic topology: it is merely the endomorphism of F whose nth power series coefficient is $\lim_m \zeta_{m,n}$. So every $\mathfrak{c}$-adic Cauchy sequence in $\operatorname{End}(F)$ converges, so $\operatorname{End}(F)$ is $\mathfrak{c}$-adically complete. □

[6]To avoid any potential confusion, we remind the reader that the endomorphism ring $\operatorname{End}(F)$ of F is a subset of $R[[X]]$, but the multiplication in $\operatorname{End}(F)$ is given by composition, and the addition in $\operatorname{End}(F)$ is given by formal addition using F. Consequently $\operatorname{End}(F)$ is not a subring of the power series ring $R[[X]]$.

Lemma 2.3.10. *Let A be a Noetherian commutative ring and let I be an ideal in A. Suppose that A is separated (but not necessarily complete) in the I-adic topology. Let R be a commutative A-algebra which is I-adically separated and complete, and let F be a formal A-module over R. Then F is the underlying formal A-module of exactly one formal $\hat{A}_I$-module. That is, the action map $\rho : A \to \mathrm{End}(F)$ extends uniquely to an action map $\tilde{\rho} : \hat{A}_I \to \mathrm{End}(F)$ making F a formal $\hat{A}_I$-module.*

Proof. Choose an element $a \in \hat{A}_I$, and for each positive integer n, let a_n be the image of a under the projection map $\hat{A}_I \to A/I^n$, and let $\tilde{a}_n$ be an element of A whose reduction modulo I^n is a_n. (In other words: choose a sequence of elements $(\tilde{a}_1, \tilde{a}_2, \dots)$ of A converging to a in the I-adic topology.) Then the sequence $(\tilde{a}_1, \tilde{a}_2, \dots)$ uniquely determines the element $a \in \hat{A}_I$, since we assumed that A is separated in the I-adic topology.

The tangent condition on ρ (that $\rho(X) \equiv X \mod X^2$) implies that the image of I^n under ρ is contained in the ideal $\mathfrak{c}_n$ of Definition-Proposition 2.3.8. By Lemma 2.3.9, $\mathrm{End}(F)$ is $\mathfrak{c}$-adically separated and complete, so the sequence $(\rho(\tilde{a}_1), \rho(\tilde{a}_2), \dots)$ in $\mathrm{End}(F)$ converges to a unique element in $\mathrm{End}(F)$. Let $\tilde{\rho}(a)$ be defined to be this element. It is elementary to check that the resulting map $\tilde{\rho} : \hat{A}_I \to \mathrm{End}(F)$ is a well-defined ring homomorphism and agrees with ρ when composed with the injection $A \hookrightarrow \hat{A}_I$. (This map $\tilde{\rho}$ is, of course, the one given by the universal property of completion, but we are giving some detail here because $\mathrm{End}(F)$ is not typically commutative and ρ does not typically have its image inside the center of $\mathrm{End}(F)$, so the situation is not exactly the textbook one encountered in algebra.) The tangent condition for $\tilde{\rho}$ is similarly easy: any element $a \in \hat{A}_I$ can be approximated arbitrarily $\mathfrak{c}$-adically closely by an element of A, and $\tilde{\rho}$ satisfies the tangent condition on elements of A since $\tilde{\rho}$ coincides with ρ on elements of A.

Consequently F is indeed the underlying formal A-module of a formal $\hat{A}_I$-module. The fact that the ring homomorphism $\tilde{\rho}$ is the *unique* extension of ρ to a ring map $\hat{A}_I \to \mathrm{End}(F)$ is as follows: any ring homomorphism $\hat{A}_I \to \mathrm{End}(F)$ extending ρ sends each I^n into $\mathfrak{c}_n$ and hence is continuous, hence is a continuous homomorphism of abelian groups; now the universal property of the completion implies that the extension $\tilde{\rho}$ is unique. □

Lemma 2.3.11. *Let A be a Noetherian commutative ring and let I be an ideal in A. Suppose that A is I-adically separated, and suppose that R is an I-adically separated and complete A-algebra. Let F, G be formal $\hat{A}_I$-modules over R. Suppose that $f : F \to G$ is a strict isomorphism of the underlying formal A-modules of F and G. Then f is also a strict isomorphism $F \to G$ of formal $\hat{A}_I$-modules.*

Proof. Let $\rho_F : \hat{A}_I \to \mathrm{End}(F)$ and $\rho_G : \hat{A}_I \to \mathrm{End}(G)$ denote the structure maps of F and G as formal $\hat{A}_I$-modules, respectively. (The existence and uniqueness of these structure maps follows from Lemma 2.3.10.) Then $\rho_F(f(a)(X)) = f(\rho_G(a)(X))$ for all $a \in A$, and we need to show that the same is true for all $a \in \hat{A}_I$. For any $a \in \hat{A}_I$, choose a sequence of elements $a_1, a_2, \dots$ in A such that $\lim_{n\to\infty} a_n = a$ in the I-adic topology. Then the fact that ρ_F and ρ_G are continuous (since each sends I into $\mathfrak{c}$) implies that

$$f(\rho_F(\lim_{n\to\infty} a_n)(X)) = f(\lim_{n\to\infty} \rho_F(a_n)(X)), \text{ and} \tag{2.13}$$

$$\rho_G(\lim_{n\to\infty} a_n)(f(X)) = \lim_{n\to\infty} \rho_G(a_n)(f(X)). \tag{2.14}$$

Now since f is a homomorphism of formal group laws, its constant coefficient is zero, and hence f is continuous in a limited sense: whenever $\xi_1(X), \xi_2(X), \dots$ is a $\mathfrak{c}$-adically convergent sequence in $\mathrm{End}(F)$ such that each power series $f(\xi_1(X)), f(\xi_2(X)), \dots$ is contained in $\mathrm{End}(G) \subseteq R[[X]]$ and $\mathfrak{c}$-adically convergent, we get an equality $\lim_{n\to\infty} f(\xi_n(X)) = f(\lim_{n\to\infty} \xi_n(X))$. Consequently (2.13) is equal to (2.14), and hence $\rho_F(f(a)(X)) = f(\rho_G(a)(X))$. □

Theorem 2.3.12. *Let A be a commutative ring which is finitely generated as a commutative ring. Let I be a maximal ideal of A, and suppose that A is separated*[7] *in the I-adic topology. Then the natural maps of graded Hopf algebroids*

$$(L^A \otimes_A \hat{A}_I, L^A B \otimes_A \hat{A}_I) \to ((L^A)\hat{}_I, (L^A B)\hat{}_I) \tag{2.15}$$

$$\to (L^{\hat{A}_I}, L^{\hat{A}_I} B) \tag{2.16}$$

are isomorphisms.

Proof. By Proposition 2.3.2, for all integers n the degree n summand L^A_n in the ring L^A is a finitely-generated A-module, and $L^A B \cong L^A B \otimes_L LB \cong L^A[t_1, t_2, \dots]$ (by Proposition 1.3.2 and Theorem 1.2.1) is also a finitely-generated A-module in each degree. Consequently Lemma 2.3.7 applies, since A is finitely generated as a commutative ring and hence Noetherian, even though L^A is typically not Noetherian. So the map (2.15) is an isomorphism.

The more substantial result is that (2.16) is also an isomorphism. By Lemma 2.3.10, Lemma 2.3.11, and the universal properties of the rings involved, the map (2.16) induces bijections

$$\hom_{\hat{A}_I-alg}(L^{\hat{A}_I}, R) \xrightarrow{\cong} \hom_{\hat{A}_I-alg}(L^A \otimes_A \hat{A}_I, R) \quad \text{and}$$
$$\hom_{\hat{A}_I-alg}(L^{\hat{A}_I}B, R) \xrightarrow{\cong} \hom_{\hat{A}_I-alg}(L^A B \otimes_A \hat{A}_I, R),$$

natural in R, for all commutative $\hat{A}_I$-algebras R which are I-adically separated and complete.

Now the Yoneda lemma tells us that the ring maps $L^A \otimes_A \hat{A}_I \to L^{\hat{A}_I}$ and $L^A B \otimes_A \hat{A}_I \to L^{\hat{A}_I} B$ are isomorphisms, as long as all four of these rings are actually objects in the category of I-adically separated and complete commutative $\hat{A}_I$-algebras! The graded ring L^{A_I} a finitely generated A-module in each degree by Corollary 2.3.3. Hence the same is true of $(L^{A_I})\hat{}_I \cong L^{A_I} \otimes_{A_I} \hat{A}_I$, hence $(L^{A_I})\hat{}_I$ is I-adically separated and I-adically complete by the same argument as in the proof of Corollary 2.3.6, and the same is true for $(L^{A_I}B)\hat{}_I \cong (L^{A_I})\hat{}_I \otimes_L LB$, by Proposition 1.3.2. On the other hand, $L^{\hat{A}_I}$ is I-adically separated and complete in each grading degree by Corollary 2.3.6, hence $L^{\hat{A}_I} B \cong L^{\hat{A}_I} \otimes_L LB$ is as well, again by Proposition 1.3.2. □

Corollary 2.3.13. *Let A be a commutative ring which is finitely generated as a commutative ring. Let I be a maximal ideal of A, and suppose that A is separated in the I-adic topology. Let M be a graded left $L^A B$-comodule which is finitely-generated as an A-module in each degree, and suppose that M is bounded-below, i.e., there exists some integer b such that M is trivial below degree b. (For example,*

[7] This separation condition is automatically satisfied if A is an integral domain, by Krull's intersection theorem.

$M = L^A$ satisfies all these conditions on M.) Then, for all integers s,t with $s \geq 0$, we have isomorphisms of $\hat{A}_I$-modules

$$\mathrm{Ext}^{s,t}_{(L^A,L^AB)}(L^A,M)\otimes_A \hat{A}_I \cong \mathrm{Ext}^{s,t}_{(L^A,L^AB)}(L^A,\hat{M}_I) \tag{2.17}$$

$$\cong \mathrm{Ext}^{s,t}_{(L^{\hat{A}_I},L^{\hat{A}_I}B)}(L^{\hat{A}_I},\hat{M}_I) \tag{2.18}$$

Proof. Let $C^\bullet_{(L^A,L^AB)}(M)$ be the cobar complex[8] of the Hopf algebroid (L^A,L^AB) with coefficients in M. Then, since L^AB is a finitely generated A-module in each degree by Proposition 2.3.2, the same is true of $L^AB\otimes_{L^A} L^AB\otimes_{L^A}\cdots\otimes_{L^A} M = (L^AB)^{\otimes_{L^A}n}\otimes_{L^A} M$. Consequently we have isomorphisms

$$\begin{aligned}(L^AB)^{\otimes_{L^A}n}\otimes_{L^A}\hat{M}_I &\cong (L^AB)^{\otimes_{L^A}n}\otimes_{L^A} M\otimes_A \hat{A}_I\\ &\cong \left(L^AB\otimes_A \hat{A}_I\right)^{\otimes_{L^A\otimes_A\hat{A}_I}n}\otimes_{L^A\otimes_A\hat{A}_I}\left(M\otimes_A\hat{A}_I\right)\\ &\cong (L^{\hat{A}_I}B)^{\otimes_{L^{\hat{A}_I}}n}\otimes_{L^{\hat{A}_I}}\hat{M}_I,\end{aligned}$$

which is the module of n-cochains $C^n_{(L^{\hat{A}_I},L^{\hat{A}_I}B)}(\hat{M}_I)$. These isomorphisms are natural, commuting with the cobar complex differentials, hence giving us isomorphism (2.18). Meanwhile, isomorphism (2.17) follows from $\hat{A}_I$ being a flat A-module (classical, as in Proposition 10.14 in [2]), hence tensoring with $\hat{A}_I$ commutes with taking cohomology of the cobar complexes. □

Corollary 2.3.14. *Let A,I,M be as in Corollary 2.3.13. Let E_0^IM denote the associated graded comodule of the I-adic filtration on M. Then there exists a conditionally convergent spectral sequence*

$$\begin{aligned}E_1^{s,t,u}\cong \mathrm{Cotor}^{s,t,u}_{L^AB}\left(L^A,E_0^IM\right) &\Rightarrow \mathrm{Cotor}^{s,t}_{L^AB}\left(L^A,M\right)\hat{}_I\\ &\cong \mathrm{Cotor}^{s,t}_{L^{\hat{A}_I}B}\left(L^{\hat{A}_I},\hat{M}_I\right)\\ d_r: E_r^{s,t,u} &\to E_r^{s+1,t,u+r}.\end{aligned} \tag{2.19}$$

Proof. This is the spectral sequence of the I-adic filtration on the cobar complex of (L^A,L^AB) with coefficients in M. The isomorphism (2.19) is due to Corollary 2.3.13. □

See section 4.1 and section 4.2 for examples of explicit calculations with the spectral sequence of Corollary 2.3.14.

2.4. **Point-detecting ideals and the fundamental functional.**

Definition 2.4.1. *Let A be a commutative ring, and let k be a field. We say that an ideal I of A* detects all k-points *I is in the kernel of every ring homomorphism $A\to k$.*

Let k be the finite field with p^n elements. Then there exists a *largest* ideal of A which detects all k-points: namely, the ideal generated by p and by the difference

[8] See Appendix 1 of [27] for the definition and basic properties of the cobar complex. All that is necessary for us to know right now about the cobar complex is that its module of n-cochains $C^n_{(L^A,L^AB)}(M)$ is isomorphic to $(L^AB)^{\otimes_{L^A}n}\otimes_{L^A}M$, and the cohomology of $C^\bullet_{(L^A,L^AB)}(M)$ is $\mathrm{Ext}^{*,*}_{(L^A,L^AB)}(L^A,M)$, i.e., $\mathrm{Cotor}^{*,*}_{L^AB}(L^A,M)$.

$a^{p^n} - a$ for each $a \in A$. We will write I_{p^n} for the largest ideal of A which detects all k-points, so that

$$I_{p^n} = \left(p, a^{p^n} - a \text{ for all } a \in A\right).$$

We call I_{p^n} the *universal* $\mathbb{F}_{p^n}$*-point-detecting ideal* of A.

Proposition 2.4.2. *Let A be a Noetherian*[9] *commutative ring. Then the quotient map $A \to A/I_p$ coincides with the universal map $A \to \prod \mathbb{F}_p$, with the product taken over all ring homomorphisms $A \to \mathbb{F}_p$. In particular,* $\operatorname{Spec} A/I_p$ *is the union of the $\mathbb{F}_p$-points of* $\operatorname{Spec} A$.

Proof. Elementary, but here is the argument: clearly A/I_{p^n} has no nonzero nilpotents, so I_p is reduced. Consequently, in its primary decomposition $I_p = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$, each primary ideal $\mathfrak{p}_j$ is prime. The quotient map $A \to A/\mathfrak{p}_j$ must factor through $A \to A/I_p$, so $A/\mathfrak{p}_j$ is an integral domain which is a quotient of A/I_p. In such an integral domain, we have $a(a^{p-1} - 1) = 0$ for all a, hence $a^{p-2} = a^{-1}$ for all nonzero a, i.e., $A/\mathfrak{p}_j$ is a field. Hence I_p is an intersection of finitely many coprime maximal ideals, so the Chinese Remainder Theorem ensures that A/I_p is a product of fields. □

Remark 2.4.3. The local zeta-function of an affine variety $V = \operatorname{Spec} A$ at some prime number p is defined as

$$Z(V,t) = exp\left(\sum_{m \geq 1} \frac{N_m}{m} t^m\right),$$

where N_m is the number of $\mathbb{F}_{p^m}$-points of V. Proposition 2.4.2 shows that the order of A/I_p is p^{N_1}. An analogous statement holds for other prime powers, with appropriate adjustments: A/I_{p^2} "counts" both the $\mathbb{F}_p$-points and the $\mathbb{F}_{p^2}$-points of $\operatorname{Spec} A$, for example.

Definition 2.4.4. *Let n be a positive integer, and let A be a commutative ring. Recall from Proposition 1.2.2 that $\overline{L}^A_{2n-2}$ is described by Drinfeld's presentation: it is generated, as an A-module, by elements γ and $\{c_a\}_{a \in A}$, subject to the relations* (1.2)*,* (1.3)*, and* (1.4)*.*

By the degree $2n-2$ fundamental functional of A*, we mean the unique A-module homomorphism $\sigma_{2n-2} : \overline{L}^A_{2n-2} \to A$ given by*

$$\sigma_{2n-2}(\gamma) = \nu(n), \quad \textit{and}$$
$$\sigma_{2n-2}(c_a) = a^n - a.$$

Proposition 2.4.5. *Let A be a commutative ring. Suppose that A has no nonzero 2-torsion. Then the degree 2 fundamental functional $\sigma_2 : \overline{L}^A_2 \to A$ is injective, and its image is the universal $\mathbb{F}_2$-point-detecting ideal I_2 of A. Consequently we have an isomorphism of A-modules $\overline{L}^A_2 \xrightarrow{\cong} I_2$.*

Proof. It is straightforward from the definition of σ_2 that its image is generated by 2 and by $a^2 - a$ for each $a \in A$, i.e., $I_2 = \operatorname{im} \sigma_2$. All that needs to be checked is that σ_2 is injective.

[9]Without the Noetherian hypothesis, it is still true that $A/I_p \to \prod \mathbb{F}_p$ is injective, but surjectivity is not guaranteed: consider the case of the ring $A = \mathbb{F}_p[x_1, x_2, \dots]$, which is countable (hence A/I_p is also countable), but which has infinitely many $\mathbb{F}_p$-points, hence $\prod \mathbb{F}_p$ is an uncountable ring.

Our argument for injectivity of σ_2 involves the Hochschild homology of the commutative ring $A/2$, with coefficients in a particular $A/2$-bimodule Ψ. To be clear, by $A/2$ we mean the quotient of the ring A by its principal ideal generated by 2. The $A/2$-bimodule Ψ is defined as follows:

- as a left $A/2$-module, Ψ is simply $A/2$ itself, i.e., free on one generator,
- and the right $A/2$-action on Ψ is given by letting $a \cdot x$ be the product $a^2 x$ in the ring $A/2$.

The cyclic bar complex (i.e., the standard resolution for calculating Hochschild homology) with coefficients in Ψ is as follows:

$$0 \leftarrow A/2 \xleftarrow{d_0} A/2 \otimes_{\mathbb{F}_2} A/2 \xleftarrow{d_1} A/2 \otimes_{\mathbb{F}_2} A/2 \otimes_{\mathbb{F}_2} A/2 \xleftarrow{d_2} \dots$$

with

$$\begin{aligned} d_0(a_0 \otimes a_1) &= (a_1^2 - a_1)a_0, \quad \text{and} \\ d_1(a_0 \otimes a_1 \otimes a_2) &= a_0 a_1^2 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1. \end{aligned}$$

Let P^A be the cokernel of the A-module homomorphism $A \to \overline{L}_2^A$ sending 1 to γ. Consider the kernel $\ker d_0$ as a left $A/2$-module. It admits an $A/2$-module homomorphism $f : \ker d_0 \to P^A$ given by $f(a_0 \otimes a_1) = a_0 c_{a_1}$. Let $\theta : P^A \to A/2$ be the $A/2$-module map given by $\theta(c_a) = a^2 - a$. It is routine to check that the resulting sequence of $A/2$-modules

$$0 \to \text{im } d_1 \to \ker d_0 \xrightarrow{f} P^A \xrightarrow{\theta} A/2$$

is exact, by checking that Drinfeld relation (1.4) is precisely the relation imposed on $\ker d_0$ by quotienting out by its submodule im d_1. Consequently the Hochschild homology group $HH_1(A/2;\Psi)$ vanishes if and only if θ is injective.

Now consider the commutative diagram with exact rows[10]

$$\begin{tikzcd} \ker\gamma \arrow[r] \arrow[d] & A \arrow[r, "\gamma"] \arrow[d, "\mathrm{id}"] & \overline{L}_2^A \arrow[r] \arrow[d, "\sigma_2"] & P^A \arrow[r] \arrow[d, "\theta"] & 0 \arrow[d] \\ 0 \arrow[r] & A \arrow[r, "2"] & A \arrow[r] & A/2 \arrow[r] & 0. \end{tikzcd} \tag{2.20}$$

A routine diagram chase in diagram (2.20) shows that σ_2 is injective if and only if θ is injective.

Finally, we invoke a 2007 calculation of Pirashvili, the main result of [22]: if n is a positive power of a prime number p, B is a commutative $\mathbb{F}_p$-algebra, and $\Phi^n(B)$ is the ring B regarded as a B-bimodule via the free B-action on the left and via the Frobenius-twisted B-action $x \cdot b := b^p x$ on the right, then the Hochschild homology groups $HH_i(B;\Phi^n(B))$ are trivial for all $i > 0$. In particular, our Hochschild group $HH_1(A/2;\Psi)$ is trivial. Hence θ is injective, hence σ_2 is injective, as desired. □

Proposition 2.4.5 admits a generalization to the higher fundamental functionals σ_n and the universal k-point-detecting ideals for all finite fields k. That generalization is not used in this paper, so we do not present it here, but it can be found in the preprint [28].

Proposition 2.4.5, and the proof we gave of it, involve the assumption that the ring A is 2-torsion-free. The author does not know whether the conclusion of

[10]This is the place where we use the hypothesis that A is 2-torsion-free.

Proposition 2.4.5 still holds even if A has nontrivial 2-torsion. Here is an amusing class of examples:

Example 2.4.6. Let m be a positive integer, and consider the situation where the ring A is $\mathbb{Z}/m\mathbb{Z}$. It is a nice exercise to calculate that, in the A-module $\overline{L}_2^A$, we have the relation $c_i = \binom{i}{2}\gamma$ for every integer i. Consequently $\overline{L}_2^A \cong A/\binom{m}{2}A$, generated by γ. Elementary calculation then shows that $\overline{L}_2^A \xrightarrow{\sigma_2} A$ is an isomorphism for all m.

It does not seem urgent at present to find out whether the conclusion of Proposition 2.4.5 still holds when A has nontrivial 2-torsion, since some of the later cohomological calculations we make (e.g. the proof of Theorem 3.4.1) also require, for other reasons, that A be torsion-free.

3. Generalities on the moduli of formal A-module 2-buds.

Now we begin to restrict our attention from formal A-modules down to only their 2-buds, in order to facilitate explicit calculations which hold for a very large class of rings A.

3.1. **Structure of the classifying ring of formal A-module 2-buds.** A formal group law 2-bud is given by a single coefficient γ, so that $F(X,Y) \equiv X + Y + \gamma XY \mod (X,Y)^3$. The A-action map $\rho : A \to \mathrm{End}(F)$ is given by $\rho_a(X) \equiv aX + c_a X^2 \mod X^3$. It is classical (see [5]) that, by elementary calculation, one finds that the relations among the elements γ and c_a imposed by associativity of F, by $\rho_{ab}(X) = \rho_a(\rho_b(X))$, by $\rho_{a+b}(X) = F(\rho_a(X), \rho_b(X))$, and by $F(\rho_a(X), \rho_a(Y)) = \rho_a F(X,Y)$, are the Drinfeld relations

$$(a^2 - a)\gamma = 2c_a \tag{3.21}$$

$$c_{a+b} - c_a - c_b = ab\gamma \tag{3.22}$$

$$ac_b + b^2 c_a = c_{ab}. \tag{3.23}$$

Definition 3.1.1. *Let Dr^A denote the A-module generated by the symbols γ and c_a for each $a \in A$, subject to the relations* (3.21) *through* (3.23)*.*

That is, $\mathrm{Dr}^A \cong \overline{L}_2^A$. It follows easily[11] that the classifying ring L^A_{2-buds} of formal A-module 2-buds is the symmetric A-algebra $\mathrm{Sym}_A(\mathrm{Dr}^A)$.

Remark 3.1.2. Among the elementary consequences of (3.22) and (3.23), we have that $c_0 = 0 = c_1$, and that $c_{-1} = \gamma$. Consequently a smaller presentation for Dr^A is possible: Dr^A is the A-module with a generator c_a for each $a \in A$, subject to the relations (3.21) through (3.23) but with c_{-1} written in place of γ throughout. This is convenient for the sake of understanding the moduli of formal A-module 2-buds,

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[11] The same claim cannot safely be made for formal module n-buds for $n > 2$. Consider the situation in the case of 3-buds: we have a parameter γ_2, and a parameter $c_{a,2}$ for each $a \in A$, which jointly determine the quadratic terms in the 3-bud. We also have a deformation parameter γ_3, and a deformation parameter $c_{a,3}$ for each $a \in A$, for the cubic terms in the 3-bud. The A-module $\overline{L}_3^A$ generated by γ_3 and $c_{a,3}$, related by the cubic Drinfeld relations, is not necessarily (i.e., not for all A) a *projective* A-module. What happens as a consequence is that the classifying ring L^A_{3-buds} of formal A-module 3-buds is not necessarily a symmetric A-algebra on $\overline{L}_2^A \oplus \overline{L}_3^A$. We have made some calculations of the ring L^A_{3-buds} as a function of A, but those calculations do not fit within the scope of this paper.

but the coincidence $c_{-1} = \gamma$ is specific to the case of 2-buds. Consequently it is easier to recognize patterns that hold for formal A-module n-buds for $n > 2$ (and for formal A-modules *simpliciter*) if we use the non-minimal presentation for Dr^A, rather than replacing γ with c_{-1}.

3.2. **Structure of the moduli stack of formal A-module 2-buds.** It follows from section 3.1 and from the structure theory in section 2 that the classifying Hopf algebroid $(L^A_{2-buds}, L^A_{2-buds}B)$ of formal A-module 2-buds has the following simple form:

$$L^A_{2-buds} \cong \mathrm{Sym}_A(\mathrm{Dr}^A),$$

$$L^A_{2-buds}B \cong \mathrm{Sym}_A(\mathrm{Dr}^A)[t], \tag{3.24}$$

$$\eta_L(x) = x \quad \text{for all } x \in L^A_{2-buds},$$

$$\eta_R(\gamma) = \gamma + 2t, \tag{3.25}$$

$$\eta_R(c_a) = c_a + (a^2 - a)t \quad \text{for all } a \in A, \tag{3.26}$$

$$\Delta(t) = t \otimes 1 + 1 \otimes t, \tag{3.27}$$

$$\epsilon(t) = 0. \tag{3.28}$$

For any formal A-module 2-bud F, the truncated power series $f(X) \equiv X + tX^2$ modulo X^3 is a strict isomorphism from F to *some* formal A-module 2-bud G. The parameter t in (3.24) is the parameter t in the strict isomorphism $f(X) = X + tX^2$ of formal A-module 2-buds. If $F(X,Y) \equiv X + Y + \gamma XY$ modulo $(X,Y)^3$, then f is a strict isomorphism from G to $G(X,Y) \equiv X + Y + (\gamma + 2t)XY$ modulo $(X,Y)^3$. This yields (3.25).

Similarly, if the action map $\rho_F : A \to \mathrm{End}(F)$ of the formal A-module 2-bud F is given by $\rho_F(a)(X) = aX + c_aX^2$ modulo X^3, then the codomain of the strict isomorphism f with domain F has A-action map $\rho_G(a)(X) = aX + (c_a + (a^2 - a)t)X^2$ modulo X^3. This yields (3.26).

The formulas (3.27) and (3.28) arise from the observation that the composite of strict isomorphisms $f(X) \equiv X + aX^2$ modulo X^3 and $g(X) \equiv X + a'X^2$ modulo X^3 is $(f \circ g)(X) \equiv X + (a + a')X^2$ modulo X^3.

Smoothness of the ring map η_L implies that the stackification of the groupoid scheme represented by $(\mathrm{Spec}\, L^A_{2-buds}, \mathrm{Spec}\, L^A_{2-buds}B)$ is an Artin stack. We write $\mathcal{M}^{2-buds}_{fmA}$ for the resulting *moduli stack of formal A-module 2-buds*. To be clear, the groupoid of R-points of $\mathcal{M}^{2-buds}_{fmA}$ is the groupoid of A-module structures, with identity element 0, on the first-order neighborhood of $\mathrm{Spec}\, R$ in $\mathbb{A}^1_R$. Such an A-module structure is slightly less data than a formal A-module 2-bud. To go from the A-module structure to a formal A-module 2-bud, one needs to specify an isomorphism of $\mathrm{Spec}\, R[X]/X^2$ with the first-order neighborhood of $\mathrm{Spec}\, R$ in $\mathbb{A}^1_R$. The situation is entirely analogous to that described in section 1.2.1.

As described in section 1.2.2, it follows from standard arguments that the flat cohomology group $H^s_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes t})$ is isomorphic to $\mathrm{Cotor}^{s,2t}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds})$. Here ω is the line bundle of differentials, which coincides with the quasicoherent $\mathcal{O}_{\mathcal{M}^{2-buds}_{fmA}}$-module corresponding to the graded L^AB-comodule $\Sigma^2 L^A$. Since L^A_{2-buds} and $L^A_{2-buds}B$ are trivial in odd degrees, the Cotor-groups $\mathrm{Cotor}^{s,t}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds})$ are trivial in odd internal degrees t. Hence each of the nontrivial Cotor-groups $\mathrm{Cotor}^{s,t}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds})$ agrees with a flat cohomology group of $\mathcal{M}^{2-buds}_{fmA}$. Consequently, throughout the rest of this paper, we will often state our results in terms of flat cohomology of the Artin stack $\mathcal{M}^{2-buds}_{fmA}$, but the proofs will be carried out in terms of Cotor over the Hopf algebroid $(L^A_{2-buds}, L^A_{2-buds}B)$. The reader who strongly prefers one perspective over the other can translate all statements about flat cohomology into statements about Cotor, or conversely.

Lemma 3.2.1. *Suppose* 2 *is a unit in the commutative ring* A*. Then the* $L^A_{2-buds}B$*-comodule* L^A_{2-buds} *is a summand of the* $L^A_{2-buds}B$*-comodule* $L^A_{2-buds}B$*.*

Proof. By the assumption $\frac{1}{2} \in A$, the right unit map

$$\eta_R : L^A_{2-buds} \to L^A_{2-buds}B = L^A_{2-buds}[t]$$

admits the retraction given by the L^A_{2-buds}-algebra map

$$\begin{aligned} r : L^A_{2-buds}B &\to L^A_{2-buds} \\ t &\mapsto \frac{d}{2}. \end{aligned}$$

It is elementary to verify that r is a left $L^A_{2-buds}B$-comodule algebra homomorphism, and that $r \circ \eta_R = \mathrm{id}_{L^A_{2-buds}}$, i.e., L^A_{2-buds} is a $L^A_{2-buds}B$-comodule retract of $L^A_{2-buds}B$. □

Here is a simple cohomological application of Theorem 2.2.1:

Proposition 3.2.2. *Let* A *be a commutative ring. Then, for all* $s > 0$ *and all* t*, the abelian group* $H^s_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes t})$ *is* 2*-power-torsion.*

Proof. By Theorem 2.2.1, we have

$$\mathrm{Cotor}^{s,t}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds})[2^{-1}] = \mathrm{Cotor}^{s,t}_{L^{A[2^{-1}]}_{2-buds}B}(L^{A[2^{-1}]}_{2-buds}, L^{A[2^{-1}]}_{2-buds}). \tag{3.29}$$

By Lemma 3.2.1, $L^{A[2^{-1}]}_{2-buds}$ is a relatively injective $L^{A[2^{-1}]}_{2-buds}B$-comodule (see appendix 1 of [27] for basic properties of relatively injective comodules). Consequently the right-hand side of (3.29) is trivial for $s > 0$. Hence $\mathrm{Cotor}^{s,t}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds})$ is killed by inverting 2 for $s > 0$, i.e., $\mathrm{Cotor}^{s,t}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds})$ is 2-power-torsion for $s > 0$. □

3.3. $H^*_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes *})$ **as the cohomology of a** $\mathbb{G}_a$**-action.** Here is a slightly more conceptual way to phrase the presentation for the Hopf algebroid given by (3.24) through (3.28). Given a Hopf algebroid (B, Γ) and a left Γ-comodule algebra C, one can form the Hopf algebroid $(C, \Gamma \otimes_B C)$ by the construction given in Proposition 1.3.1. We will refer to $(C, \Gamma \otimes_B C)$ as the *one-sided base-change Hopf algebroid* of (B, Γ) along $B \to C$. The left Γ-coaction on C is necessary in order to define the right unit map on the one-sided base change Hopf algebroid, and making different choices of left Γ-coaction on C will yield different Hopf algebroids $(C, \Gamma \otimes_B C)$. Specializing to the case of formal A-module 2-buds: consider the situation where the Hopf algebroid (B, Γ) is the Hopf $\mathbb{Z}$-algebr$a$ representing the additive group scheme $\mathbb{G}_a$ over A. Concretely, (B, Γ) is the Hopf algebra $(A, A[t])$, with t primitive. We have a left Γ-coaction on the ring L^A_{2-buds} given by the A-algebra map

$$\begin{aligned}\psi : L^A_{2-buds} &\to A[t] \otimes_A L^A_{2-buds} \\ d &\mapsto 1 \otimes d + 2t \otimes 1 \\ c_a &\mapsto 1 \otimes c_a + (a^2 - a)t \otimes 1.\end{aligned} \tag{3.30}$$

The map ψ is simply the ring map classifying the underlying formal A-module of the *target* of the universal strict isomorphism of formal A-modules. From (3.24) through (3.28), we see that $(L^A_{2-buds}, L^A_{2-buds}B)$ is simply the one-sided base change of $(A, A[t])$ along the map ψ.

One useful consequence, which we use in computations throughout the rest of this paper, is an isomorphism in Cotor: by the change-of-rings isomorphism of Proposition 1.3.1, the identification of $(L^A_{2-buds}, L^A_{2-buds}B)$ as a one-sided base change Hopf algebroid yields isomorphisms of bigraded A-modules

$$H^s_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes t}) \cong \mathrm{Cotor}^{s,2t}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds}) \tag{3.31}$$

$$\cong \mathrm{Cotor}^{s,2t}_{A[t]}(A, L^A_{2-buds}). \tag{3.32}$$

3.4. **Calculation of $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes *})$.** Now we begin to make cohomological calculations. The global sections (i.e., H^0) of the line bundles $\omega^{\otimes *}$ over $\mathcal{M}^{2-buds}_{fmA}$ are already nontrivial and interesting. The graded abelian group

$$\Gamma(\omega^{\otimes *}, \mathcal{M}^{2-buds}_{fmA}) \cong H^0_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes *}) \cong \mathrm{Cotor}^{0,2*}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds})$$

is simply the cotensor product $L^A_{2-buds} \square_{L^A_{2-buds}B} L^A_{2-buds}$, i.e., the kernel of the difference

$$\eta_R - \eta_L : L^A_{2-buds} \to L^A_{2-buds}B$$

of the unit maps on the Hopf algebroid $(L^A_{2-buds}, L^A_{2-buds}B)$. Let A be a commutative ring of characteristic zero. It follows from the presentation for $(L^A_{2-buds}, L^A_{2-buds}B)$ given in (3.24) through (3.28) that we have a commutative square

$$\begin{tikzcd} L^A_{2-buds} \arrow[r, "\eta_R - \eta_L"] \arrow[d, "\cong"] & L^A_{2-buds}B \arrow[d, "\cong"] \\ \mathrm{Sym}_A(\mathrm{Dr}^A) \arrow[r, "\delta"] & \mathrm{Sym}_A(\mathrm{Dr}^A)[t] \end{tikzcd}$$

where δ is the A-module homomorphism which is given as follows:

- On the zeroth symmetric power $\mathrm{Sym}^0_A(\mathrm{Dr}^A) = A$ of Dr^A, δ is the zero map.
- On the first symmetric power $\mathrm{Sym}^1_A(\mathrm{Dr}^A) = \mathrm{Dr}^A$ of Dr^A, δ is the *fundamental functional* σ_2, defined in Definition 2.4.4,
$$\sigma_2 : \mathrm{Dr}^A \to A = A\{t\} \subseteq \mathrm{Sym}^0_A(\mathrm{Dr}^A)[t] \subseteq \mathrm{Sym}_A(\mathrm{Dr}^A)[t].$$
That is, $\delta(\gamma) = 2t$ and $\delta(c_a) = (a^2 - a)t$.
- On the second symmetric power $\mathrm{Sym}^2_A(\mathrm{Dr}^A)$ of Dr^A, δ is given by
$$\delta(xy) = ((\sigma_2 x)y + x(\sigma_2 y))\,t + (\sigma_2 x)(\sigma_2 y)t^2, \tag{3.33}$$
where $x, y \in \mathrm{Dr}^A$.

- On the third symmetric power $\mathrm{Sym}^3_A(\mathrm{Dr}^A)$ of Dr^A, δ is given by

$$\begin{aligned}\delta(xyz) = &((\sigma_2 x)yz + x(\sigma_2 y)z + xy(\sigma_2 z))\,t \\ &+ ((\sigma_2 x)(\sigma_2 y)z + (\sigma_2 x)y(\sigma_2 z) + x(\sigma_2 y)(\sigma_2 z))\,t^2 \\ &+ (\sigma_2 x)(\sigma_2 y)(\sigma_2 z)t^3,\end{aligned} \tag{3.34}$$

  where $x, y, z \in \mathrm{Dr}^A$.
- More generally[12], on the nth symmetric power $\mathrm{Sym}^n_A(\mathrm{Dr}^A)$ of Dr^A, δ is given by

$$\delta(x_1 \dots x_n) = \sum_{i=1}^n \sum_{U \in P_i(\{1,\dots,n\})} \sigma_2^U(x_1,\dots,x_n)t^i, \tag{3.35}$$

  where $P_i(\{1,\dots,n\})$ is the set of i-element subsets of $\{1,\dots,n\}$, and where $\sigma_2^U(x_1,\dots,x_n)$ is the product of the elements $x_1 \dots x_n$ with σ_2 applied to each x_j such that $j \in U$.

Hence our task is to calculate the kernel of δ. We accomplish this in Theorem 3.4.1. First, we need to define a certain function τ_n. Suppose that I is an ideal in a commutative ring A. Write $\iota : I \to A$ for the inclusion map. For each positive integer n, we can consider the nth symmetric power $\mathrm{Sym}^n_A(I)$. Given an element $x_1 \cdot x_2 \cdot \dots \cdot x_n$ of $\mathrm{Sym}^n_A(I)$, we could take one of the elements $x_j \in I$ and regard it as an element of A, with the idea that $x_1 \cdot \ldots x_{j-1} \cdot \iota(x_j)x_{j+1} \cdot \ldots x_n$ is then an element of $\mathrm{Sym}^{n-1}_A(I)$. That construction would not quite yield a map $\mathrm{Sym}^n_A(I) \to \mathrm{Sym}^{n-1}_A(I)$: the trouble is that we cannot single out a particular factor x_j in the product $x_1 \cdot x_2 \cdot \dots \cdot x_n$, since $x_1 \cdot x_2 \cdot \dots \cdot x_n$ is an element of the *symmetric* power $\mathrm{Sym}^n_A(I)$. To get a well-defined, natural map $\mathrm{Sym}^n_A(I) \to \mathrm{Sym}^{n-1}_A(I)$, we must sum over the values $j = 1, 2, \dots, n$, yielding the A-module morphism

$$\begin{aligned}\tau_n^I : \mathrm{Sym}^n_A(I) &\to \mathrm{Sym}^{n-1}_A(I) \\ x_1 \cdot \dots \cdot x_n &\mapsto \iota(x_1)x_2 \cdot \dots \cdot x_n \\ &\quad + \iota(x_2)x_1 \cdot x_3 \cdot \dots \cdot x_n \\ &\quad + \dots + \iota(x_n)x_1 \cdot \dots \cdot x_{n-1}.\end{aligned} \tag{3.36}$$

Theorem 3.4.1. *Let A be a torsion-free commutative ring. Write I_2 for the universal $\mathbb{F}_2$-point-detecting ideal $(2, a^2 - a \text{ for all } a \in A) \subseteq A$ in A, as in section 2.4. Then we have an isomorphism of A-modules:*

$$H^0_{fl}(\mathcal{M}^{2\text{-}buds}_{fmA}; \omega^{\otimes n}) \cong \begin{cases} 0 & \text{if } n < 0 \\ A & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ \ker \tau_n^{I_2} & \text{if } n > 1, \end{cases}$$

where $\tau_n^{I_2}$ is the A-module morphism defined in (3.36).

Proof. By Proposition 2.4.5, σ_2 yields an isomorphism of A-modules $\tilde{\sigma_2} : \mathrm{Dr}^A \xrightarrow{\cong} I_2$. The projection of $\delta(x_1 \dots x_n)$ to the $(n-1)$st symmetric power $\mathrm{Sym}^{n-1}_A(\mathrm{Dr}^A)\{t\} \subseteq \mathrm{Sym}_A(\mathrm{Dr}^A)[t]$ is given by the formula $\delta(x_1 \dots x_n) = \tau_n^{I_2}(x_1 \dots x_n)t$.

[12] Of course the formula (3.35) subsumes formulas (3.33) and (3.34). We have chosen to present the ideas in this redundant way, because it is easier to read formulas (3.33) and (3.34), and to grasp the simple pattern they fit into, than to parse the general formula (3.35).

Consequently, for an element of $\mathrm{Sym}_A^n(\mathrm{Dr}^A)$ to be in the kernel of δ, that element must be in the kernel of $\tau_n^{I_2}$. We claim that the converse is also true. That is, we claim that the kernel of $\delta\mid_{\mathrm{Sym}_A^n(\mathrm{Dr}^A)}$ is precisely the kernel of the map $\tau_n^{I_2}$. To prove this claim, suppose that $x \in \mathrm{Sym}_A^n(\mathrm{Dr}^A)$ is in the kernel of $\tau_n^{I_2}$. We will have cause to consider a more general class of A-module homomorphisms, defined as follows for each positive integer n and each integer i such that $0 \le i \le n$:

$$\begin{aligned} \text{\textcircled{$\scriptstyle\triangle$}}_i^n : \mathrm{Sym}_A^n(\mathrm{Dr}^A) &\to \mathrm{Sym}_A^{n-i}(\mathrm{Dr}^A) \\ x_1 \dots x_n &\mapsto \sum_{U \in P_i(\{1,\dots,n\})} \sigma_2^U(x_1,\dots,x_n). \end{aligned}$$

We introduce the functions $\text{\textcircled{$\scriptstyle\triangle$}}_i^n$ because of the following three observations:

(1) The formula (3.35) for δ is equivalent to

$$\delta(x_1 \dots x_n) = \sum_{i=1}^{n} \text{\textcircled{$\scriptstyle\triangle$}}_i^n(x_1 \dots x_n) t^i. \tag{3.37}$$

(2) By a simple combinatorial argument, the composite

$$\text{\textcircled{$\scriptstyle\triangle$}}_k^{i-j} \circ \text{\textcircled{$\scriptstyle\triangle$}}_j^i : \mathrm{Sym}_A^i(\mathrm{Dr}^A) \to \mathrm{Sym}_A^{i-j-k}(\mathrm{Dr}^A)$$

is equal to the binomial coefficient $\binom{j+k}{j}$ times the map $\text{\textcircled{$\scriptstyle\triangle$}}_{j+k}^i$.

(3) $\text{\textcircled{$\scriptstyle\triangle$}}_1^n = \tau_n^{I_2}$.

Consequently $\text{\textcircled{$\scriptstyle\triangle$}}_i^n$ is equal to a product of nonzero binomial coefficients times the composite $\text{\textcircled{$\scriptstyle\triangle$}}_1^{i+1} \circ \dots \circ \text{\textcircled{$\scriptstyle\triangle$}}_1^{n-1} \circ \text{\textcircled{$\scriptstyle\triangle$}}_1^n$. Since A is torsion-free and since x was assumed to be in the kernel of $\tau_n^{I_2} = \text{\textcircled{$\scriptstyle\triangle$}}_1^n$, we now have that x is in the kernel of $\text{\textcircled{$\scriptstyle\triangle$}}_i^n$ for each i. Formula (3.37) now yields that x is in the kernel of δ. Hence $\mathrm{Cotor}^{0,2n}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds}) \cong \ker\delta$ coincides with the kernel of $\tau_n^{I_2}$, as claimed. □

In particular, if A is torsion-free, then $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes 2})$ is isomorphic to the kernel of the canonical multiplication map

$$\mathrm{Sym}_A^2(I_2) \to I_2^2, \tag{3.38}$$

i.e., the kernel of the canonical comparison map from the symmetric square of I_2 to the Rees module $\mathrm{Rees}_A^2(I_2)$. If A is a domain, then $I_2^2 \subseteq A$ is torsion-free, so the kernel of the multiplication map (3.38) must be torsion. Torsion in $\mathrm{Sym}_A(I)$ is well-studied in commutative algebra: see [10] for a nice entry-point into the literature. The kernel of the multiplication map $\mathrm{Sym}_A^2(I) \to I^2$, in particular, coincides with the *delta-invariant* of a finitely-generated ideal I: see [20], or Corollary 1.2 of [31]. Since the delta-invariant $\delta(I)$ is known to agree with the second Andre-Quillen homology group $H_2(A, A/I; A/I)$ of A/I regarded as an A-algebra, with coefficients in A/I, we have:

Corollary 3.4.2. *Let A be a Noetherian integral domain of characteristic zero. Then the following A-modules are isomorphic:*

- *The sections $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes 2})$ of the second tensor power $\omega^{\otimes 2}$ of the bundle of invariant differentials on $\mathcal{M}^{2-buds}_{fmA}$.*
- *The Andre-Quillen homology group $H_2(A, A/I_2; A/I_2)$.*

Since at least the 1980s, the kernel of the map $\mathrm{Sym}^n_A(I) \to I^n$ has drawn attention in commutative algebra[13], especially in the case $n = 2$. For example, an ideal I in an integral domain R is called *syzygetic* if the map $\mathrm{Sym}^2_A(I) \to I^2$ is injective (equivalently, an isomorphism); see [4] for some discussion and relevant results. We have:

Corollary 3.4.3. *Let A be a Noetherian integral domain of characteristic zero. Then $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes 2})$ vanishes if and only if the universal $\mathbb{F}_2$-point-detecting ideal of A is syzygetic.*

If I_2 can be generated by a regular sequence, or more generally a d-sequence in the sense of Huneke [10], then $\mathrm{Sym}^n_A(I_2) \to I_2^n$ is an isomorphism for *all* n, and consequently $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes n})$ vanishes for $n > 0$. This explains why, in Theorem 3.2 of [26], Ravenel obtained the vanishing of $\mathrm{Cotor}^{0,n}_{L^AB}(L^A, L^A)$ for all $n \neq 0$ and all number rings A: the same would be true for any regular integral domain of characteristic zero. More generally:

Theorem 3.4.4. *If A is a Cohen-Macaulay integral domain of characteristic zero, then $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes n})$ is trivial for all $n \neq 0$.*

Proof. Since A is Cohen-Macaulay, the ideal I_2 can be generated by a regular sequence, so the multiplication map

$$\mathrm{Sym}^n_A(I_2) \to I_2^n \tag{3.39}$$

is injective for all n. The map (3.39) is equal to the function $\text{\textcircled{$\scriptstyle\triangle$}}^n_n$ defined in the proof of Theorem 3.4.1. In that proof, it was shown that $\text{\textcircled{$\scriptstyle\triangle$}}^n_n$ factors as a product of nonzero binomial coefficients times the composite $\text{\textcircled{$\scriptstyle\triangle$}}^2_1 \circ \dots \circ \text{\textcircled{$\scriptstyle\triangle$}}^{n-1}_1 \circ \text{\textcircled{$\scriptstyle\triangle$}}^n_1$. Hence the injectivity of (3.39) implies the injectivity of the composite

$$\text{\textcircled{$\scriptstyle\triangle$}}^{i+1}_1 \circ \dots \circ \text{\textcircled{$\scriptstyle\triangle$}}^{n-1}_1 \circ \text{\textcircled{$\scriptstyle\triangle$}}^n_1 \tag{3.40}$$

for each $i = 1, \dots, n-1$. The composite (3.40) is equal to a product of nonzero binomial coefficients times $\text{\textcircled{$\scriptstyle\triangle$}}^n_i$, so since A is an integral domain of characteristic zero, $\text{\textcircled{$\scriptstyle\triangle$}}^n_i$ is injective for all j. Hence, by equation (3.37), the cobar complex differential morphism $\delta : \mathrm{Sym}_A(\mathrm{Dr}^A) \to \mathrm{Sym}_A(\mathrm{Dr}^A)[t]$ is injective in positive internal degrees > 0, so $\mathrm{Cotor}^{0,n}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds})$ vanishes for $n > 0$. □

3.5. **Consequences for torsion in L^A.** To date, there is no known example of an integral domain A such that the classifying rings L^A or L^A_{2-buds} have nontrivial A-torsion. The H^0 calculations from section 3.4, together with Theorem 2.2.1, yield some insight about when and why L^A_{2-buds} is torsion-free. In this section, we obtain the first known example of an integral domain A such that L^A_{2-buds} is *not* torsion-free.

We begin with a simple observation for the case where A is a field:

Proposition 3.5.1. *Let K be a field of characteristic zero. Then $H^0_{fl}(\mathcal{M}^{2-buds}_{fmK};\omega^{\otimes n})$ is trivial for $n \neq 0$.*

[13] For example, in the paper [12], it is remarked that "Finding the defining equations of Rees rings is a classical problem in elimination theory that amounts to determining the kernel $\mathcal{A}$ of the natural map from the symmetric algebra $\mathrm{Sym}(I)$ onto $\mathcal{R}$."

Proof. Since K is a field, the symmetric powers $\mathrm{Sym}^n_K(I_2)$ are all free K-modules, hence torsion-free. Consequently $\tau_n^{I_2}$ is injective. Now Theorem 3.4.1 yields the result. □

Proposition 3.5.2. *Let A be an integral domain of characteristic zero. Then, for each $n > 0$, the A-module $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes n})$ is A-torsion, and an A-submodule of L^A_{2-buds}.*

Proof. We have isomorphisms

$$\begin{aligned} H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes *}) &\cong \mathrm{Cotor}^{0,2*}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds}) \\ &\cong L^A_{2-buds} \,\square_{L^A_{2-buds}B}\, L^A_{2-buds} \subseteq L^A_{2-buds}, \end{aligned}$$

so $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes n})$ is an A-submodule of the degree n summand of L^A_{2-buds}.

Write K for the field of fractions of A. From the localization theorem (Theorem 2.2.1) and Proposition 3.5.1, we have isomorphisms

$$\begin{aligned} 0 &\cong H^0_{fl}(\mathcal{M}^{2-buds}_{fmK};\omega^{\otimes n}) \\ &\cong H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes n}) \otimes_A K \end{aligned}$$

for $n \neq 0$. Hence $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes n})$ must be torsion for $n \neq 0$. □

Corollary 3.4.3 and Proposition 3.5.2 now yield:

Corollary 3.5.3. *Let A be an integral domain of characteristic zero. If the ideal I_2 in A is not syzygetic, then L^A_{2-buds} is not torsion-free.*

Example 3.5.4. Let $A = \mathbb{Z}[a,b,c,x]/(2a - (x^2 - x)b,\ 2b - (x^2 - x)c)$. Write $\hat{x}$ as shorthand for $x^2 - x$. Then $\hat{x}a$ and $\hat{x}b$ and $\hat{x}c$ are elements of the ideal I_2 of A. It is routine to check that the product

$$\hat{x}a \cdot \hat{x}c - \hat{x}b \cdot \hat{x}b \in \mathrm{Sym}^2_A(I_2)$$

is nonzero, and furthermore that

$$2(\hat{x}a \cdot \hat{x}c - \hat{x}b \cdot \hat{x}b) = \hat{x}b \cdot 2\hat{x}b - 2\hat{x}b \cdot \hat{x}b = 0.$$

Consequently there is a nonzero 2-torsion element of $\mathrm{Sym}^2_A(I_2) \cong H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes 2})$, hence also a nonzero 2-torsion element in L^A_{2-buds}. To the author's knowledge, this is the first known example of an integral domain A such that L^A_{2-buds} has nontrivial torsion.

4. Cohomology of $\mathcal{M}^{2-buds}_{fm\mathbb{Z}}$ in low degrees.

We consider the moduli stack $\mathcal{M}^{2-buds}_{fmA}$ of formal A-module 2-buds in the case $A = \mathbb{Z}$. We will calculate the first cohomology group of the moduli stack of formal group 2-buds, i.e., $\mathrm{Cotor}^{1,*}_{L^{\mathbb{Z}}_{2-buds}B}(L^{\mathbb{Z}}_{2-buds}, L^{\mathbb{Z}}_{2-buds})$. The author does not know where this specific calculation appears in the literature, but the author does not believe that this particular calculation should be seen as especially new: it is quite similar to 2-primary calculations of portions of the Adams-Novikov spectral sequence E_2-term, and also to various standard 2-primary calculations in Iwasawa theory. Nevertheless it is worth the effort to present the calculation here, as the base case $A = \mathbb{Z}$ will be used in section 5 as input for the extension-of-formal-multiplications spectral sequence which converges to $H^*_{fl}(\mathcal{M}_{fmA};\omega^{\otimes *})$ for rings A other than $\mathbb{Z}$.

4.1. **Construction of the 2-adic spectral sequence.** The $\mathbb{Z}$-module $\mathrm{Dr}^{\mathbb{Z}}$ is free on the generator γ, and consequently the $\mathbb{Z}[t]$-comodule algebra $L^{\mathbb{Z}}_{2-buds}$ is isomorphic to $\mathbb{Z}[\gamma]$, with the coaction

$$\psi : \mathbb{Z}[\gamma] \to \mathbb{Z}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[\gamma]$$
$$\psi(\gamma) = 1 \otimes \gamma + t \otimes 2, \tag{4.41}$$

as described above in (3.30). The ideal (2) of $L^{\mathbb{Z}}_{2-buds}$ is closed under the $\mathbb{Z}[t]$-coaction, i.e., the (2)-adic filtration

$$L^{\mathbb{Z}}_{2-buds} \supseteq (2) \supseteq (2)^2 \supseteq (2)^3 \supseteq \dots$$

of $L^{\mathbb{Z}}_{2-buds}$ is a multiplicative filtration and also a filtration by subcomodules. Writing $E_0^{(2)} L^{\mathbb{Z}}_{2-buds}$ for the associated graded comodule of the 2-adic filtration on $L^{\mathbb{Z}}_{2-buds}$, we get a multiplicative conditionally convergent spectral sequence[14]

$$E_1^{p,q,u} \cong \mathrm{Cotor}^{p,q,u}_{\mathbb{Z}[t]}(\mathbb{Z}, E_0^{(2)} L^{\mathbb{Z}}_{2-buds}) \Rightarrow \mathrm{Cotor}^{p,q}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})\hat{}_2 \tag{4.42}$$
$$\cong H^p_{fl}(\mathcal{M}_{fm\mathbb{Z}}; \omega^{\otimes q/2})\hat{}_2 \tag{4.43}$$
$$d_r : E_r^{p,q,u} \to E_r^{p+1,q,u+r}.$$

We will refer to spectral sequence (4.42) as the 2*-adic spectral sequence.* To be clear about the notation: the abutment $\mathrm{Cotor}^{*,*}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})\hat{}_2$ is the 2-adic completion of $\mathrm{Cotor}^{*,*}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})$. The spectral sequence is trivial for odd q, so the tensor power $\omega^{\otimes q/2}$ is well-defined.

The associated graded $\mathbb{Z}[t]$-comodule algebra $E_0^{(2)} L^{\mathbb{Z}}_{2-buds}$ of the 2-adic filtration on $L^{\mathbb{Z}}_{2-buds}$ is isomorphic to $\mathbb{F}_2[\tilde{2}, \gamma]$, with $\tilde{2}$ in 2-adic filtration degree 1, and with trivial coaction. Consequently the E_1-term of the 2-adic spectral sequence is isomorphic to

$$\mathrm{Cotor}^{*,*,*}_{\mathbb{Z}[t]}\left(\mathbb{Z}, \mathbb{F}_2[\tilde{2}, \gamma]\right) \cong \mathrm{Cotor}^{*,*,*}_{\mathbb{F}_2[t]}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tilde{2}, \gamma]$$
$$\cong \mathbb{F}_2[\eta, P\eta, P^2\eta, P^3\eta, \dots] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tilde{2}, \gamma], \tag{4.44}$$

with tridegrees $\tilde{2} \in E_1^{0,0,1}$, and $\gamma \in E_1^{0,2,0}$, and $P^j\eta \in E_1^{1,2^{j+1},0}$. Isomorphism (4.44) is due to the isomorphism

$$\mathrm{Cotor}^{*,*}_{\mathbb{F}_2[t]}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[\eta, P\eta, P^2\eta, P^3\eta, \dots], \tag{4.45}$$

where

- $P^n\eta \in \mathrm{Cotor}^{1,2^{n+1}}_{\mathbb{F}_2[t]}(\mathbb{F}_2, \mathbb{F}_2)$,
- $P^n\eta$ is represented in the cobar complex of $(\mathbb{F}_2, \mathbb{F}_2[t])$ by the 1-cocycle t^{2^n},
- and P is the algebraic Steenrod operation P^0 which operates in Cotor by applying the Frobenius operation to cocycle representatives in the cobar complex. For these ideas, see the material on algebraic Steenrod operations in Appendix 1 of [27], or [19].

[14]It is automatic that this spectral sequence converges to $\mathrm{Cotor}^{p,q}_{\mathbb{Z}[t]}\left(\mathbb{Z}, (L^{\mathbb{Z}}_{2-buds})\hat{}_2\right)$. The fact that it also converges to $\mathrm{Cotor}^{p,q}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})\hat{}_2$ is a consequence of the finite generation of $L^{\mathbb{Z}}_{2-buds}$ and of $L^{\mathbb{Z}}_{2-buds}B$ in each degree. Similar spectral sequence convergence results for rings A other than $\mathbb{Z}$, and for the full moduli stack of formal A-modules and not merely of formal A-module 2-buds, and for various filtrations including (but not limited to) the 2-adic filtration, follow from the finiteness and completion results in section 2.3.

A straightforward way to see isomorphism (4.45) is to observe that $\mathbb{F}_2[t]$ splits, *as a coalgebra*, as the tensor product of the tensor factors $\mathbb{F}_2[t^{2^n}]/(t^{2^n})^2$ over all $n \geq 0$. The Cotor-algebra of $\mathbb{F}_2[\epsilon]/\epsilon^2$ is polynomial on a single generator in cohomological degree 1, represented by the 1-cocycle ϵ in the cobar complex of $\mathbb{F}_2[\epsilon]/\epsilon^2$. The cobar complex multiplication is concatenation of tensors, and in the case of $\mathbb{F}_2[t]$, it is routine to make the cocycle-level calculation to verify that the coalgebra splitting

$$\mathbb{F}_2[t] \cong \bigotimes_{n\geq 0} \mathbb{F}_2[t^{2^n}]/(t^{2^n})^2 \tag{4.46}$$

induces a ring isomorphism

$$\mathrm{Cotor}^*_{\mathbb{F}_2[t]}(\mathbb{F}_2,\mathbb{F}_2) \cong \bigotimes_{n\geq 0} \mathrm{Cotor}^*_{\mathbb{F}_2[t^{2^n}]/(t^{2^n})^2}(\mathbb{F}_2,\mathbb{F}_2)$$

despite (4.46) not respecting the ring structure.

4.2. **Running the 2-adic spectral sequence:** $H^n_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}};\omega^{\otimes *})$ **for** $n = 0,1$. As the 2-adic spectral sequence is a multiplicative spectral sequence, to calculate the d_1-differential it suffices to calculate the d_1 differential on the generators $\tilde{2},\gamma$, and $P^j\eta$ for each $j = 0,1,2,\dots$. We accomplish this using cocycle representatives for each such generator in the cobar complex for $\mathbb{Z}[t]$ with coefficients in $E_0^{(2)}L^{\mathbb{Z}}_{2-buds}$:

- Since $\tilde{2}$ is represented by the 0-cocycle $\tilde{2}$, which lifts to the 0-cocycle 2 in the cobar complex for $\mathbb{Z}[t]$ with coefficients in $L^{\mathbb{Z}}_{2-buds}$, we have
$$d_1(\tilde{2}) = 0,$$
and in fact $\tilde{2}$ is an infinite cocycle.
- The generator γ is represented by the 0-cocycle γ, which lifts to the 0-cochain γ in the cobar complex for $\mathbb{Z}[t]$ with coefficients in $L^{\mathbb{Z}}_{2-buds}$. We have
$$\delta(\gamma) = t\otimes 2 \in \mathbb{Z}[t]\otimes_{\mathbb{Z}} L^{\mathbb{Z}}_{2-buds}$$
in that cobar complex, and since $t\otimes 2$ represents a cocycle representative for $\tilde{2}\eta$ in the cobar complex for $\mathbb{Z}[t]$ with coefficients in $E_0^{(2)}L^{\mathbb{Z}}_{2-buds}$, we have
$$d_1(\gamma) = \tilde{2}\eta.$$
- The generator $P^j\eta$ is represented by the 1-cocycle $t^{2^j}\otimes 1$, which lifts to the 1-cochain $t^{2^j}\otimes 1$ in the cobar complex for $\mathbb{Z}[t]$ with coefficients in $L^{\mathbb{Z}}_{2-buds}$. We have
$$\delta(t^{2^j}\otimes 1) = \sum_{i=1}^{2^j}\binom{2^j}{i} t^{2^j-i}\otimes t^i\otimes 1$$
in that cobar complex, whose unique term of least 2-adic filtration is $\binom{2^j}{2^{j-1}}t^{2^{j-1}}\otimes t^{2^{j-1}}\otimes 1$, since the central binomial coefficient $\binom{2^j}{2^{j-1}}$ has 2-adic valuation 1. Consequently
$$d_1(P^j\eta) = \frac{\binom{2^j}{2^{j-1}}}{2}\tilde{2}(P^{j-1}\eta)^2.$$

- Consequently, by the Leibniz rule, we have

$$d_1\left(\tilde{2}^h\gamma^i P^j\eta\right) = \tilde{2}^{h+1}\gamma^{i-1}\left(i\eta P^j\eta + \gamma\frac{\binom{2^j}{2^{j-1}}}{2}(P^{j-1}\eta)^2\right),$$

with the understanding that negative powers of γ and of P are zero. Consequently the Cotor^1-line in the E_2-page of the 2-adic spectral sequence consists of the $\mathbb{F}_2[\tilde{2}]$-linear combinations of the elements $\gamma^{2i}\eta$ and the elements $\gamma^{2i}(\gamma\eta - P\eta)$, for $i \geq 0$. We write $Q\eta$ as shorthand for $\gamma\eta - P\eta$, i.e., the cohomology class of the 1-cocycle $t\otimes\gamma - t^2\otimes 1$ in the cobar complex $C^\bullet_{\mathbb{Z}[t]}(E_0^{(2)}L^{\mathbb{Z}}_{2-buds})$ for $\mathbb{Z}[t]$ with coefficients in $E_0^{(2)}L^{\mathbb{Z}}_{2-buds}$.

In the remaining calculations in the 2-adic spectral sequence, we will be sloppy about the distinction between $t\otimes\gamma - t^2\otimes 1$ and $t\otimes\gamma + t^2\otimes 1$, since they represent the same class in the associated graded of the 2-adic filtration.

We can calculate the Cotor^0 and Cotor^1 lines on later pages by similar arguments, together with the calculations

$$\begin{aligned}
d_2(\gamma^2) &= \left[(t\otimes 2 + 1\otimes\gamma)^2 - 1\otimes\gamma^2\right]\\
&= \left[4(t^2\otimes 1 + t\otimes\gamma)\right]\\
&\sim \tilde{2}^2 Q\eta \text{ in the cobar complex } C^\bullet_{\mathbb{Z}[t]}(E_0^{(2)}L^{\mathbb{Z}}_{2-buds}),\\
d_3(\gamma^4) &= \left[(t\otimes 2 + 1\otimes\gamma)^4 - 1\otimes\gamma^4\right]\\
&= \left[2^3(2t^4\otimes 1 + 2^2t^3\otimes\gamma + 3t^2\otimes\gamma^2 + t\otimes\gamma^3)\right]\\
&\sim \tilde{2}^3\gamma^2 Q\eta \text{ in the cobar complex } C^\bullet_{\mathbb{Z}[t]}(E_0^{(2)}L^{\mathbb{Z}}_{2-buds}),\\
d_4(\gamma^8) &\sim \tilde{2}^4\gamma^6 Q\eta \text{ in the cobar complex } C^\bullet_{\mathbb{Z}[t]}(E_0^{(2)}L^{\mathbb{Z}}_{2-buds}),
\end{aligned}$$

and in general,

$$d_{r+1}(\gamma^{2^r}) \sim \tilde{2}^{r+1}\gamma^{2^r-2}Q\eta \text{ in the cobar complex } C^\bullet_{\mathbb{Z}[t]}(E_0^{(2)}L^{\mathbb{Z}}_{2-buds}). \tag{4.47}$$

We are using the symbol ~ to denote the equivalence relation "is cohomologous to, modulo terms of higher 2-adic filtration."

Differential formula (4.47) yields the following description of the Cotor^0 and Cotor^1 lines on each page of the 2-adic spectral sequence:

(4.48)

r	Cotor^0-line in the E_r-page	Cotor^1-line in the E_r-page
1	$\mathbb{F}_2[\tilde{2},\gamma]$	$\mathbb{F}_2[\tilde{2},\gamma]\left\{\eta, P\eta, P^2\eta,\dots\right\}$
2	$\mathbb{F}_2[\tilde{2},\gamma^2]$	$\mathbb{F}_2[\tilde{2},\gamma^2]\left\{\eta, Q\eta\right\}/\tilde{2}\eta$
3	$\mathbb{F}_2[\tilde{2},\gamma^4]$	$\frac{\mathbb{F}_2[\tilde{2},\gamma^4]\{\eta,Q\eta,\gamma^2\eta,\gamma^2Q\eta\}}{(\tilde{2}\eta,\tilde{2}^2Q\eta,\tilde{2}\gamma^2\eta)}$
4	$\mathbb{F}_2[\tilde{2},\gamma^8]$	$\frac{\mathbb{F}_2[\tilde{2},\gamma^8]\{\eta,Q\eta,\gamma^2\eta,\gamma^2Q\eta,\gamma^4\eta,\gamma^4Q\eta,\gamma^6\eta,\gamma^6Q\eta\}}{(\tilde{2}\eta,\tilde{2}^2Q\eta,\tilde{2}\gamma^2\eta,\tilde{2}^3\gamma^2Q\eta,\tilde{2}\gamma^4\eta,\tilde{2}^2\gamma^4Q\eta,\tilde{2}\gamma^6\eta)}$,

and in the limit, the Cotor^0-line in the E_∞-page is $\mathbb{F}_2[\tilde{2}]$, while the Cotor^1-line in the E_∞-page is

$$\mathbb{F}_2[\tilde{2}]\left\{\gamma^{2n}\eta, \gamma^{2n}Q\eta\ \forall n\geq 0\right\}/\left(\tilde{2}\gamma^{2n}\eta, \tilde{2}^{1+\nu_2(2n+2)}\gamma^{2n}Q\eta\ \forall n\geq 0\right).$$

Resolving the extension problems to pass from the E_∞-page to the abutment $\mathrm{Cotor}^{*,*}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})\hat{}_2$, we have

$$\begin{aligned}\mathrm{Cotor}^0_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})\hat{}_2 &\cong \hat{\mathbb{Z}}_2,\\ \mathrm{Cotor}^1_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})\hat{}_2 &\cong \frac{\mathbb{Z}\left\{\gamma^{2n}\eta, \gamma^{2n}Q\eta\ \forall n \geq 0\right\}}{\left(2\gamma^{2n}\eta, 2^{1+\nu_2(2n+2)}\gamma^{2n}Q\eta\ \forall n \geq 0\right)}.\end{aligned}$$

From the 2-adic completions of these Cotor-groups, it is easy to make the cocycle-level calculations and to use the finiteness results from section 2.3 to deduce that *before* 2-adic completion, we must have

$$\begin{aligned}\mathrm{Cotor}^0_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds}) &\cong \mathbb{Z},\\ \mathrm{Cotor}^1_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds}) &\cong \frac{\mathbb{Z}\left\{\gamma^{2n}\eta, \gamma^{2n}Q\eta\ \forall n \geq 0\right\}}{\left(2\gamma^{2n}\eta, 2^{1+\nu_2(2n+2)}\gamma^{2n}Q\eta\ \forall n \geq 0\right)}.\end{aligned}$$

4.3. **Running the 2-adic spectral sequence: $H^2_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}}; \omega^{\otimes n})$ for $n \leq 3$.** For later spectral sequence calculations in section 5.2, it will be useful to have calculated $\mathrm{Cotor}^{2,2n}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds}) \cong H^2_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}}; \omega^{\otimes n})$ for a few small values of n. We do this by running the 2-adic spectral sequence in internal[15] degrees ≤ 6. In principle, there is no reason that similar calculations could not be done for a much wider range of internal and cohomological degrees. However, in order to keep this paper at a manageable length, we confine our attention to only the most immediately relevant calculations.

4.3.1. *Internal degrees $q < 4$.* In these internal degrees, the 2-adic spectral sequence has no summands which contribute to Cotor^2.

4.3.2. *Internal degrees $q = 4, 5$.* In internal degree 4, the 2-adic spectral sequence E_1-page is straightforwardly calculated. The charts are as follows:

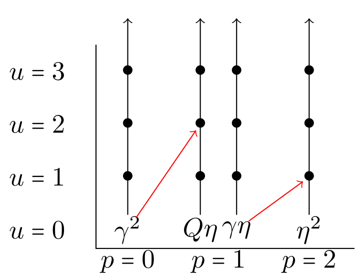


Figure 1. 2-adic SS
E_1-page, $q = 4$
$d_1(\gamma\eta) = \tilde{2}\eta^2$
$d_2(\gamma^2) = \tilde{2}^2 Q\eta$

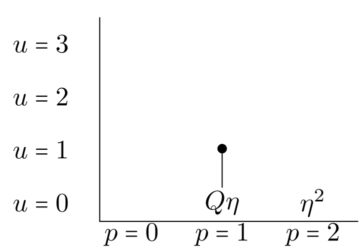


Figure 2. 2-adic SS
$E_3 \cong E_\infty$-page, $q = 4$

[15] The spectral sequence of a filtered cochain complex of abelian groups is bigraded: it has the cohomological degree, and the filtration degree. If the cochain complex is furthermore a filtered cochain complex of *graded* abelian groups, then the spectral sequence has a third grading, traditionally called the *internal* grading. For example, since $(L^A, L^A B)$ is a *graded* Hopf algebroid, the cobar complex of $(L^A, L^A B)$ is a cochain complex of *graded* A-modules, and the 2-adic spectral sequence has an *internal* grading as a consequence. The internal degree is the degree q in (4.42).

Empty bidegrees are understood to be trivial. Each nontrivial element name is understood to be an $\mathbb{F}_2$-linear basis element. Vertical black arrows indicate towers of multiplications by $\tilde{2}$, so for example, the E_1-page is a free $\mathbb{F}_2[\tilde{2}]$-algebra on the four elements $\gamma^2, Q\eta, \gamma\eta$, and η^2. The E_∞-page is $\mathbb{F}_2[\tilde{2}]\{Q\eta,\eta^2\}/\left(\tilde{2}^2 Q\eta, \tilde{2}\eta^2\right)$. Resolving the extensions, we have that

$$\begin{aligned}\operatorname{Cotor}^{n,4}_{(L^{\mathbb{Z}}_{2-buds}, L^{\mathbb{Z}}_{2-buds}B)}\left(L^{\mathbb{Z}}_{2-buds}, L^{\mathbb{Z}}_{2-buds}\right) &\cong H^n(\mathcal{M}^{2-buds}_{fm\mathbb{Z}};\omega^{\otimes 2}) \\ &\cong \begin{cases} 0 & \text{if } n=0 \\ \mathbb{Z}/4\mathbb{Z}\{Q\eta\} & \text{if } n=1 \\ \mathbb{Z}/2\mathbb{Z}\{\eta^2\} & \text{if } n=2 \\ 0 & \text{if } n>2. \end{cases}\end{aligned} \tag{4.49}$$

Since $L^{\mathbb{Z}}_{2-buds}$ is concentrated in even internal degrees, the Cotor-groups $\operatorname{Cotor}^{*,*}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})$ vanish in internal degree 5.

4.3.3. *Internal degrees* $q=6$. The 2-adic spectral sequence charts in internal degree 6 are as follows:

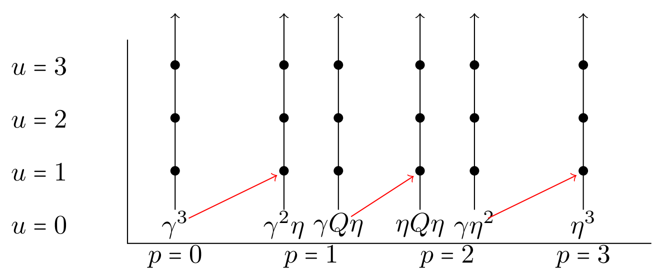


FIGURE 3. 2-adic SS
E_1-page, $q=6$
$d_1(\gamma^3) = \tilde{2}\gamma^2\eta$
$d_1(Q\eta\cdot\gamma) = \tilde{2}\eta\cdot Q\eta$
$d_1(\eta^2\cdot\gamma) = \tilde{2}\eta^3$

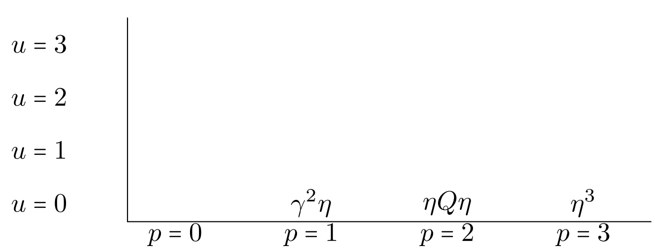


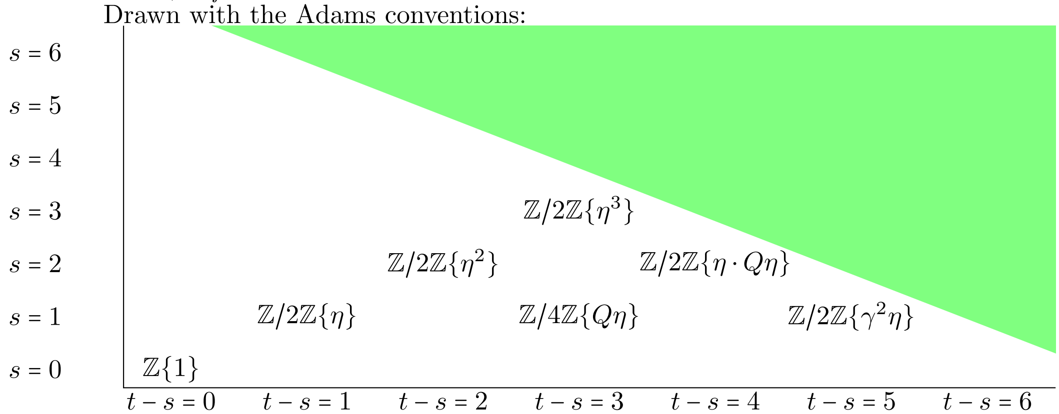
FIGURE 4. 2-adic SS
$E_3 \cong E_\infty$-page, $q=6$

There is no room for nontrivial extensions, so from the E_∞-page we have that

$$\operatorname{Cotor}^{n,6}_{(L^{\mathbb{Z}}_{2-buds}, L^{\mathbb{Z}}_{2-buds}B)}\left(L^{\mathbb{Z}}_{2-buds}, L^{\mathbb{Z}}_{2-buds}\right) \cong H^n(\mathcal{M}^{2-buds}_{fm\mathbb{Z}}; \omega^{\otimes 3})$$

$$\cong \begin{cases} 0 & \text{if } n = 0 \\ \mathbb{Z}/2\mathbb{Z}\{\gamma^2\eta\} & \text{if } n = 1 \\ \mathbb{Z}/2\mathbb{Z}\{\eta \cdot Q\eta\} & \text{if } n = 2 \\ \mathbb{Z}/2\mathbb{Z}\{\eta^3\} & \text{if } n = 3 \\ 0 & \text{if } n > 3. \end{cases}$$

Remark 4.3.1. As a consequence of the calculations in this section, we have that the bigraded ring $\coprod_{s,t} H^s_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}}; \omega^{\otimes t})$ is 2-locally isomorphic in the range $t-s \leq 3$ to $\coprod_{s,t} H^s_{fl}(\mathcal{M}_{fm\mathbb{Z}}; \omega^{\otimes t})$, i.e., the input for the Adams-Novikov spectral sequence. Drawn with the Adams conventions:



The green-shaded region lies beyond what we have just calculated. Of course it is possible to run the 2-adic spectral sequence in higher internal degrees for a more far-reaching comparison of $H^*_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}}; \omega^{\otimes *})$ with the E_2-term $H^*_{fl}(\mathcal{M}_{fm\mathbb{Z}}; \omega^{\otimes *})$ of the Adams-Novikov spectral sequence, but in this paper our priority is on making calculations of $H^*_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{A}}; \omega^{\otimes *})$ in low bidegrees for a very wide range of rings A. Those calculations begin in the next section, and the low-degree calculations of $H^*_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}}; \omega^{\otimes *})$ here are only in the service of those in the next section.

5. H^0 and H^1 of the moduli of formal A-module 2-buds.

5.1. **The symmetric filtration on symmetric powers.** In section 5.2, we will construct and use a spectral sequence which passes from the cohomology of the moduli stack of formal $\mathbb{Z}$-module 2-buds to the cohomology of the moduli stack of formal A-module 2-buds. The construction of this spectral sequence relies on a certain filtration of symmetric powers. This subsection is about that filtration. The filtration is constructed in Definition 5.1.4, and its associated graded is calculated in Proposition 5.1.5.

The filtration is, at least under certain hypotheses, quite well-known. Given a commutative ring R and a short exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0, \tag{5.50}$$

we ask for an increasing R-module filtration on the symmetric algebra $\operatorname{Sym}^*_R(M)$ such that the associated graded R-algebra $E^0 \operatorname{Sym}^*_R(M)$ is isomorphic to

$\mathrm{Sym}^*_R(M') \otimes_R \mathrm{Sym}^*_R(M'')$. It is standard that such a filtration exists, *if the R-modules M, M', M'' are projective*; see exercise 5.16 in chapter II of [7], for example.

However, to build the spectral sequence that we will use in section 5.2, it will be necessary to relax those hypotheses slightly. The essential condition is that each of the maps $s_1, s_2, \dots$ in a certain sequence, (5.56), are one-to-one. It turns out that this condition is satisfied in the case of interest in section 5.2, even though certain of the R-modules involved are not projective, and even though the short exact sequence (5.50) will not split.

The results in this section are elementary, but technical, involving colimits over certain "truncated-cube-shaped" diagrams. The author apologizes for not being able to find a simpler way to present the ideas. Surely these ideas cannot be new, and must be well-known within some circles, but we were unable to find a reference in the literature.

Now we begin the relevant definitions. First we must introduce the indexing categories for certain colimits.

Definition 5.1.1.

- *Let $\mathcal{I}$ denote the category with*
  - *two objects,* 0 *and* 1*,*
  - *a single homomorphism* $0 \to 1$*,*
  - *and no non-identity endomorphisms.*

  In other words, $\mathcal{I}$ is the partially-ordered set $\{0,1\}$, regarded as a category.
- *Let n be a nonnegative integer. Let $\mathcal{I}^n$ be the n-fold Cartesian product category $\mathcal{I} \times \dots \times \mathcal{I}$. That is, $\mathcal{I}^n$ is the partially-ordered set of n-tuples $\{0,1\} \times \dots \times \{0,1\}$, regarded as a category. The relevant partial ordering is the one in which $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ if and only if $a_i \leq b_i$ for all i.*
- *Let i, n be nonnegative integers, with $i \leq n$. Let $\mathcal{I}^n_i$ denote the full subcategory of $\mathcal{I}^n$ containing precisely those objects $(a_1, \dots, a_n)$ such that $\sum_j a_j \leq i$.*

For example, when $n = 2$, we have the following pictures of $\mathcal{I}^2_0, \mathcal{I}^2_1$, and $\mathcal{I}^2_2 = \mathcal{I}^2$:

$$(0,0)$$

FIGURE 5. $\mathcal{I}^2_0$

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \\ (0,1) & \end{tikzcd}$$

FIGURE 6. $\mathcal{I}^2_1$

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\ (0,1) \arrow[r] & (1,1) \end{tikzcd}$$

FIGURE 7. $\mathcal{I}^2_2 = \mathcal{I}^2$

When $n = 3$, we have the following pictures:

$$
\begin{tikzcd}
(0,0,0)
\end{tikzcd}
$$

FIGURE 8. $\mathcal{I}_0^3$

$$
\begin{tikzcd}
(0,0,0) \arrow[r] \arrow[d] \arrow[ddr] & (1,0,0) \\
(0,1,0) & \\
 & (0,0,1)
\end{tikzcd}
$$

FIGURE 9. $\mathcal{I}_1^3$

$$
\begin{tikzcd}
(0,0,0) \arrow[r] \arrow[d] \arrow[ddr] & (1,0,0) \arrow[d] \arrow[ddr] & \\
(0,1,0) \arrow[r] \arrow[ddr] & (1,1,0) & \\
 & (0,0,1) \arrow[r] \arrow[d] & (1,0,1) \\
 & (0,1,1) &
\end{tikzcd}
$$

FIGURE 10. $\mathcal{I}_2^3$

$$
\begin{tikzcd}
(0,0,0) \arrow[r] \arrow[d] \arrow[ddr] & (1,0,0) \arrow[d] \arrow[ddr] & \\
(0,1,0) \arrow[r] \arrow[ddr] & (1,1,0) \arrow[ddr] & \\
 & (0,0,1) \arrow[r] \arrow[d] & (1,0,1) \arrow[d] \\
 & (0,1,1) \arrow[r] & (1,1,1)
\end{tikzcd}
$$

FIGURE 11. $\mathcal{I}_3^3 = \mathcal{I}^3$

Definition 5.1.2. *Let R be a commutative ring, let n be a nonnegative integer, let M_0, M_1 be R-modules, and let $f : M_0 \to M_1$ be an R-module homomorphism.*

- *Let $\mathcal{F} : \mathcal{I}^n \to \mathrm{Mod}(R)$ be the functor given by sending $(a_1, \dots, a_n)$ to $M_{a_1} \otimes_R \dots \otimes_R M_{a_n}$.*
- *For each nonnegative integer $i \leq n$, we have the restriction $\mathcal{F} \mid_{\mathcal{I}_i^n} : \mathcal{I}_i^n \to \mathrm{Mod}(R)$ of $\mathcal{F}$ to the full subcategory $\mathcal{I}_i^n$ of $\mathcal{I}^n$.*
- *For each nonnegative integer $i < n$, the restriction functor $\mathrm{res}_i : \mathrm{Mod}(R)^{\mathcal{I}_{i+1}^n} \to \mathrm{Mod}(R)^{\mathcal{I}_i^n}$ admits a left adjoint. We write L_i for this left adjoint.*

Of course $\mathrm{colim}\, \mathcal{F}$ is simply the n-fold tensor power of M_1, and using the natural action of Σ_n on $\mathcal{I}_n$, we have $(\mathrm{colim}\, \mathcal{F})_{\Sigma_n} \cong \mathrm{Sym}_R^n(M_1)$. The idea of introducing the subcategories $\mathcal{I}_i^n$ of $\mathcal{I}^n$ is to obtain a useful filtration of the symmetric power $\mathrm{Sym}_R^n(M_1)$.

Lemma 5.1.3. *Let $R, n, M_0, M_1, f, \mathcal{F}$ be as in Definition 5.1.2. Let $\tilde{M}_0$ denote M_0, and let $\tilde{M}_1$ denote the cokernel of $f : M_0 \to M_1$. For each nonnegative integer $i < n$, the cokernel of the counit map*

$$L_i \, \mathrm{res}_i(\mathcal{F} \mid_{\mathcal{I}_{i+1}^n}) \to \mathcal{F} \mid_{\mathcal{I}_{i+1}^n} \tag{5.51}$$

is the functor $\mathcal{I}_{i+1}^n \to \mathrm{Mod}(R)$ that sends $(a_1, \dots, a_n)$ to 0 if $\sum_j a_j \leq i$, and sends $(a_1, \dots, a_n)$ to $\tilde{M}_{a_1} \otimes \dots \otimes \tilde{M}_{a_n}$ if $\sum_j a_j = i + 1$.

Proof. By the pointwise formula for Kan extensions (classical; see [18] for example), $L_i \operatorname{res}_i(\mathcal{F}\mid_{\mathcal{I}^n_{i+1}})$ is given as follows:

(5.52)
$$L_i \operatorname{res}_i(\mathcal{F}\mid_{\mathcal{I}^n_{i+1}})(a_1,\dots,a_n) = \begin{cases} \mathcal{F}(a_1,\dots,a_n) & \text{if } \sum_j a_j \leq i \\ \operatorname{colim}_{(b_1,\dots,b_n)<(a_1,\dots,a_n)} \mathcal{F}(b_1,\dots,b_n) & \text{if } \sum_j a_j = i+1. \end{cases}$$

using the partial ordering on $\mathcal{I}^n$ from Definition 5.1.1.

Cokernels in functor categories are computed levelwise, so the fact that (5.52) coincides with $\mathcal{F}$ if $\sum_j a_j \leq i$ tells us that the cokernel $c_i : \mathcal{I}^n_{i+1} \to \operatorname{Mod}(R)$ of the map (5.51) vanishes on all tuples $(a_1,\dots,a_n)$ such that $\sum_j a_j \leq i$, as claimed.

As for those tuples $(a_1,\dots,a_n)$ such that $\sum_j a_j = i+1$: let $\mathcal{I}_{<(a_1,\dots,a_n)}$ denote the full subcategory of $\mathcal{I}^n_{i+1}$ consisting of those tuples $(b_1,\dots,b_n)$ satisfying $(b_1,\dots,b_n) < (a_1,\dots,a_n)$. Then $\mathcal{I}_{<(a_1,\dots,a_n)}$ is isomorphic to $\mathcal{I}^{i+1}_i$. Let $\tilde{\mathcal{F}}_{(a_1,\dots,a_n)} : \mathcal{I}_{<(a_1,\dots,a_n)} \to \operatorname{Mod}(R)$ denote the constant functor taking the value $\mathcal{F}(a_1,\dots,a_n)$. We have a natural map $\mathcal{F}\mid_{\mathcal{I}_{<(a_1,\dots,a_n)}} \to \tilde{\mathcal{F}}_{(a_1,\dots,a_n)}$ which is an isomorphism when evaluated on $(a_1,\dots,a_n)$. Again using the fact that cokernels are calculated levelwise in functor categories, the value of the cokernel of (5.51) at $(a_1,\dots,a_n)$ agrees with the value of the cokernel of the composite map

$$L_i \operatorname{res}_i(\mathcal{F}\mid_{\mathcal{I}^n_{i+1}})\mid_{\mathcal{I}_{<(a_1,\dots,a_n)}} \to \mathcal{F}\mid_{\mathcal{I}_{<(a_1,\dots,a_n)}} \to \tilde{\mathcal{F}}_{(a_1,\dots,a_n)} \tag{5.53}$$

at $(a_1,\dots,a_n)$.

The cokernel of the composite map (5.53) is the functor which sends $(b_1,\dots,b_n)$ to the cokernel of the map $\mathcal{F}(b_1,\dots,b_n) \to \mathcal{F}(a_1,\dots,a_n)$. In the case that $\sum_j b_j = i$, this cokernel is precisely $\tilde{M}_{b_1} \otimes_R \dots \otimes_R \tilde{M}_{b_n}$. It is routine to verify that the colimit of the cokernel of (5.53) is consequently $\tilde{M}_{a_1} \otimes_R \dots \otimes_R \tilde{M}_{a_n}$. Consequently the cokernel of (5.53) sends $(a_1,\dots,a_n)$ to $\tilde{M}_{a_1} \otimes_R \dots \otimes_R \tilde{M}_{a_n}$, as claimed. □

Definition 5.1.4. *Let $R, n, M_0, M_1, f, \mathcal{F}$ be as in Definition 5.1.2. Let n be a nonnegative integer.*

- *Given a nonnegative integer $i < n$, write $\tilde{\operatorname{res}}_i$ for the restriction functor $\operatorname{Mod}(R)^{\mathcal{I}^n} \to \operatorname{Mod}(R)^{\mathcal{I}^n_i}$, i.e., the composite of the functors $\operatorname{res}_i \circ \dots \circ \operatorname{res}_{n-2} \circ \operatorname{res}_{n-1}$ from Definition 5.1.2. Write $\tilde{L}_i$ for its left adjoint, i.e., the composite of the functors $\operatorname{res}_{n-1} \circ \dots \circ \operatorname{res}_{i+1} \circ L_i$, also from Definition 5.1.2.*
- *Given a positive integer $i < n$, we have the natural transformation of functors $\mathcal{I}^n \to \operatorname{Mod}(R)$*

$$\tilde{L}_{i-1}\tilde{\operatorname{res}}_{i-1}\mathcal{F} \to \tilde{L}_i\tilde{\operatorname{res}}_i\mathcal{F}. \tag{5.54}$$

We write s_i for the induced map of R-modules

$$\left(\operatorname{colim} \tilde{L}_{i-1}\tilde{\operatorname{res}}_{i-1}\mathcal{F}\right)_{\Sigma_n} \to \left(\operatorname{colim} \tilde{L}_i\tilde{\operatorname{res}}_i\mathcal{F}\right)_{\Sigma_n}. \tag{5.55}$$

By the symmetric layer sequence *we mean the sequence of R-module maps*

$$\begin{tikzcd}
\mathrm{Sym}^n_R(M_0) \arrow[r, "="] & \left(\operatorname{colim} \tilde{L}_0 \tilde{\mathrm{res}}_0 \mathcal{F}\right)_{\Sigma_n} \arrow[d, "s_1"] & \\
& \left(\operatorname{colim} \tilde{L}_1 \tilde{\mathrm{res}}_1 \mathcal{F}\right)_{\Sigma_n} \arrow[d, "s_2"] & \\
& \vdots \arrow[d, "s_{n-1}"] & \\
& \left(\operatorname{colim} \tilde{L}_{n-1} \tilde{\mathrm{res}}_{n-1} \mathcal{F}\right)_{\Sigma_n} \arrow[d, "s_n"] & \\
& \left(\operatorname{colim} \mathcal{F}\right)_{\Sigma_n} \arrow[r, "="] & \mathrm{Sym}^n_R(M_1).
\end{tikzcd} \tag{5.56}$$

- *If each of the maps s_i is injective, then the symmetric layer sequence* (5.56) *is a filtration of* $\mathrm{Sym}^n_R(M_1)$*, and we call this filtration the* symmetric filtration.

Using Lemma 5.1.3 to identify the cokernels of the maps in (5.54), we have:

Proposition 5.1.5. *Let $R, n, M_0, M_1, f, \mathcal{F}$ be as in Definition 5.1.2. Suppose that each of the maps s_i in the symmetric layer sequence* (5.56) *is injective. Then the symmetric filtration is an increasing filtration of* $\mathrm{Sym}^n_R(M_1)$ *whose associated graded R-module is isomorphic to the direct sum*

$$\coprod_{i=0}^{n} \mathrm{Sym}^i_R(M_0) \otimes_R \mathrm{Sym}^{n-i}_R(\operatorname{coker} f).$$

If each of the maps s_i in the symmetric layer sequence (5.56) *is injective* for all nonnegative integers n, *then, in particular, f is injective, so we regard M_0 as a submodule of M_1, and we have an isomorphism of graded R-algebras*

$$E_0 \mathrm{Sym}^*_R(M_1) \cong \mathrm{Sym}^*_R(M_0) \otimes_R \mathrm{Sym}^*_R(M_1/M_0).$$

5.2. **The extension-of-formal-multiplications (EFM) spectral sequence.** We now use the calculations from section 4 to obtain calculations of $H^1_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes *})$ for a broad class of rings A, not just the base case $A = \mathbb{Z}$. The main tool is a spectral sequence which allows us to pass from $H^*_{fl}(\mathcal{M}^{2-buds}_{fm\mathbb{Z}}; \omega^{\otimes *})$ to $H^*_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega^{\otimes *})$. As this is a matter of passing from the cohomology of the moduli of formal group law 2-buds with a small ring of formal multiplications (namely, $\mathbb{Z}$) to the cohomology of the moduli of formal group 2-buds with a larger ring of formal multiplications (namely, A), we call this spectral sequence the *extension-of-formal-multiplications spectral sequence,* or for short, "EFM spectral sequence."

Theorem 5.2.1. *Let A be a torsion-free commutative ring. Let $A/2$ denote the reduction of the ring A modulo the ideal* (2). *Let $\tilde{\Omega}^A$ denote the free bigraded*

$A/2$-module[16] *on the set of generators $\{c_a : a \in A\}$ modulo the relations*

$$\begin{aligned} c_{a+b} &= c_a + c_b \quad \text{for all } a, b \in A, \\ c_{ab} &= ac_b + b^2 c_a \quad \text{for all } a, b \in A, \\ c_a &= 0 \quad \text{for all } a \in \mathbb{Z}. \end{aligned}$$

Then there exists a conditionally convergent spectral sequence[17]

$$(5.57) \quad \begin{aligned} E_1^{p,q,u} &\cong \begin{cases} A \otimes_{\mathbb{Z}} \operatorname{Cotor}^{p,q}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds}) & \text{if } u = 0 \\ \coprod_{i \geq 0} \operatorname{Cotor}^{p,q-2(i+u)}_{\mathbb{F}_2[t]}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \operatorname{Sym}^u_{A/2}(\tilde{\Omega}^A)\{\gamma^i\} & \text{if } u > 0 \end{cases} \\ &\Rightarrow \operatorname{Cotor}^{p,q}_{A[t]}(A, L^A_{2-buds}) \\ &\cong \operatorname{Cotor}^{p,q}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds}) \\ d_r : E_r^{p,q,u} &\to E_r^{p+1,q,u-r} \end{aligned}$$

with tridegrees as follows:

Coh. class	*Coh. degree* (p)	*Int. degree* (q)	*Filt. degree* (u)
γ	0	2	0
c_a	0	2	1
$P^n\eta$	1	2^{n+1}	0,

where the elements $\eta, P\eta, P^2\eta, \dots$ are the generators of $\operatorname{Cotor}^{1,*}_{\mathbb{F}_2[t]}(\mathbb{F}_2, \mathbb{F}_2)$, as in (4.45) in section 4.2, and η is also the generator of $\operatorname{Cotor}^{1,*}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})$, as in section 4.2.

Proof. Consider the symmetric layer sequence, as in (5.56), arising from the A-module homomorphism $A \otimes_{\mathbb{Z}} \mathrm{Dr}^{\mathbb{Z}} \hookrightarrow \mathrm{Dr}^A$. Since $A \otimes_{\mathbb{Z}} \mathrm{Dr}^{\mathbb{Z}}$ is a free A-module of rank 1, each of its symmetric powers is also a free A-module of rank 1, and the natural map

$$\operatorname{Sym}^n_A(A \otimes_{\mathbb{Z}} \mathrm{Dr}^{\mathbb{Z}}) \to \operatorname{Sym}^n_A(\mathrm{Dr}^A)$$

is simply the inclusion of the A-submodule of $(L^A)^{2n}$ generated by γ^n. Hence the maps in the symmetric layer sequence are injective. The resulting symmetric filtration of $\operatorname{Sym}^*_A(\mathrm{Dr}^A) \cong L^A_{2-buds}$ is the increasing filtration

$$F_0 L^A_{2-buds} \subseteq F_1 L^A_{2-buds} \subseteq F_2 L^A_{2-buds} \subseteq \dots$$

on L^A_{2-buds} given by letting $F_0 L^A_{2-buds}$ be the A-subalgebra of L^A_{2-buds} generated by γ, and letting $F_n L^A_{2-buds}$ be the $F_0 L^A_{2-buds}$-submodule of L^A_{2-buds} generated by all products of up to n elements c_a with $a \in A$. By the coaction formulas given in (3.30), this is a filtration by $A[t]$-subcomodules of L^A_{2-buds}. The filtration is

[16]The $A/2$-module $\tilde{\Omega}^2$ is a module of "twisted Kähler differential forms" in the sense of [11]. This is intriguing, but we do not know of any general theorems about modules of twisted Kähler differentials which give us any leverage here. Perhaps this is a reasonable direction for later investigations.

[17]The author admits to finding it difficult to visualize the spectral sequence's E_1-term merely from the description given in (5.57). We find the charts drawn below, starting in Figure 13, much more helpful for visualizing the spectral sequence.

multiplicative, exhaustive, complete, and separated, hence we have a conditionally convergent multiplicative spectral sequence

$$\begin{aligned} E_1^{p,q,u} \cong \operatorname{Cotor}^{p,q}_{A[t]}\left(A, F_u L^A_{2-buds}/F_{u-1}L^A_{2-buds}\right) &\Rightarrow \operatorname{Cotor}^{p,q}_{A[t]}(A, L^A_{2-buds}) \\ d_r : E_r^{p,q,u} &\to E_r^{p+1,q,u-r}. \end{aligned} \tag{5.58}$$

Since $\tilde{\Omega}^A \cong \mathrm{Dr}^A/(A\otimes_{\mathbb{Z}} \mathrm{Dr}^{\mathbb{Z}})$, by Proposition 5.1.5 the associated graded $A[t]$-comodule $E^0L^A_{2-buds} = \coprod_u F_u L^A_{2-buds}/F_{u-1}L^A_{2-buds}$ of the symmetric filtration is isomorphic to

$$\operatorname{Sym}^*_A(A\otimes_{\mathbb{Z}} \mathrm{Dr}^{\mathbb{Z}})\otimes_A \operatorname{Sym}^*_A(\tilde{\Omega}^A). \tag{5.59}$$

To more explicitly identify the E_1-term of spectral sequence (5.58), we need to know what happens when we apply the functor $\operatorname{Cotor}^*_{A[t]}(A,-)$ to (5.59). In general there is no particularly nice Künneth-like formula for Cotor applied to a tensor product of comodules, but in this case, we are fortunate: if $u > 0$, then the $A[t]$-coaction on $F_uL^A_{2-buds}$ lands in $F_{u-1}L^A_{2-buds}$. Consequently the $A[t]$-coaction on $\coprod_{u>0} F_uL^A_{2-buds}/F_{u-1}L^A_{2-buds}$ is trivial. Hence, for positive u, we use Proposition 5.1.5 to obtain isomorphisms

$$\begin{aligned}
\operatorname{Cotor}^{p,q}_{A[t]}\left(A, \frac{F_uL^A_{2-buds}}{F_{u-1}L^A_{2-buds}}\right) &\cong \operatorname{Cotor}^{p,q}_{A[t]}\left(A, \coprod_{i\geq 0}\operatorname{Sym}^i_A(A\otimes_{\mathbb{Z}}\mathrm{Dr}^{\mathbb{Z}})\otimes_A \operatorname{Sym}^u_A(\tilde{\Omega}^A)\right)\\
&\cong \operatorname{Cotor}^{p,q}_{A[t]}\left(A, \coprod_{i\geq 0}\operatorname{Sym}^i_A(A\otimes_{\mathbb{Z}}\mathrm{Dr}^{\mathbb{Z}})\otimes_A \operatorname{Sym}^u_{A/2}(\tilde{\Omega}^A)\right)\\
&\cong \Bigg(\coprod_{i\geq 0}\operatorname{Cotor}^{p,*}_{A/2[t]}(A/2, A/2)\\
&\qquad \otimes_{A/2}\left(A/2\otimes_{\mathbb{Z}}\operatorname{Sym}^i_{\mathbb{Z}}(\mathrm{Dr}^{\mathbb{Z}})\right)\otimes_{A/2}\operatorname{Sym}^u_{A/2}(\tilde{\Omega}^A)\Bigg)^q\\
&\cong \left(\coprod_{i\geq 0}\operatorname{Cotor}^{p,*}_{\mathbb{F}_2[t]}(\mathbb{F}_2,\mathbb{F}_2)\otimes_{\mathbb{Z}}\operatorname{Sym}^i_{\mathbb{Z}}(\mathrm{Dr}^{\mathbb{Z}})\otimes_{\mathbb{Z}}\operatorname{Sym}^u_{A/2}(\tilde{\Omega}^A)\right)^q.
\end{aligned}$$

□

To avoid any potential confusion: in the statement of Theorem 5.2.2, A/I_2^2 denotes the quotient of A by the square of its ideal I_2.

Theorem 5.2.2. *Let A be a torsion-free commutative Noetherian*[18] *ring. Then we have isomorphisms of A-modules*

$$H^s_{fl}(\mathcal{M}^{2-buds}_{fmA};\mathcal{O}) \cong \begin{cases} A & \text{if } s = 0 \\ 0 & \text{if } s \neq 0, \end{cases}$$

$$H^s_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega) \cong \begin{cases} A/I_2 & \text{if } s = 1 \\ 0 & \text{if } s \neq 1, \end{cases}$$

$$H^s_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes 2}) \cong \begin{cases} \delta(I_2) & \text{if } s = 0 \\ A/I_2^2 & \text{if } s = 1 \\ A/I_2 & \text{if } s = 2 \\ 0 & \text{otherwise}, \end{cases}$$

where $\delta(I_2)$ is the delta-invariant of the universal $\mathbb{F}_2$-point-detecting ideal I_2 of A, i.e., $\delta(I_2)$ is isomorphic to the Andre-Quillen homology group $H_2(A, A/I; A/I)$, as explained in Corollary 3.4.2.

The claim that $H^1_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes n})$ vanishes for $n \leq 0$ is quite trivial, since the Hopf algebroid $(L^A_{2-buds}, L^A_{2-buds}B)$ vanishes in negative degrees, and is easily seen to have no coalgebroid primitives in degree zero. It is the calculation of $H^1_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega)$ and of $H^1_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes 2})$ which takes a bit more work. We carry out this calculation with by running spectral sequence (5.58) in the necessary bidegrees.

Throughout the calculation, we will often begin with an element $a \in A$, and then need to consider the element $a^2 - a \in A$. We adopt the following notation, which serves to streamline the discussion: given an element $a \in A$, we will write $\hat{a}$ for the element $a^2 - a \in A$. It will become convenient later to have taken note of a few basic properties of $\hat{a}$:

$$\begin{aligned} \widehat{ab} &= \hat{a}\hat{b} + a^2 b + ab^2, \quad \text{and} \\ \hat{a}c_b + \hat{b}c_a &= 2c_{ab} - 2bc_a - 2ac_b \\ &= \hat{a}\hat{b}\gamma. \end{aligned} \tag{5.60}$$

The EFM spectral sequence differentials preserve the internal degree q, so it is convenient to carry out the further calculations in the spectral sequence by proceeding one internal degree at a time.

5.2.1. *Internal degree $q = 0$.* In negative degrees and in odd internal degrees, everything is trivial, since L^A and $L^A B$ vanish in those internal degrees. In internal degree $q = 0$, the spectral sequence is as depicted:

[18]The assumption that A is Noetherian is used only to ensure that I_2 is finitely generated, so that the results on the delta-invariant from [20] and [3] apply. If A is not assumed Noetherian, the theorem holds as stated, except for the identification of the summand $H^0_{fl}(\mathcal{M}^{2-buds}_{fmA};\omega^{\otimes 2})$ in terms of the delta-invariant.

$$\begin{array}{r|lll}
u=3 & & & \\
u=2 & & & \\
u=1 & & & \\
u=0 & A\{1\} & & \\
\hline
 & p=0 & p=1 & p=2
\end{array}$$

FIGURE 12. EFM SS
$E_1 \cong E_\infty$-page, $q = 0$
No differentials.

Bidegrees left blank are understood to be zero. Consequently we have

$$\mathrm{Cotor}^{0,0}_{A[t]}(A, L^A_{2\text{-}buds}) \cong H^0_{fl}(\mathcal{M}^{2\text{-}buds}_{fmA}; \mathcal{O}) \cong A\{1\}$$

trivially, for degree reasons.

5.2.2. *Internal degree $q = 2$.* The EFM spectral sequence is as depicted:

$$\begin{array}{r|lll}
u=3 & & & \\
u=2 & & & \\
u=1 & \tilde{\Omega}_A & & \\
u=0 & & A/2\{\eta\} & \\
\hline
 & p=0 & p=1 & p=2
\end{array}$$

FIGURE 13. EFM SS
E_1-page, $q = 2$
$d_1(c_a) = \hat{a}\eta$

$$\begin{array}{r|lll}
u=3 & & & \\
u=2 & & & \\
u=1 & & & \\
u=0 & & A/I_2\{\eta\} & \\
\hline
 & p=0 & p=1 & p=2
\end{array}$$

FIGURE 14. EFM SS
$E_2 \cong E_\infty$-page, $q = 2$

For degree reasons, $d_r(\eta) = 0$ for all r. By the coaction map (3.30), we have the differential $d_1(c_a) = \hat{a}\eta$ in the spectral sequence; this is the differential drawn in red in the $q = 2$ diagram in Figure 13. Consequently $\mathrm{Cotor}^{1,2}_{A[t]}(A, L^A_{2\text{-}buds})$ is a free A/I_2-module on the generator η. We also have that $\mathrm{Cotor}^{0,2}_{A[t]}(A, L^A_{2\text{-}buds})$ is the kernel of that d_1-differential, but this Cotor-group was calculated already in section 3.4.

5.2.3. *Internal degree $q = 4$.* We use isomorphism (4.49), obtained from the 2-adic spectral sequence, to identify the $u = 0$-line in the EFM spectral sequence E_1-term:

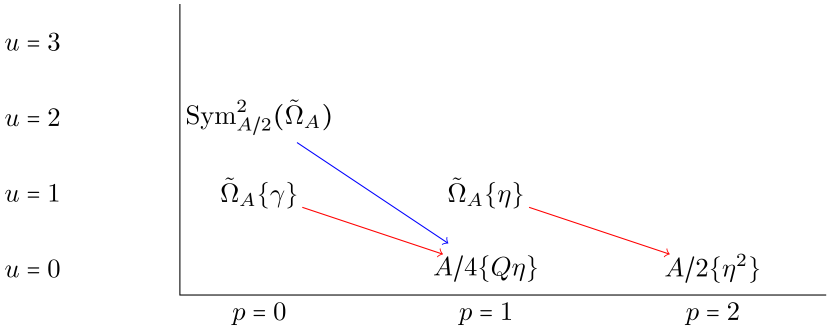


FIGURE 15. EFM SS E_1-page, $q = 4$
$d_1(c_a\eta) = \hat{a}\eta^2$ $\quad d_1(c_a\gamma) = 2\hat{a}Q\eta$
$d_1(c_ac_b) = 0$ $\quad d_2(c_ac_b) = \hat{a}\hat{b}Q\eta$

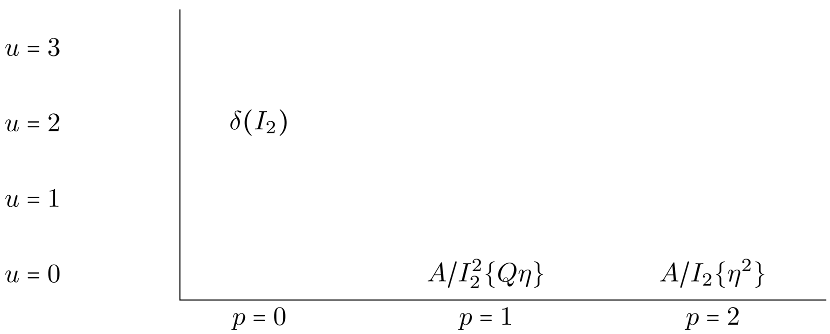


FIGURE 16. EFM SS E_∞-page, $q = 4$

The differential $d_1 : \mathrm{Sym}^2_{A/2}(\tilde{\Omega}_A) \to \tilde{\Omega}_A\{\eta\}$ vanishes due to the equalities

$$\begin{aligned} d_1(c_ac_b) &= (\hat{a}c_b + \hat{b}c_a)\eta \\ &= ((a^2c_b + bc_a) + (ac_b + b^2c_a))\eta \\ &= (c_{ab} + c_{ab})\eta \\ &= 0 \end{aligned} \tag{5.61}$$

in the associated graded $E^0L^A_{2-buds}$ of the symmetric filtration on L^A_{2-buds}. Meanwhile, by the Leibniz rule, the differential $d_1 : \tilde{\Omega}_A\{\eta\} \to A \otimes_{\mathbb{Z}} \mathrm{Cotor}^{2,4}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})$ is merely η times the $q = 2$ d_1-differential $\tilde{\Omega}_A \to A \otimes_{\mathbb{Z}} \mathrm{Cotor}^{1,2}_{\mathbb{Z}[t]}(\mathbb{Z}, L^{\mathbb{Z}}_{2-buds})$. Since $t\otimes t\otimes 1$ is not a coboundary in the cobar complex of $\mathbb{Z}[t]$ with coefficients in $L^{\mathbb{Z}}_{2-buds}$, η^2 is nonzero, although $2\eta^2 = 0$. Hence the kernel of $d_1 : E_1^{1,4,1} \to E_1^{2,4,0}$ is the kernel of the A-module homomorphism

$$\begin{aligned} \tilde{\Omega}_A\{\eta\} &\to A/2\{\eta^2\} \\ c_a\eta &\mapsto \hat{a}\eta^2. \end{aligned}$$

Since $d_1(c_ac_b)$ was already shown to vanish, the kernel of $d_1 : E_1^{1,4,1} \to E_1^{2,4,0}$ is $E_2^{1,4,1}$. Hence we have an isomorphism

$$\begin{aligned} E_2^{0,2,1} &\xrightarrow{\cong} E_2^{1,4,1} \\ x &\mapsto x\eta. \end{aligned}$$

Since $E_2^{0,2,1} \cong \mathrm{Cotor}^{0,2}_{L^A_{2-buds}B}(L^A_{2-buds}, L^A_{2-buds})$ is trivial by Theorem 3.4.1, $E_2^{1,4,1}$ also is trivial.

We have the d_2-differential $E_2^{0,4,2} \to E_2^{1,4,0}$ given by the cobar complex calculation

$$d_2 : \mathrm{Sym}^2_{A/2}(\tilde{\Omega}_A) \to E_2^{1,4,0}$$

$$\begin{aligned}
d_2(c_ac_b) &= \left[(c_a \otimes 1 + \hat{a} \otimes t)(c_b \otimes 1 + \hat{b} \otimes t) - c_ac_b \otimes 1\right] && (5.62)\\
&= \left[(\hat{a}c_b + \hat{b}c_a) \otimes t + \hat{a}\hat{b} \otimes t^2\right] \\
&= \hat{a}\hat{b}\left[\gamma \otimes t + 1 \otimes t^2\right] && (5.63)\\
&= \hat{a}\hat{b}Q\eta,
\end{aligned}$$

with (5.63) a consequence of (5.60). Hence we have isomorphisms

$$\begin{aligned} \mathrm{Cotor}^{1,4}_{A[t]}(A, L^A_{2-buds}) &\cong E_3^{1,4,0} \\ &\cong E_\infty^{1,4,0} \\ &\cong A/I_2^2\{Q\eta\}. \end{aligned}$$

As a corollary of Remark 2.4.3 and Theorem 5.2.2, we have:

Corollary 5.2.3. *Let A be a Noetherian integral domain of characteristic zero. Then $H^1_{fl}(\mathcal{M}^{2-buds}_{fmA}; \omega)$ is a finite abelian group of order equal to 2^{N_1}, where N_1 is the number of $\mathbb{F}_2$-points of* Spec A*, i.e., the logarithmic derivative of the 2-local zeta-function $Z(\operatorname{Spec} A, t)$ evaluated at $t = 0$.*

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